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POSITIVE SOLUTIONS OF NONLINEAR FRACTIONAL THREE-POINT BOUNDARY-VALUE PROBLEM

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In this paper, we study the existence of positive solutions to the boundary-value problem with fractional order

$$({}_{a}^{C}D^{\alpha}y)(t)+q(t)f(y)=0, \quad 0\leq a<t<b, \quad 1<\alpha<2,$$

$$y(a)=0, \quad y(b)=\beta y(\eta),$$

where $a<\eta<b$ and $\beta(\eta-a)-b+a\neq 0$. We prove the existence of at least one positive solution when f is either superlinear or sublinear using the well-known Guo-Lakshmikantham fixed point theorem in cones. Moreover, the convexity and concavity of the solutions are investigated with respect to the behavior of the function q.

1. Introduction

In the last decades, the investigation of multi-point boundary value problem for linear second order ordinary differential equations was begun by Il'in and Moiseev [10, 11]. The study of three-point BVPs for nonlinear integer-order ordinary differential equations was initiated by Gupta [7]. Many authors since

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To the memory of Prof. Maryam Mirzakhani

then considered the existence and multiplicity of solutions (or positive solutions) of three-point BVPs for nonlinear integer-order ordinary differential equations. To identify a few, we refer the reader to [15, 16, 24] and the references therein.

In 2000, using the fixed point index theorems, Leray-Schauder degree and upper and lower solutions, Ma [15] investigated the following second-order three-point boundary value problem

$$\begin{gathered} u'' + \lambda h(t) f(u) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad cu(\eta) = u(1), \end{gathered} \tag{1}$$

where

(A) λ is a positive parameter; $\eta \in (0,1)$ and $0 < c\eta < 1$;

(B) $h : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subset of positive measure;

(C) $f : [0,\infty) \to [0,\infty)$ is continuous;

(D) $f_\infty := \lim_{u\to\infty} \dfrac{f(u)}{u} = \infty$.

In the result of He and Ge [8], utilizing Leggett-Williams fixed-point theorem [13], the multiplicity of positive solutions of the following problem has been concerned:

$$\begin{gathered} u'' + f(t,u) = 0, \ t \in (0,1), \\ u(0) = 0, \quad cu(\eta) = u(1), \end{gathered}$$

where $0 < \eta < 1$, $c > 0$ and $0 < c\eta < 1$. Moreover, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, and $f(t,\cdot)$ does not vanish identically on any subset of $[0,1]$ with positive measure.

In the last decades, fractional calculus and fractional differential equations have attracted much attention, we refer for instance to [1, 2, 14, 18, 19, 26] and references therein. It is found that many phenomena can be modeled with the aid of fractional derivatives or integrals, such as fractional Brownian motion [3], anomalous diffusion [9, 17], etc. This motivates us to remodel the problem (1) by a fractional order and study on it.

Throughout this paper, we consider the existence of positive solutions to the three-point boundary value problem consisting by the fractional differential equation

$$({}^C_aD^\alpha y)(t) + q(t) f(y) = 0, \quad 0 \le a < t < b, \tag{2}$$

where ${}_a^C D^\alpha$ is the Caputo fractional derivative of order $1 < \alpha < 2$, subject to the boundary conditions

$$y(a) = 0, \quad y(b) = \beta y(\eta), \quad a < \eta < b, \tag{3}$$

where f, q satisfy

(H1) $f \in C([0,\infty),[0,\infty))$;

(H2) $q \in C([a,b],[0,\infty))$.

By taking

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u},$$

we set $f_0 = 0$ and $f_\infty = \infty$ corresponding to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ corresponding to the sublinear case. Here, in this paper, our goal is to present some existence results for positive solutions to (2)-(3), assuming that f is either superlinear or sublinear. The technique of proof of our main result is based upon the well-known Guo-Lakshmikantham fixed point theorem [6] in a cone.

Theorem 1.1. *[6] Let E be a Banach space, and let $K \subseteq E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that:

(i) $\|Au\| \le \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \ge \|u\|, \quad u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \ge \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \le \|u\|, \quad u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries and auxiliary facts

For completeness, in this section, we gather some fundamental definitions of Caputo's derivatives of fractional order which can be found in ([12], [20], [21]) together with some simple crucial lemmas which will be needed further on.

