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$K_n \square P$ IS RADIO GRACEFUL

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For G a simple, connected graph, a vertex labeling $f : V(G) \to \mathbb{Z}_+$ is called a *radio labeling of* G if it satisfies $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$ for all distinct vertices $u, v \in V(G)$. If a bijective radio labeling onto $\{1, 2, \ldots, |V(G)|\}$ exists, G is called a *radio graceful* graph. In this paper, we show $K_n \square P$ is radio graceful.

1. Introduction

For a positive integer k and G a simple, connected graph with vertex set $V(G)$, a vertex labeling $f : V(G) \to \mathbb{Z}_+$, is a *$k$-radio labeling of* G if it satisfies

$$|f(u) - f(v)| \geq k + 1 - d(u, v)$$

for all distinct $u, v \in V(G)$. This labeling[1] was first defined in [2]. It is a generalization of labelings that had been defined previously and continue to be studied, most of which were developed in response to the problem of optimally assigning frequencies to radio transmitters. Originally posed as a graph theory problem by Hale in 1980 ([4]), transmitters are modeled by vertices and the labeling represents the assignment of frequencies. This definition would then require that the

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[1] We use a codomain of $\mathbb{Z}_+$ for k-radio labelings, while some authors use a codomain of $\mathbb{Z}_+ \cup \{0\}$. Labelings and results under one convention can be easily converted to the other convention by the appropriate shift by 1.

difference in frequency increases as the distance between transmitters decreases, which is the desirable relationship to avoid interference between transmitters[2]. In this paper we establish an important example of k-radio labeling.

Specializations of this definition of k-radio labeling include vertex coloring (1-radio labeling), $L(2,1)$-labeling (2-radio labeling, [3]), $L(3,2,1)$-labeling (3-radio labeling, [10]), and radio labeling, which is k-radio labeling with $k = \text{diam}(G)$ ([1]). Note the following requirement of any k-radio labeling f. For any pair of distinct vertices $u, v \in V(G)$ with $d(u,v) \leq k$, it is in particular the case that $f(u) \neq f(v)$. Since k has this natural association with distance in the graph, we consider $k = \text{diam}(G)$ to be the maximum choice for k. This is the case for radio labeling, which is the labeling we work with in this paper.

Remark 1.1. For graphs of diameter 3, like the graphs in this paper, radio labeling and $L(3,2,1)$-labeling coincide. The main result is first framed in the context of radio labeling, then restated in the language of $L(3,2,1)$-labeling.

Definition 1.2. Let G be a simple, connected graph. A vertex labeling f of G, $f : V(G) \to \mathbb{Z}_+$, is a *radio labeling of* G if it satisfies

$$|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u,v) \tag{1}$$

for all distinct $u, v \in V(G)$.

Inequality (1) is called the *radio condition*. The largest element in the image of a labeling f is called the *span of* f.

Definition 1.3. For a graph G, the minimal span over all radio labelings of G is called the *radio number of* G, denoted $\text{rn}(G)$.

Formulas, or bounds, for the radio numbers of graphs are sought. One general bound is $\text{rn}(G) \geq |V(G)|$, which follows from the injectivity of radio labeling. We are most interested in finding graphs for which equality holds: $\text{rn}(G) = |V(G)|$; we will call such graphs *radio graceful*, a term first introduced in [11]. For any radio labeling $f : V(G) \to \mathbb{Z}_+$ of G, it is true that $|f(V(G))| = |V(G)|$. However, for the span of f to be $|V(G)|$, the image $f(V(G))$ must be the set of consecutive integers $\{1, 2, \ldots, |V(G)|\}$. This is a very restrictive condition for a radio labeling to satisfy, and it leads to the following definition.

Definition 1.4. A radio labeling f of a graph G is a *consecutive radio labeling of* G if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. A graph for which a consecutive radio labeling exists is called *radio graceful*.

