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EXISTENCE RESULTS FOR A CLASS OF NON-UNIFORMLY ELLIPTIC EQUATIONS

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We present existence results for a class of non-uniformly elliptic problems whose prototype is

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(a(x)+u)^{\alpha}}\right) = u^s + f & \text{in } \Omega \\ u(x) \geq 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 3$, $a(x)$ is a measurable function belonging to $L^\infty(\Omega)$ such that $0 < a_1 \leq a(x) \leq a_2$, for a. e. $x \in \Omega$ with a_1 and a_2 positive constants. Moreover we assume that α and s are real numbers such that $0 \leq \alpha < 1$ and $0 \leq s < 1 - \alpha$. Finally we assume that the datum f belongs to Lebesgue spaces $L^m(\Omega)$ where m varies in suitable intervals. We further present an existence result for nontrivial solutions to problem (0.1) when $f \equiv 0$.

1. Introduction.

We present some recent results proved in paper [18]. They concern with the existence of solutions to the following elliptic problem

$$\begin{cases} -\operatorname{div}(A(x,u)\nabla u) = u^s + f(x) & \text{in } \Omega \\ u(x) \geq 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 3$ and $A(x, t)$ is a matrix whose coefficients are Carathéodory functions $A_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ belonging to $L^\infty(\Omega \times \mathbb{R})$. Moreover we assume that the matrix $A(x, t)$ satisfies the following ellipticity condition

$$\frac{c_0}{(a(x) + |t|)^\alpha}|\xi|^2 \leq \langle A(x, t)\xi, \xi\rangle \leq c_1|\xi|^2, \tag{1.2}$$

for a. e. $x \in \Omega$, $\forall t \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, for some constants $c_0 > 0$, $c_1 > 0$ for a constant α such that

$$0 \leq \alpha < 1, \tag{1.3}$$

and for a function $a(x) \in L^\infty(\Omega)$ which satisfies the condition

$$0 < a_1 \leq a(x) \leq a_2, \quad \text{a. e. } x \in \Omega, \quad a_1,\ a_2 > 0\,. \tag{1.4}$$

Furthermore we assume that s is a real number such that

$$0 \leq s < 1 - \alpha, \tag{1.5}$$

and the datum f is a nonnegative function on Ω belonging to some Lebesgue space, i.e.

$$f \in L^m(\Omega), \qquad f(x) \geq 0 \quad \text{a. e. } x \in \Omega, \tag{1.6}$$

for suitable values of m which will be specified later.

We are interested in existence results for problem (1.1) when $f \not\equiv 0$ or when $f \equiv 0$.

The main features of problem (1.1) are the non-uniformly ellipticity condition (1.2), which produces a lack of coercivity when u is large, and the presence of the semilinear term u^s. We explicitely remark that the operator $-\text{div}(A(x, u)\nabla u)$ though well-defined between $W_0^{1,2}(\Omega)$ and $W^{-1,2}(\Omega)$ is not coercive in $W_0^{1,2}(\Omega)$ when u is large. Evidently if u is bounded then the operator becomes coercive and classical theory can be applied in order to prove existence of a weak solution. However in general the boundedness of u or is not true either couldn't be guaranteed *a priori*.

In [18] we prove three existence results for nonhomogeneus problem (1.1). Depending on the summability of the datum f, we prove the existence of a weak solution u such that u^σ belongs to $W_0^{1,2}(\Omega)$ for a suitable σ, the existence of a solution in distributional sense which in general belongs to a suitable

Sobolev space larger then $W_0^{1,2}(\Omega)$ and the existence of a renormalized solution. In Section 2 we will present the first and the second existence results. The definition of renormalized solution has been introduced by P.-L. Lions and F. Murat ([16], [19]), while an equivalent definition of solution, the entropy solution, has been introduced in [6].

In the case where $f \equiv 0$, the model problem of the general setting considered above is the following one

$$\tag{1.7} \begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(a(x)+u)^\alpha}\right) = u^s & \text{in } \Omega \\ u(x) \geq 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a(x)$ is a function belonging to $L^\infty(\Omega)$ such that $0 < a_1 \leq a(x) \leq a_2$ for suitable positive constants a_1 and a_2. In such a case the matrix $A(x,t) \equiv \big(A_{ij}(x,t)\big)$ is given by

$$A(x,t) = \frac{1}{(a(x)+t)^\alpha} I_{N\times N} \ .$$

When $\alpha = 0$ the elliptic problem (1.7) has a variational structure and both existence and uniqueness of a nontrivial solution are well-known (see, for example, [2] and some extensions in [1]). Actually a variational approch to problem (1.7) is also possible when $\alpha > 0$, but such variational formulation does not hold for general operators $-\operatorname{div}((A(x,u)\nabla u))$ and a different approach is needed. In Section 3 we will present our existence result for problem (1.1) when f is identically zero.