Definition 2.1. Let $\alpha \ge 0$ and f be a real function defined on $[a,b]$. The *Riemann-Liouville fractional integral* of order α for a continuous function $f : (a,\infty) \to \mathbb{R}$ is defined by $({}_aI^0 f)(x) = f(x)$ and

$$({}_aI^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \quad t \in [a,b],$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. For a continuous function $f:(a,\infty)\to\mathbb{R}$ the Riemann-Liouville fractional derivative of fractional order $\alpha>0$ is defined by

$$^{RL}D^{\alpha}_{a+}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t(t-s)^{n-\alpha-1}f(s)ds,\ \ n=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α.

For $\alpha<0$, we use the convention that $D^{\alpha}y=I^{-\alpha}y$. Also for $\beta\in[0,\alpha)$, it is valid that $D^{\beta}I^{\alpha}y=I^{\alpha-\beta}y$.

Definition 2.3. The *Caputo fractional derivative* of order $\alpha\geq 0$ is given by $({}^C_aD^0f)(t)=f(t)$ and $({}^C_aD^{\alpha}f)(t)=({}_aI^{m-\alpha}D^mf)(t)$ for $\alpha>0$, where m is the smallest integer greater or equal to α. Besides, it can be formulated by

$$^{C}D^{\alpha}_{a+}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds,\ \ n=[\alpha]+1,\quad f\in AC^n([a,b]),$$

where $\alpha\notin\mathbb{N}_0$ and $AC^n([a,b])$ represents the space of all absolutely continuous functions having absolutely continuous derivative up to $(n-1)$ (see also [12]).

The Green function for the BVP (2)-(3) can be obtained by using an important lemma derived by Zhang [25] as follows:

Lemma 2.4. *Let $\alpha>0$, then in $C(0,T)\cap L(0,T)$ the differential equation*

$$^{C}D^{\alpha}_{0+}u(t)=0$$

has solutions $u(t)=c_0+c_1t+c_2t^2+\cdots+c_nt^{n-1}$, $c_i\in\mathbb{R}$, $i=0,1,\cdots,n$, $n=[\alpha]+1$.

Moreover, it has been proved that $I^{\alpha}_{0+}D^{\alpha}_{0+}u(t)=u(t)+c_0+c_1t+c_2t^2+\cdots+c_nt^{n-1}$ for some $c_i\in\mathbb{R}$, $i=0,1,\cdots,n$, $n=[\alpha]+1$ (see Lemma 2.3 in [25]).

In the following we present a pivotal lemma which will play major role in our next analysis and concern a linear variant of problem (2)-(3).

Lemma 2.5. *For $g\in C([a,b],[0,\infty))$, the problem*

$$({}^C_aD^{\alpha}y)(t)+g(t)=0,\qquad 0\leq a<t<b,\tag{4}$$

with order $1<\alpha<2$ and the boundary condition (3) has a unique solution

$$\begin{aligned}y(t)=&-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}g(s)ds\\&+\frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b(b-s)^{\alpha-1}g(s)ds-\beta\int_a^{\eta}(\eta-s)^{\alpha-1}g(s)ds\right).\end{aligned}$$

Proof. Applying the Riemann-Liouville fractional integral ${}_aI^{\alpha}$ for (4)-(3) and the imposed boundary conditions together with a fact from fractional calculus theory we see that $y \in C[a,b]$ is a solution of (4)-(3) if and only if

$$y(t) = c_0 + c_1(t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds \tag{5}$$

for some real constants c_0 and c_1 (see Lemma 2.4). Since $y(a) = 0$ we get immediately that $c_0 = 0$. Now,

$$\begin{aligned} y(b) = \beta y(\eta) \;\; &\Leftrightarrow \;\; c_1(b-a) - \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} g(s)ds \\ &= c_1\beta(\eta-a) - \frac{\beta}{\Gamma(\alpha)} \int_a^\eta (\eta-s)^{\alpha-1} g(s)ds \\ &\Leftrightarrow \;\; c_1 = \frac{1}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \bigg(\int_a^b (b-s)^{\alpha-1} g(s)ds \\ &\qquad - \beta \int_a^\eta (\eta-s)^{\alpha-1} g(s)ds \bigg). \end{aligned}$$

Hence, equality (5) becomes

$$\begin{aligned} y(t) \;\; = \;\; & -\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds + \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \times \\ & \times \bigg(\int_a^b (b-s)^{\alpha-1} g(s)ds - \beta \int_a^\eta (\eta-s)^{\alpha-1} g(s)ds \bigg). \end{aligned}$$