[2]While the historical origin of the labeling is still mathematically relevant, k-radio labeling is now studied independently, and considers examples that are not practical as frequency assignment models.

The complete graphs K_n are a trivial family of radio graceful examples; trivial because any injective labeling of K_n satisfies the radio condition. In particular, any labeling of consecutive integers of K_n is a consecutive radio labeling. A well-known, nontrivial radio graceful example is the Petersen graph P.

The first infinite, nontrivial family of radio graceful examples was given in [9]; in this paper we give another infinite family of examples. The result was inspired by two former results, both involving the Cartesian product of graphs: Tomova and Wyels found that $P \square P$ is radio graceful ([12]), and Niedzialomski found that $K_n \square K_n$ for $n \geq 3$ is radio graceful ([9]).

Theorem 1.5. *Let $n \in \mathbb{Z}_+$. $K_n \square P$ is radio graceful.*

Remark 1.6. Theorem 1.5 could be restated as the following. Let $n \in \mathbb{Z}_+$. The $L(3,2,1)$-labeling number[3] of $K_n \square P$ is $10n$.

The Cartesian product has proven to be a helpful tool in constructing radio graceful graphs, but it has some definite limitations. For example, one may wonder from these results if G and H radio graceful implies $G \square H$ is radio graceful, but this is not true ([9]). Results pertaining to radio labeling Cartesian graph products include [5], [7]-[9], and [12]; see [6] for related results specific to $L(3,2,1)$-labeling.

2. Preliminaries

Graphs are assumed simple and connected. We denote the distance between vertices x and y in a graph G by $d_G(x,y)$, or, if G is clear from context, by $d(x,y)$. We use the convention that $a \pmod n \in \{1,2,\ldots,n\}$ throughout.

Diameter affects what is required for a graph to be radio graceful, which the below proposition illustrates. If f is a consecutive radio labeling of G, then there is an ordering[4] of its vertices $x_1, x_2, \ldots, x_n$ such that $f(x_i) = i$ for all $i \in \{1,2,\ldots,n\}$. Using this, a straightforward rearrangement of the radio condition gives the following.

Proposition 2.1. *A graph G is radio graceful if and only if there is an ordering $x_1, x_2, \ldots, x_n$ of its vertices such that*

$$d(x_i, x_{i+\Delta}) \geq \operatorname{diam}(G) - \Delta + 1 \tag{2}$$

for all $\Delta \in \{1,2,\ldots,\operatorname{diam}(G)-1\}$, $i \in \{1,2,\ldots,n-\Delta\}$.

[3] The $L(3,2,1)$-labeling number of a graph G is analogous to the radio number of G (and equal to the radio number when $\operatorname{diam}(G) = 3$). It is defined as the minimal span over all $L(3,2,1)$-labelings of G.

[4] An ordering of $V(G)$ is an ordered list of the vertices of G without repetition or exclusion.

We call inequality (2) the *radio graceful condition*. Observe that the radio graceful condition is automatically satisfied if $\operatorname{diam}(G) \leq 1$, which is another way of understanding the triviality of the K_n example. As diameter increases, more values of Δ must be considered. Figure 1 shows an ordering $w_1, w_2, \ldots, w_{10}$ of the vertices of the Petersen graph that satisfies the radio graceful condition.

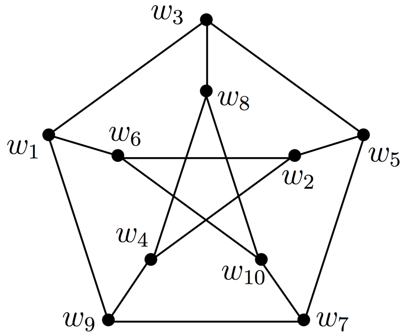

Figure 1: P with an ordering of its vertices satisfying the radio graceful condition.