When the term u^s does not appear in (1.1), this type of problems has been studied by many authors. In [5] and [10] the authors prove both existence and regularity (depending on the summability of the datum f) of weak solutions, while in [4] the existence and regularity of weak solutions and entropy solutions in a nonlinear case are proved. Existence and uniqueness results for renormalized solution for the class of problems (1.1) in the case where the term u^s does not appear have been proved , for example, in [7], [8] or in [20].

We also mention the papers [13] (where the higher integrability of the gradients is studied), [14] (where the case of the datum in divergence form is considered), [11] and [17] (where noncoercive functionals related to such type of equations are studied).

Finally we briefly make some remarks on the bounds on N and α. We assume that $N \geq 3$; the case $N = 2$ is excluded, for simplicity, since it leads to technicalities due to the fact that Sobolev embedding Theorem have to be replaced by Moser-Trudinger Theorem. Moreover we assume that $\alpha < 1$. Such a condition on α is not restrictive since, when $\alpha = 1$, then $s = 0$, i.e. $u^s \equiv 1$, and existence results for problem (1.1) in such a case are proved in [5] and [10].

2. Existence results for nonhomogeneus problem.

In paper [18] we prove three existence results for the problem (1.1) when the datum f is not identically zero according to the values of the summability of f, i.e. $m \geq m_1 = \frac{2N}{N+2-\alpha(N-2)}$, $m_2 = \frac{N(2-\alpha)}{N+2-N\alpha} \leq m < m_1$ and $1 \leq m < m_2$.

The first existence result is given by Theorem 1 below, which concerns the existence of a *weak solution* to problem (1.1), i.e. a nonnegative function u belonging to $W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \langle A(x,u)\nabla u, \nabla \phi\rangle \, dx = \int_\Omega u^s \phi \, dx + \int_\Omega f\phi \, dx, \qquad \forall \phi \in W_0^{1,2}(\Omega). \tag{2.1}$$

Under the assumption that the datum f, belongs to the Lebesgue space $L^m(\Omega)$ for the values of $m \geq m_1 = \frac{2N}{N+2-\alpha(N-2)}$, Theorem 1 below states the existence of a weak solution u which further verifies $u^\sigma \in W_0^{1,2}(\Omega)$ for a suitable value of σ (estimates for $|\nabla u^\sigma|$ are proved in [15] for solutions to a class of quasilinear elliptic problems).

Theorem 1. *Assume that* (1.2)–(1.6) *holds true with* $f \not\equiv 0$ *and*

$$m \geq m_1, \quad \textit{with } m_1 = \frac{2N}{N+2-\alpha(N-2)}. \tag{2.2}$$

Then problem (1.1) *has at least a weak solution u which further satisfies*

$$u^\sigma \in W_0^{1,2}(\Omega), \tag{2.3}$$

where

$$\sigma = \begin{cases} \dfrac{N-2}{2}\dfrac{m(1-\alpha)}{N-2m}, & \textit{if } m_1 \leq m < \frac{N}{2}, \\ \textit{any } r, \ r \geq 1, & \textit{if } m \geq \frac{N}{2}. \end{cases}$$

The second existence result is given by Theorem 2 below. Under the assumption that the datum f belongs to the Lebesgue space $L^m(\Omega)$ with $m_2 = \frac{N(2-\alpha)}{N+2-N\alpha} \leq m < m_1 = \frac{2N}{N+2-\alpha(N-2)}$, it states the existence of a solution in distributional sense. Such a solution in general does not belong to the energy space $W_0^{1,2}(\Omega)$, but it belongs to the larger Sobolev space $W_0^{1,q}(\Omega)$, where q is defined in (2.5) below.