□

Lemma 2.6. *Suppose that $g \in C^2([a,b];\mathbb{R})$ and $g(a) \geq 0$.*

(a) *If g is convex, then the unique solution of* (4)-(3) *is concave.*

(b) *If g is concave, then the unique solution of* (4)-(3) *is convex.*

Proof. In order to prove the validity of (a), first, by the definition of the Caputo's derivative, it is easily seen from (4)-(3) that

$$I_a^{2-\alpha}(y''(t)) = -g(t).$$

Then it follows that

$$I_a^{\alpha}(I_a^{2-\alpha}(y''(t))) = -I_a^{\alpha}(g(t)).$$

That is,

$$I_a^2(y''(t)) = -I_a^{\alpha}(g(t)).$$

Hence, we can obtain

$$y''(t) = -\frac{d^2}{dt^2} I_a^\alpha(g(t)) = -I_a^\alpha(g''(t)) = -{}^{RL}D_a^{2-\alpha} g(t).$$

On the other hand, from the fractional calculus we know that

$${}^C D_a^\alpha g(t) = {}^{RL}D_a^\alpha g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}, \quad (n = \lfloor \alpha \rfloor + 1),$$

see also [12]. Since $0 < 2-\alpha < 1$ then we get

$${}^C D_a^{2-\alpha} g(t) = {}^{RL}D_a^{2-\alpha} g(t) - \frac{g(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}, \qquad a < t \le b,$$

which implies that

$$\begin{aligned} y''(t) &= -\left({}^C D_a^{2-\alpha} g(t) + \frac{g(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}\right) \\ &= -\left(I_a^\alpha(g''(t)) + \frac{g(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}\right) \end{aligned}$$

which is obviously non-positive for all $t \in (a,b]$ and so the solution of (4)-(3) is concave. The proof of the second part is quite similar. □

Lemma 2.7. *Let $0 < \beta\eta < b$ and $g \in C^2([a,b];\mathbb{R})$ be a convex function with $g(a) \ge 0$. Then the unique solution of the problem* (4)-(3) *satisfies $y(t) \ge 0$ for all $t \in [a,b]$ and is concave.*

Proof. Following Lemma 2.6 we see that $y(t)$ is concave down on (a,b). If $y(b) \ge 0$, then the concavity of y and the boundary condition $y(a) = 0$ yield $y(t) \ge 0$ for all $t \in [a,b]$. Otherwise, letting $y(b) < 0$, we have $y(\eta) < 0$ and

$$y(b) = \beta y(\eta) > \frac{b}{\eta} y(\eta),$$

which contradicts the concavity of y and the proof is complete. □

Proposition 2.8. *Suppose that $\beta\eta > b$ and $g \in C^2([a,b];\mathbb{R})$ is a convex function with $g(a) \ge 0$. Then the problem* (4)*-(3) has no positive solution.*

Proof. Suppose the contrary, (4)-(3) has a positive solution y. If $y(b) > 0$, then $y(\eta) > 0$ and

$$y(b) = \beta y(\eta) > \frac{b}{\eta} y(\eta),$$

which contradicts the concavity of y, since g is convex. Now, let $y(b)=0$ and $y(r)>0$ for some $r\in(a,b)$, then

$$y(\eta)=y(b)=0, \qquad \eta\neq r.$$

This together with the condition $y(a)=0$ implies that y is not concave. Indeed,

$$\begin{aligned} &r\in(a,\eta)\Longrightarrow y(r)<y(\eta)=y(b)=0\\ &r\in(\eta,b)\Longrightarrow y(r)>y(\eta)=y(b)=0\Longrightarrow y(a)<0 \end{aligned}$$

which both cases show a contradiction using the concavity of y. □

Lemma 2.9. *Let $0<\beta\eta<b$, $\beta(a-\eta)+b-a\neq 0$, and $g\in C^2([a,b];\mathbb{R})$ be a convex function with $g(a)\geq 0$. Then the solution of Eq.* (4)-(3) *satisfies*

$$\min_{t\in[\eta,b]} y(t)\geq\gamma\|y\|$$

where

$$\gamma=\min\left\{\frac{\beta(b-\eta)}{\beta(a-\eta)+b-a},\frac{\beta\eta}{b},\frac{\eta}{b}\right\}. \tag{6}$$

Proof. We split the proof into the following cases.