Definition 2.2. The *Cartesian product of graphs G and H*, denoted $G\Box H$, has vertex set $V(G)\times V(H)$, and has edges defined by the following property. Vertices $(u,v),(u',v') \in V(G\Box H)$ are adjacent if

1. $u = u'$ and v is adjacent to v' in H, or
2. $v = v'$ and u is adjacent to u' in G.

For a vertex $x_i = (u,v) \in V(G\Box H)$, we will refer to (u,v) as the *coordinate representation* of x_i.

Distance in $G\Box H$ is dependent on corresponding distances in G and H:

$$d_{G\Box H}((u,v),(u',v')) = d_G(u,u') + d_H(v,v').$$

It follows that $\operatorname{diam}(G\Box H) = \operatorname{diam}(G) + \operatorname{diam}(H)$. For the majority of this paper, we will be investigating $K_n\Box P$ for $n \geq 3$, in which case $\operatorname{diam}(K_n\Box P) = \operatorname{diam}(K_n) + \operatorname{diam}(P) = 3$. (The $n = 1$ case is trivial, and the $n = 2$ case is handled separately, at the end of Section 5.)

Our method for proving the main result will be to define a list of vertices of $K_n\Box P$, prove that the list is an ordering of the vertices of $K_n\Box P$, then prove that the ordering satisfies the radio graceful condition, in which case $K_n\Box P$ is radio graceful by Proposition 2.1.

3. Definition of a list $x_1, x_2, \ldots, x_{10n}$ of vertices of $K_n \square P$

Let $w_1, w_2, \ldots, w_{10}$ be the ordering of the vertices of P given in Figure 1, and let $v_1, v_2, \ldots, v_n$ be any ordering of the vertices of K_n, $n \geq 3$. We will define a list $x_1, x_2, \ldots, x_{10n}$ of the vertices of $K_n \square P$, organized as rows of 10×2 matrices. The first of these matrices is

$$A_1 = \begin{bmatrix} v_1 & w_1 \\ \sigma(v_1) & w_2 \\ \sigma^2(v_1) & w_3 \\ \sigma^3(v_1) & w_4 \\ \sigma^4(v_1) & w_5 \\ \sigma^5(v_1) & w_6 \\ \sigma^6(v_1) & w_7 \\ \sigma^7(v_1) & w_8 \\ \sigma^8(v_1) & w_9 \\ \sigma^9(v_1) & w_{10} \end{bmatrix}$$

where $\sigma \in S_{V(K_n)}$ is the n-cycle $(v_1 \ v_2 \ \cdots \ v_n)$. We will define a total of n matrices. This task will be helped by thinking of each of these matrices in terms of its columns. Let $A_1 = \begin{bmatrix} C_1 & D_1 \end{bmatrix}$. For $1 < k \leq n$, let

$$A_k = \begin{bmatrix} C_k & D_k \end{bmatrix} = \begin{cases} \begin{bmatrix} \sigma^{11}(C_{k-1}) & D_1 \end{bmatrix} & \text{if } k \equiv 1 \ \left(\text{mod } \frac{1}{10}\text{lcm}(n,10)\right) \\ \\ \begin{bmatrix} \sigma^{10}(C_{k-1}) & D_1 \end{bmatrix} & \text{otherwise} \end{cases}.$$

Notice that the second columns of all n matrices are identical. The matrices give the list of vertices in the natural way: Let $A_k = [a^k_{i,j}]$. If $h = 10b + c$, where $c \in \{1, 2, \ldots, 10\}$, then x_h is $\left(a^{b+1}_{c,1}, a^{b+1}_{c,2}\right)$. See Table 1 for an example of this list for $K_4 \square P$.

4. The List $x_1, x_2, \ldots, x_{10n}$ is an ordering

In Section 3, we defined n matrices (each of dimensions 10×2) which gave us a list with $10n$ vertices. As $K_n \square P$ also has $10n$ vertices, we only need to check that this list has no repetition to show that it is an ordering of $V(K_n \square P)$. First, three observations from the definition of $x_1, x_2, \ldots, x_{10n}$.