Theorem 2. *Assume that* (1.2)–(1.6) *holds true with* $f \not\equiv 0$ *and*

$$m_2 \leq m < m_1, \quad \text{with} \quad m_2 = \frac{N(2-\alpha)}{N+2-N\alpha}. \tag{2.4}$$

Then problem (1.1) *has at least a solution in the sense of distribution belonging to* $W_0^{1,q}(\Omega)$ *with*

$$q = \frac{N(2-\alpha)}{N-\alpha}\,. \tag{2.5}$$

Remark 1. We explicitly remark that, since $\alpha < 1$, it results $q > 1$. Moreover, since $N > 2$, then $q < 2$, i.e. the Sobolev space $W_0^{1,q}(\Omega)$ is larger then the energy space $W_0^{1,2}(\Omega)$. Observe also that q does not depends on the summability of the datum f. Actually, using the same arguments of [10] or [5] we could prove that there exists a solution in the sense of distribution to the problem (1.1) which belongs to a Sobolev space smaller then $W_0^{1,q}(\Omega)$ for the values of s in a suitable interval.

Finally in [18] we prove a third existence result for problem (1.1) when the datum f belongs to the Lebesgue space $L^m(\Omega)$ with m in the interval $[1, m_2[$. In such a case we have to change framework and we prove the existence of a renormalized solution for problem (1.1) (see [18], Section 6).

The proofs of the existence results given by Theorems 1 and 2 follow the same scheme. We begin by defining the sequence of "approximated problems",

$$\begin{cases} -\text{div}(A(x, T_n(u_n))\nabla u_n) = (T_n(u_n))^s + T_n(f) & \text{in } \Omega \\ u_n(x) \geq 0 & \text{in } \Omega \\ u_n(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

where, for any $n > 0$, $T_n : \mathbb{R} \to \mathbb{R}$ denotes the usual truncation at level n, that is

$$T_n(s) = \begin{cases} s & |s| \leq n, \\ n\,\text{sign}(s) & |s| > n, \end{cases}$$

for all $s \in \mathbb{R}$. Such approximated problems have a weak solution (even bounded) u_n; the existence of such a solution is a consequence, for example, of a result proved in [3]. Then we prove some *a priori* estimates: they are *a priori* estimates for $\||\nabla u_n|\|_{L^2(\Omega)}$ and for $\||\nabla (u_n)^\sigma|\|_{L^2(\Omega)}$ in the proof of Theorem 1; they are *a priori* estimates for $\||\nabla u_n|\|_{L^q(\Omega)}$ in the proof of Theorem 2. Finally we pass to the limit in problem (2.6) and we prove that the limit of u_n is a solution to problem (1.1).

3. Existence result for the homogeneus problem.

Consider problem (1.1) when the datum f is identically zero, i.e.

$$\begin{cases} -\mathrm{div}(A(x,u)\nabla u) = u^s & \text{in } \Omega \\ u(x) \geq 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Theorem 3 below states the existence of a nontrivial weak solution to the homogeneous problem (3.1) under the assumptions (1.2)–(1.5) and the further assumptions

$$A(x,t) \text{ is a simmetric matrix,} \tag{3.2}$$

$$|A(x,t_1) - A(x,t_2)| \leq L(|t_1 - t_2|), \tag{3.3}$$

for almost everywhere $x \in \Omega$ and for every $t_1, t_2 \in \mathbb{R}$, where $L : \mathbb{R} \to \mathbb{R}$ is a function which satisfies the following conditions

$$L(t) \text{ is a nondecreasing function,} \tag{3.4}$$

$$L(0) = 0, \tag{3.5}$$

$$\int_{0^+} \frac{dt}{L(t)} = +\infty\,. \tag{3.6}$$

Theorem 3. *Under the assumptions* (1.2)–(1.5), (3.2)–(3.6)*, problem* (3.1) *has at least a nontrivial weak solution.*

The proof of Theorem 3 is done by several steps (cf. [3]).

The first step consists in proving the existence of a convenient sub-solution ϕ to the problem (3.1). Indeed the assumptions (1.2)–(1.5), (3.2)–(3.6) allows to apply Theorem 1 in [9] which implies that, for every fixed $n \in \mathbb{N}$ and $r > 0$, there exists an eigenvalue $\lambda_{n,r}$ with corresponding positive eigenfunction $v_{n,r} \in W_0^{1,2}(\Omega)$ such that

$$\begin{cases} -\mathrm{div}(A(x,T_n(v_{n,r}))\nabla v_{n,r}) = \lambda_{n,r} v_{n,r} & \text{in } \Omega \\ v_{n,r}(x) > 0 & \text{in } \Omega \\ v_{n,r}(x) = 0 & \text{on } \partial\Omega \\ ||v_{n,r}||_{L^2(\Omega)} = r\,. \end{cases} \tag{3.7}$$