Case 1. We encounter with the case $0<\beta<1$. Following Lemma 2.7 and initial conditions we know that $y(\eta)\geq y(b)$. Now, let $y(\hat{t})=\|y\|$ for some $\hat{t}\in(a,b]$. Assume that $\hat{t}\leq\eta<b$, then

$$\min_{t\in[\eta,b]} y(t)=y(b). \tag{7}$$

On the other hand, from the concavity of the solution y we see

$$\frac{y(\eta)-y(\hat{t})}{\eta-\hat{t}}\geq\frac{y(b)-y(\eta)}{b-\eta}$$

which shows that

$$\begin{aligned} y(\hat{t})&\leq\frac{(b-\eta)+(1-\beta)(\eta-\hat{t})}{\beta(b-\eta)}y(b)\\ &\leq\frac{(b-\eta)+(1-\beta)(\eta-a)}{\beta(b-\eta)}y(b)\\ &=\frac{\beta(a-\eta)+b-a}{\beta(b-\eta)}y(b). \end{aligned}$$

This together with (7) yields that

$$\min_{t\in[\eta,b]} y(t) \geq \frac{\beta(b-\eta)}{\beta(a-\eta)+b-a}\|y\|.$$

Now, let us take $\eta < \hat{t} < b$, then

$$\min_{t\in[\eta,b]} y(t) = y(b). \tag{8}$$

Using the concavity of y we conclude

$$\frac{y(\eta)}{\eta} \geq \frac{y(\hat{t})}{\hat{t}}.$$

This together with the boundary condition $y(b) = \beta y(\eta)$ implies that

$$\frac{y(b)}{\beta\eta} \geq \frac{y(\hat{t})}{\hat{t}} > \frac{1}{b}\|y\|$$

which means

$$\min_{t\in[\eta,b]} y(t) > \frac{\beta\eta}{b}\|y\|.$$

Case 2. Suppose that $1 \leq \beta < \dfrac{b}{\eta}$. Then we have $y(\eta) \leq y(b)$. Now, by setting $y(\hat{t}) = \|y\|$ we see that $\eta \leq \hat{t} \leq b$. We notice that if $a < \hat{t} < \eta$, then the point $P_\eta = (\eta, y(\eta))$ is below the straight line given by the points $P_b = (b, y(b))$ and $P_{\hat{t}} = (\hat{t}, y(\hat{t}))$ and this contradicts the concavity of y. The recent facts guarantee the following equality:

$$\min_{t\in[\eta,b]} y(t) = y(\eta).$$

Similar to the former case and using Lemma 2.7 we obtain

$$\frac{y(\eta)}{\eta} \geq \frac{y(\hat{t})}{\hat{t}}$$

which implies

$$\min_{t\in[\eta,b]} y(t) \geq \frac{\eta}{b}\|y\|$$

and the consequence follows. □

3. Main result

Based on the lemmas presented in previous section we derive our main result as follows.

Theorem 3.1. *Assume that (H1) and (H2) hold. Then the problem (2)-(3) has at least one positive solution in the case*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear) or*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. Let us first consider the case (i):
Superlinear case. Suppose then that $f_0 = 0$ and $f_\infty = \infty$. We want to establish the existence of a positive solution of (2)-(3). Following the proof of Lemma 2.5, problem (2)-(3) has a solution $y = y(t)$ if and only if y solves the operator equation

$$\begin{aligned} y(t) = &-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)f(y(s))ds + \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\times \\ &\times\left(\int_a^b (b-s)^{\alpha-1}q(s)f(y(s))ds - \beta\int_a^\eta (\eta-s)^{\alpha-1}q(s)f(y(s))ds\right) \\ &\stackrel{\text{def}}{=} Ay(t). \end{aligned} \tag{9}$$

Set

$$K := \{y \mid y \in C[a,b],\ y \geq 0,\ \min_{\eta \leq t \leq b} y(t) \geq \gamma\|y\|\}, \tag{10}$$

where γ is given by (6). It is clear that K is a cone in $C[a,b]$. Moreover, by Lemma 2.9, $AK \subset K$. It is also easy to see that $A : K \to K$ is completely continuous.