1. For fixed k, A_k can have no repeated rows because the definition of its second column D_k.

$x_1 = (v_1, w_1)$	$x_{11} = (v_3, w_1)$	$x_{21} = (v_2, w_1)$	$x_{31} = (v_4, w_1)$
$x_2 = (v_2, w_2)$	$x_{12} = (v_4, w_2)$	$x_{22} = (v_3, w_2)$	$x_{32} = (v_1, w_2)$
$x_3 = (v_3, w_3)$	$x_{13} = (v_1, w_3)$	$x_{23} = (v_4, w_3)$	$x_{33} = (v_2, w_3)$
$x_4 = (v_4, w_4)$	$x_{14} = (v_2, w_4)$	$x_{24} = (v_1, w_4)$	$x_{34} = (v_3, w_4)$
$x_5 = (v_1, w_5)$	$x_{15} = (v_3, w_5)$	$x_{25} = (v_2, w_5)$	$x_{35} = (v_4, w_5)$
$x_6 = (v_2, w_6)$	$x_{16} = (v_4, w_6)$	$x_{26} = (v_3, w_6)$	$x_{36} = (v_1, w_6)$
$x_7 = (v_3, w_7)$	$x_{17} = (v_1, w_7)$	$x_{27} = (v_4, w_7)$	$x_{37} = (v_2, w_7)$
$x_8 = (v_4, w_8)$	$x_{18} = (v_2, w_8)$	$x_{28} = (v_1, w_8)$	$x_{38} = (v_3, w_8)$
$x_9 = (v_1, w_9)$	$x_{19} = (v_3, w_9)$	$x_{29} = (v_2, w_9)$	$x_{39} = (v_4, w_9)$
$x_{10} = (v_2, w_{10})$	$x_{20} = (v_4, w_{10})$	$x_{30} = (v_3, w_{10})$	$x_{40} = (v_1, w_{10})$

Table 1: The list $x_1, x_2, \dots, x_{40}$ for $K_4 \Box P$.

2. If two rows in matrices A_j and A_k are identical, they must be located as the same row of both matrices because they have the same second columns ($D_j = D_k = D_1$).

3. Further, if two rows in matrices A_j and A_k are identical, then $A_j = A_k$ because of the cyclical nature of their first columns.

Therefore, our task reduces to proving that no two matrices A_j and A_k share their first entries. In light of this, we create a vector A consisting of only these entries,

$$A = [a_i] \text{ where } a_i = \begin{cases} v_1 & \text{if } i = 1 \\ \sigma^{11}(a_{i-1}) & \text{if } i \equiv 1 \pmod{\frac{1}{10}\mathrm{lcm}(n, 10)}, \\ \sigma^{10}(a_{i-1}) & \text{otherwise} \end{cases}$$

and we set out to prove A has no repeated entries. We start by dividing A into "blocks" that allow us to further understand the patterns within A.

Definition. Let $\ell = \frac{1}{10}\mathrm{lcm}(n, 10)$. A *block* is any one of the vectors

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_\ell \end{bmatrix}, \begin{bmatrix} a_{\ell+1} \\ a_{\ell+2} \\ \vdots \\ a_{2\ell} \end{bmatrix}, \dots, \begin{bmatrix} a_{n-\ell+1} \\ a_{n-\ell+2} \\ \vdots \\ a_n \end{bmatrix}.$$

We call $\begin{bmatrix} a_{(k-1)\ell+1} \\ a_{(k-1)\ell+2} \\ \vdots \\ a_{k\ell} \end{bmatrix}$ the k^{th} block, denoted β^k.

This divides A into n/ℓ blocks, each of size ℓ.

Proposition 4.1. *If the first entry of β^k is b_1^k, then* $\beta^k = \begin{bmatrix} b_1^k \\ \sigma^{10}(b_1^k) \\ \sigma^{20}(b_1^k) \\ \vdots \\ \sigma^{10(\ell-1)}(b_1^k) \end{bmatrix}$.