Moreover, it results

$$\frac{c_0}{(\beta+n)^{2\theta}}\mu_1 \le \lambda_{n,r} \le \frac{\mu_1}{\alpha^{2\theta}}, \tag{3.8}$$

where μ_1 denotes the first eigenvalue of Laplace operator with Dirichlet boundary datum on $\partial\Omega$. We prove that for any $n > 0$ and $r > 0$ the functions $v_{n,r}$ belongs to $L^\infty(\Omega)$ and an apriori estimate is proved in such a space. Then we prove that for any fixed $n > 0$, there exists a suitable $r > 0$ such that the function $\phi = v_{n,r}$ satisfies

$$||\phi||_{L^\infty(\Omega)} \le \min\left\{\left(\frac{\alpha^{2\theta}}{\mu_1}\right)^{\frac{1}{1-s}}, n\right\}. \tag{3.9}$$

Such a function $\phi = v_{n,r}$ is a (bounded) sub-solution to problem (3.1), i.e.

$$-\text{div}(A(x,\phi)\nabla\phi) \le \phi^s \text{ a.e. in } \Omega \text{ and in } \mathcal{D}'(\Omega).$$

The second step in the proof of Theorem 3 consists in proving the existence of a super-solution ψ to the problem (3.1). Actually such a super-solution is a solution to the problem

$$\begin{cases} -\text{div}(A(x,\psi)\nabla\psi) = P\psi^t + Q & \text{in } \Omega \\ \psi \ge 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

with suitable constants $P > 0$, $Q \ge ||\phi||^s_{L^\infty(\Omega)}$ and $0 \le s < t < 1-\alpha$.

The third step in the proof of Theorem 3 is the proof of a comparison result given by Theorem 4 below.

Theorem 4. *Assume that the matrix $A(x,s)$ verifies the assumptions* (1.2)–(1.4) *and* (3.2)–(3.6). *Moreover assume that the functions $u \in W_0^{1,2}(\Omega)$ and $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfy*

$$\begin{cases} -\text{div}(A(x,v)\nabla v) \le g & \text{in } \Omega \\ v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{3.11}$$

$$\begin{cases} -\text{div}(A(x,u)\nabla u) \ge f & \text{in } \Omega \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3.12}$$

where f and g are elements of the dual space $W^{-1,2}(\Omega)$ such that

$$f \ge g \quad \text{in } \mathcal{D}'(\Omega). \tag{3.13}$$

Then $u \ge v$ almost everywhere in Ω.

Such a result extend the comparison result proved by Artola and Boccardo in [3] to our context of lack of coerciveness; its proof is obtained by adapting the method used by Artola in [2] to prove the uniqueness result.

In the fourth step of the proof of Theorem 3 we introduce an iteration argument. We define the sequence of functions $\{u_k\}$ solutions to problem

$$(3.14) \qquad \begin{cases} -\mathrm{div}(A(x, u_{k+1})\nabla u_{k+1}) = u_k^s & \text{in } \Omega \\ u_{k+1}(x) \geq 0 & \text{in } \Omega \\ u_{k+1}(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where u_1, the first function of the sequence, is the solution to the problem

$$(3.15) \qquad \begin{cases} -\mathrm{div}(A(x, u_1)\nabla u_1) = \phi^s & \text{in } \Omega \\ u_1(x) \geq 0 & \text{in } \Omega \\ u_1(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

We prove that the functions $u_k \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfy

$$(3.16) \qquad 0 < \phi(x) \leq u_1(x) \leq u_2(x) \leq \cdots \leq u_k(x) \leq \cdots \leq \psi(x) \text{ a.e. in } \Omega.$$

and then that the function

$$u(x) = \lim_{k\to\infty} u_k(x) \text{ a.e. in } \Omega,$$

is a nontrivial weak solution to problem (3.1).

Finally we remark that, under the assumptions of Theorem 4, if we assume also that the super-solution u belongs to $L^\infty(\Omega)$, then Theorem 4 gives an uniqueness result, since in such a case we can change the role of u and v. Such uniqueness result coincides, in the case where $\alpha = 0$, with the uniqueness result of Brezis and Oswald proved in [12].

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