Now since $f_0 = 0$, we may take $r_1 > 0$ such that $f(y) \leq \varepsilon y$, for $0 < y < r_1$, where $\varepsilon > 0$ satisfies

$$\frac{\varepsilon(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)ds\right) < 1. \tag{11}$$

Hence, if $y \in K$ and $\|y\| = r_1$, then following (9) and (11), we derive

$$\begin{aligned} Ay(t) &\leq \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)f(y(s))ds\right) \\ &\leq \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)\varepsilon y(s)ds\right) \\ &\leq \frac{\varepsilon(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)\|y\|ds\right) \\ &= \frac{\varepsilon r_1(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)ds\right). \end{aligned} \tag{12}$$

Now if we set

$$\Omega_1 = \{y \in C[a,b] \mid \|y\| < r_1\}, \tag{13}$$

then (12) yields that $\|Ay\| \le \|y\|$, for all $y \in K \cap \partial\Omega_1$. Moreover, since $f_\infty = \infty$, there exists $\hat{r}_2 > 0$ so that $f(u) \ge \rho u$ for all $u \ge \hat{r}_2$ where $\rho > 0$ is taken so that

$$\frac{\rho\gamma(\eta-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_\eta^b (b-s)^{\alpha-1}q(s)ds \ge 1. \tag{14}$$

Suppose $r_2 = \max\{2r_1, \hat{r}_2\gamma^{-1}\}$ and $\Omega_2 = \{y \in C[a,b] \mid \|y\| < r_2\}$, then $y \in K$ with $\|y\| = r_2$ yields

$$\min_{\eta \le t \le b} y(t) \ge \gamma\|y\| \ge \hat{r}_2,$$

and hence

$$\begin{aligned}
Ay(\eta) &= -\frac{1}{\Gamma(\alpha)}\int_a^\eta (\eta-s)^{\alpha-1}q(s)f(y(s))ds \\
&\quad + \frac{\eta-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\bigg(\int_a^b (b-s)^{\alpha-1}q(s)f(y(s))ds \\
&\quad - \beta\int_a^\eta (\eta-s)^{\alpha-1}q(s)f(y(s))ds\bigg) \\
&= -\frac{1}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\bigg((b-a)\int_a^\eta (\eta-s)^{\alpha-1}q(s)f(y(s))ds \\
&\quad - (\eta-a)\int_a^b (b-s)^{\alpha-1}q(s)f(y(s))ds\bigg) \\
&= -\frac{1}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\bigg(\int_a^\eta \Big[(b-a)(\eta-s)^{\alpha-1} \\
&\quad - (\eta-a)(b-s)^{\alpha-1}\Big]q(s)f(y(s))ds \\
&\quad - (\eta-a)\int_\eta^b (b-s)^{\alpha-1}q(s)f(y(s))ds\bigg).
\end{aligned} \tag{15}$$

On the other hand, by the fact that

$$0 \le \frac{\eta-s}{b-s} \le \left(\frac{\eta-s}{b-s}\right)^{\alpha-1} \le \frac{\eta-a}{b-a} < 1, \quad a \le s \le \eta < b, \quad 1 < \alpha < 2,$$

we see that

$$Ay(\eta) \ge \frac{\eta-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_\eta^b (b-s)^{\alpha-1}q(s)f(y(s))ds.$$

Therefore, for $y \in K \cap \partial\Omega_2$,

$$\|Ay\| \geq \frac{\rho\gamma(\eta-a)\|y\|}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_\eta^b (b-s)^{\alpha-1}q(s)ds \geq \|y\|.$$

Consequently, by the first part of the Guo-Lakshmikantham fixed point theorem, it follows that A has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$ such that $r_1 \leq \|y\| \leq r_2$. This finishes the proof of superlinear part of the theorem.

Now we consider the case (ii):

Sublinear case. Suppose then that $f_0 = \infty$ and $f_\infty = 0$. Let us first take $r_3 > 0$ such that $f(y) \geq \mu y$ for $0 < y < r_3$, where

$$\frac{\mu\gamma(\eta-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_\eta^b (b-s)^{\alpha-1}q(s)ds \geq 1. \tag{16}$$

Utilizing the same technique as used in (15), one can obtain that

$$\begin{aligned}
Ay(\eta) &= -\frac{1}{\Gamma(\alpha)}\int_a^\eta (\eta-s)^{\alpha-1}q(s)f(y(s))ds \\
&\quad + \frac{\eta-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)f(y(s))ds\right. \\
&\quad \left. -\beta\int_a^\eta (\eta-s)^{\alpha-1}q(s)f(y(s))ds\right) \\
&\geq \frac{\eta-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_\eta^b (b-s)^{\alpha-1}q(s)f(y(s))ds \\
&\geq \frac{\mu(\eta-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_\eta^b (b-s)^{\alpha-1}q(s)y(s)ds.
\end{aligned} \tag{17}$$

Therefore, we may set $\Omega_3 = \{y \in C[a,b] \mid \|y\| < r_3\}$ such that $\|Ay\| \geq \|y\|$ for $y \in K\cap\partial\Omega_3$.