Proof. Let $\beta^k = \begin{bmatrix} b_1^k \\ b_2^k \\ \vdots \\ b_\ell^k \end{bmatrix}$, which by definition equals $\begin{bmatrix} a_{(k-1)\ell+1} \\ a_{(k-1)\ell+2} \\ \vdots \\ a_{k\ell} \end{bmatrix}$. Consider b_i^k, $k \in \{2,\dots,\ell\}$, which equals $a_{(k-1)\ell+i}$. Since $(k-1)\ell+i \equiv i \pmod{\ell}$, $a_{(k-1)\ell+i} = \sigma^{10}(a_{(k-1)\ell+i-1})$ by definiton of A. This gives the desired result. $\Box$

Proposition 4.2. *The first entry of β^k is v_k for all $1 \le k \le n/\ell$.*

Proof. Let $\beta^k = \begin{bmatrix} b_1^k \\ b_2^k \\ \vdots \\ b_\ell^k \end{bmatrix}$. We proceed by induction, noting that $b_1^1 = v_1$ by definition. Suppose $b_1^k = v_k$. Then, by Proposition 4.1,

$$\beta^k = \begin{bmatrix} v_k \\ \sigma^{10}(v_k) \\ \sigma^{20}(v_k) \\ \vdots \\ \sigma^{10(\ell-1)}(v_k) \end{bmatrix}. \tag{3}$$

By the definition of block, $b_1^{k+1} = a_{k\ell+1}$, which equals $\sigma^{11}(a_{k\ell})$ by the definition of A. Again by the definition of block, $\sigma^{11}(a_{k\ell}) = \sigma^{11}(b_\ell^k)$, which is $\sigma^{11}(\sigma^{(10(\ell-1)}(v_k)) = \sigma^{10\ell+1}(v_k)$ by (3).

Since $10\ell+1 = 10 \cdot \frac{1}{10}\mathrm{lcm}(n,10)+1 \equiv 1 \pmod{n}$ and since σ is an n-cycle,

$$\sigma^{10\ell+1}(v_k) = \sigma(v_k) = v_{k+1}.$$

Therefore, $b_1^{k+1} = v_{k+1}$. $\Box$

Proposition 4.3. *All entries of β^k have the form v_q where $q \equiv k \pmod{n/\ell}$.*

Proof. From Propositions 4.1 and 4.2, we know

$$\beta^k = \begin{bmatrix} v_k \\ \sigma^{10}(v_k) \\ \sigma^{20}(v_k) \\ \vdots \\ \sigma^{10(\ell-1)}(v_k) \end{bmatrix}, \text{ or equivalently, } \begin{bmatrix} v_k \\ v_{k+10 \pmod n} \\ v_{k+20 \pmod n} \\ \vdots \\ v_{k+10(\ell-1) \pmod n} \end{bmatrix}.$$

So all entries of β^k have the form v_q with

$$q \in \{k, k+10 \pmod n, k+20 \pmod n, \ldots, k+10(\ell-1) \pmod n\}.$$

We make two notes.

1. Because n/ℓ is a divisor of n, $[x \pmod n] \pmod{n/\ell} = x \pmod{n/\ell}$.
2. The number n/ℓ is also a divisor of 10. Indeed, $10 \div n/\ell = \frac{\operatorname{lcm}(n,10)}{n} \in \mathbb{Z}$.

It follows from these two notes that $q \equiv k \pmod{n/\ell}$ for all

$$q \in \{k, k+10 \pmod n, k+20 \pmod n, \ldots, k+10(\ell-1) \pmod n\}.$$

□

Proposition 4.4. *The entries of β^k are all distinct.*

Proof. In search of contradiction, suppose there exist i and j, $i \neq j$, with $b_i^k = b_j^k$. Then $b_i^k = v_{k+10a \pmod n}$ and $b_j^k = v_{k+10b \pmod n}$ for distinct $a, b \in \{0, 1, \ldots, \ell-1\}$ (Propositions 4.1 and 4.2), and

$$k + 10a \equiv k + 10b \pmod n. \tag{4}$$

Because of (4), $10(a-b) \equiv 0 \pmod n$, and there exists an integer c such that

$$10(a-b) = cn. \tag{5}$$

We also know that $(a-b) \in \{1, 2, \ldots, \ell-1\}$ since $a, b \in \{0, 1, \ldots, \ell-1\}$ are distinct, and therefore $10(a-b) \in \{10, 20, \ldots, 10(\ell-1)\}$. Using (5) and the definition $\ell = \frac{1}{10}\operatorname{lcm}(n, 10)$, we can rewrite this as

$$cn \in \{10, 20, \ldots, \operatorname{lcm}(n,10) - 10\}. \tag{6}$$

However, as the set $\{10, 20, \ldots, \operatorname{lcm}(n,10) - 10\}$ contains only positive multiples of 10 less than $\operatorname{lcm}(n, 10)$, it cannot also contain a multiple of n. Therefore $cn \notin \{10, 20, \ldots, \operatorname{lcm}(n,10) - 10\}$, which is a contradiction. □

To summarize, recall that blocks β^k are defined for $1 \le k \le n/\ell$; it then follows immediately from Proposition 4.3 that there are no repeated entries between different blocks. Proposition 4.4 shows that there are no repeated entries within a single block. Therefore, A has no repeated entries. As we argued at the beginning of this section, this implies $x_1, x_2, \ldots, x_{10n}$ is an ordering of the vertices of $K_n \square P$.

Theorem 4.5. *The list $x_1, x_2, \ldots, x_{10n}$ defined in Section 3 is an ordering of the vertices of $K_n \square P$, $n \ge 3$.*

5. $K_n \square P$ is radio graceful

Now to show our ordering $x_1, x_2, \ldots, x_{10n}$ does satisfy the radio graceful condition, which is done over several cases.

Theorem 5.1. *Let $n \in \mathbb{Z}_+$, $n \ge 3$. Then $K_n \square P$ is radio graceful.*

Proof. Let $n \ge 3$, let $x_1, x_2, \ldots, x_{10n}$ be the ordering of $V(K_n \square P)$ from Section 4, and let (a_i, b_i) be the coordinate representation of x_i for all i. We will show $d(x_i, x_{i+\Delta}) \ge 4 - \Delta$ for all $\Delta \in \{1, 2\}$, $i \in \{1, 2, \ldots, 10n - \Delta\}$, and therefore $K_n \square P$ is radio graceful by Proposition 2.1. We refer again to the matrix A_k as defined in Section 3. Let x_i be a row in A_k. Since $\Delta \le 2$, $x_{i+\Delta}$ is either a row in A_k, or it is a row in A_{k+1}.

Case 1. $\Delta = 1$, $x_{i+\Delta}$ in A_k.

Suppose $\Delta = 1$, and $x_{i+\Delta}$ is also a row in A_k. Then $a_{i+\Delta} = \sigma(a_i)$, and $d_{K_n}(a_i, \sigma(a_i)) = 1$. By the definition of A_k, $d_P(b_i, b_{i+\Delta}) = 2$, so $d_{K_n \square P}(x_i, x_{i+\Delta}) = 3 = 4 - \Delta$.

Case 2. $\Delta = 1$, $x_{i+\Delta}$ in A_{k+1}.

Suppose $\Delta = 1$, and $x_{i+\Delta}$ is a row in A_{k+1}. Then $a_{i+\Delta} = \sigma(a_i)$ or $a_{i+\Delta} = \sigma^2(a_i)$. Either way, $d_{K_n}(a_i, a_{i+\Delta}) = 1$ since $n \ge 3$. Also, as $b_i = w_{10}$ and $b_{i+\Delta} = w_1$, $d_P(b_i, b_{i+\Delta}) = 2$. Then $d_{K_n \square P}(x_i, x_{i+\Delta}) = 3 = 4 - \Delta$.