On the other hand, since $f_\infty = 0$ then there is $\hat{r}_4 > 0$ such that $f(y) \leq \xi y$ for $y \geq \hat{r}_4$ where $\xi > 0$ enjoys

$$\frac{\xi(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)ds\right) \leq 1. \tag{18}$$

Now, we must consider two distinct cases as follows:

Case (I). Let us assume that f is bounded, say $f(y) \leq M$ for all $y \geq 0$. For this case, we set

$$r_4 = \max\left\{2r_3, \frac{M(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_a^b (b-s)^{\alpha-1}q(s)ds\right\}$$

such that for $y \in K$ with $\|y\| = r_4$ we get

$$\begin{aligned}
Ay(t) &= -\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s) f(y(s)) ds \\
&\quad + \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \bigg(\int_a^b (b-s)^{\alpha-1} q(s) f(y(s)) ds \\
&\quad - \beta \int_a^\eta (\eta-s)^{\alpha-1} q(s) f(y(s)) ds \bigg) \\
&\leq \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \int_a^b (b-s)^{\alpha-1} q(s) f(y(s)) ds \\
&\leq \frac{M(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \int_a^b (b-s)^{\alpha-1} q(s) ds \\
&\leq r_4
\end{aligned}$$

which yields $\|Ay\| \leq \|y\|$.

Case (II). Now, suppose that f is unbounded, then we derive from (H1) that there exists r_4 such that

$$r_4 > \max\left\{2r_3, \frac{\hat{r}_4}{\gamma}\right\} \quad \text{s.t.} \quad f(y) \leq f(r_4) \quad \text{for} \quad 0 < y \leq r_4$$

and it would be possible because f is unbounded. Using (18), for any $y \in K$ with $\|y\| = r_4$ we conclude that

$$\begin{aligned}
Ay(t) &\leq \frac{b-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \int_a^b (b-s)^{\alpha-1} q(s) f(r_4) ds \\
&\leq \frac{(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \int_a^b (b-s)^{\alpha-1} q(s) \xi r_4 ds \\
&\leq r_4.
\end{aligned}$$

Hence, in any case we may set

$$\Omega_4 = \{y \in C[a,b] \ | \ \|y\| < r_4\},$$

and then we may obtain $\|Ay\| \leq \|y\|$. Based on the second part of Guo-Lakshmikantham fixed point theorem, it follows that BVP (2)-(3) has a positive solution and the consequence follows. □

4. A concrete example

Concerning with the existence of positive solution of BVP (2)-(3), we now give an example to illustrate the effciency of our main result. Let us first recall some auxiliary facts as follows.

As we know, analytic solutions to fractional-order differential equations are often expressed in terms of the Mittag-Leffler function. The Mittag-Leffler function $E_{\alpha,\beta}$ is a special function, a complex function which relates to two complex parameters α and β (it is also worth mentioning that it was firstly introduced as a one-parameter function). The Mittag-Leffler function is considered as a generalization of the exponential function. It may be given by the following series when the real part of α is strictly positive

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C},$$

which is of great importance for the fractional calculus. In the case α and β are real and positive, the series converges for all values of the argument z, so the Mittag-Leffler function is an entire function.

Example 4.1. Consider the following boundary-value problem with fractional order

$$\begin{aligned} &({}_0^C D^{1.5} y)(t) + q(t)\sqrt[3]{y^2} = 0, \qquad 0 < t < 1, \\ &y(0) = 0, \qquad y(1) = \sqrt{3}y(0.5) \end{aligned} \tag{19}$$

where

$$q(t) = \frac{4\pi^{11/6} t^{5/6} \sum_{k=0}^{\infty} \frac{(\frac{-4\pi^2 t^2}{9})^k}{(4k+3)!!}}{27\left(\frac{2}{9}\sum_{k=0}^{\infty} \frac{(\frac{-4\pi^2 t^2}{9})^k}{(2k+2)!}\right)^{\frac{1}{3}}} < \infty, \qquad t \in (0,1)$$

and $n!!$ is called the double factorial and given by $n!! = n(n-2)(n-4)\cdots 5\cdot 3\cdot 1$ for odd $n > 0$ and $n!! = n(n-2)(n-4)\cdots 6\cdot 4\cdot 2$ for even $n > 0$.