Case 3. $\Delta = 2$, $x_{i+\Delta}$ in A_k.

Suppose $\Delta = 2$, and $x_{i+\Delta}$ is also a row in A_k. Then $a_{i+\Delta} = \sigma^2(a_i)$, and since $n \ge 3$, $d_{K_n}\left(a_i, \sigma^2(a_i)\right) = 1$. No entries in the second column of A_k are repeated, so $d_P(b_i, b_{i+\Delta}) \ge 1$. Hence, $d_{K_n \square P}(x_i, x_{i+\Delta}) \ge 2 = 4 - \Delta$.

Case 4. $\Delta = 2$, $x_{i+\Delta}$ in A_{k+1}, $n \ge 4$.

Suppose $\Delta = 2$, $x_{i+\Delta}$ is a row in A_{k+1}, and $n \ge 4$. Then $a_{i+\Delta} = \sigma^2(a_i)$ or $a_{i+\Delta} = \sigma^3(a_i)$. Either way, $d_{K_n}(a_i, a_{i+\Delta}) = 1$ since $n \ge 4$. As for the second coordinate, either $b_i = w_{10}$ and $b_{i+\Delta} = w_2$, or $b_i = w_9$ and $b_{i+\Delta} = w_1$. So, $d_P(b_i, b_{i+\Delta}) \ge 1$. Then $d_{K_n \square P}(x_i, x_{i+\Delta}) \ge 2 = 4 - \Delta$.

Case 5. $\Delta = 2$, $x_{i+\Delta}$ in A_{k+1}, $n = 3$.

Suppose $\Delta = 2$, $x_{i+\Delta}$ is a row in A_{k+1}, and $n = 3$. Recall, $\ell = \frac{1}{10}\text{lcm}(n,10) = 3$ in this case. So, in constructing each A_k matrix for $K_3\square P$, $k \not\equiv 1 \pmod{\ell}$ for all $k > 1$. Therefore, $a_{i+\Delta} = \sigma^2(a_i)$ and $d_{K_n}(a_i, a_{i+\Delta}) = 1$ since $n = 3$. For the same reason as in Case 4, $d_P(b_i, b_{i+\Delta}) \geq 1$. Therefore, $d_{K_n\square P}(x_i, x_{i+\Delta}) \geq 2 = 4 - \Delta$. $\square$

As can be seen multiple times in this proof, the $n \geq 3$ hypothesis is used. The method described in Section 3 to produce an ordering of the vertices of $K_n\square P$ for $n \geq 3$ which satisfies the radio graceful condition does not work when $n = 2$. However, $K_2\square P$ is radio graceful, as shown by the ordering in Figure 2. We are very appreciative of Dr. Jason DeVito, who found and shared this ordering with us. With the $n = 2$ case settled, Theorem 1.5 is established.

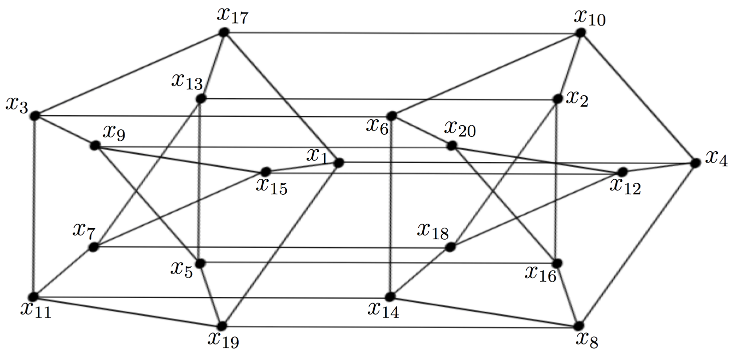

Figure 2: Ordering of $K_2\square P$ satisfying the radio graceful condition.

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