First we note that $f(u) = \sqrt[3]{u^2}$ is a sublinear function. We claim that BVP (19) has a solution $y = \sin(\frac{\pi t}{3})$ which is concave on $[0,1]$. In order to prove it, bring in mind that

$${}_0^C D^{\alpha} \sin\lambda t = -\frac{1}{2} i(i\lambda)^n t^{n-\alpha} (E_{1,n-\alpha+1}(i\lambda t) - (-1)^n E_{1,n-\alpha+1}(-i\lambda t)) \tag{20}$$

such that $\lambda \in \mathbb{C}, \alpha \in \mathbb{R}, n \in \mathbb{N}$, and $n-1 < \alpha < n$. The formula as above also can be represented in the terms of the so-called hypergeometric functions (sometimes called the Kummer or confluent functions), see also [4]. Based on (20), we obtain

$$
\begin{aligned}
{}_0^C D^{\frac{3}{2}} \sin\frac{\pi t}{3} &= -\frac{1}{2} i (\frac{\pi}{3} i)^2 \sqrt{t} \left(E_{1,\frac{3}{2}}(\frac{i\pi t}{3}) - E_{1,\frac{3}{2}}(-\frac{i\pi t}{3}) \right) \\
&= \frac{\pi^2 i}{18} \sqrt{t} \left(\sum_{k=0}^{\infty} \frac{(\frac{i\pi t}{3})^k}{\Gamma(k+\frac{3}{2})} - \sum_{k=0}^{\infty} \frac{(\frac{-i\pi t}{3})^k}{\Gamma(k+\frac{3}{2})} \right) \\
&= \frac{\pi^2}{18} \sqrt{t} \left(\sum_{k=0}^{\infty} \frac{i(\frac{i\pi t}{3})^k}{\Gamma(k+\frac{3}{2})} \left(1 - (-1)^k \right) \right) \\
&= \frac{\pi^2}{18} \sqrt{t} \left(\sum_{k=1}^{\infty} \frac{i(\frac{i\pi t}{3})^{2k-1}}{\Gamma(2k-1+\frac{3}{2})} \left(1 - (-1)^{2k-1} \right) \right) \\
&= \frac{\pi}{3\sqrt{t}} \left(\sum_{k=1}^{\infty} \frac{(-\frac{\pi^2 t^2}{9})^k}{\Gamma(2k+\frac{1}{2})} \right) \\
&= -\frac{\pi^3 t^{\frac{3}{2}}}{27} \left(\sum_{k=0}^{\infty} \frac{(-\frac{\pi^2 t^2}{9})^k}{\Gamma(2k+\frac{5}{2})} \right),
\end{aligned}
$$

making use of the fact that

$$
\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \qquad n = 0,1,2,3,\cdots,
$$

we get the following formula

$$
{}_0^C D^{3/2} \sin\frac{\pi t}{3} = -\frac{4\pi^{5/2} t^{3/2}}{27} \left(\sum_{k=0}^{\infty} \frac{(-\frac{4\pi^2 t^2}{9})^k}{(4k+3)!!} \right) < \infty.
$$

Moving forward, using the series expansion of cosine we derive

$$
\begin{aligned}
q(t)\sqrt[3]{\sin^2\frac{\pi t}{3}} &= \frac{4\pi^{11/6} t^{5/6} \sum_{k=0}^{\infty} \frac{(\frac{-4\pi^2 t^2}{9})^k}{(4k+3)!!}}{27\left(\frac{2}{9}\sum_{k=0}^{\infty} \frac{(\frac{-4\pi^2 t^2}{9})^k}{(2k+2)!}\right)^{\frac{1}{3}}} \times \sqrt[3]{\frac{2\pi^2 t^2}{9}} \left(\sum_{k=0}^{\infty} \frac{(-\frac{4\pi^2 t^2}{9})^k}{(2k+2)!} \right)^{\frac{1}{3}} \\
&= -{}_0^C D^{\frac{3}{2}} \sin\frac{\pi t}{3}
\end{aligned}
$$

which means $y = \sin\frac{\pi t}{3}$ is the solution of BVP (19).

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