LE MATEMATICHE
Vol. LXXIII (2018) – Fasc. I, pp. 155–160
doi: 10.4418/2018.73.1.11

ON COMPLEX H-TYPE LIE ALGEBRAS

NATHANIEL ELDREDGE

Let $\mathfrak{g}$ be a complex nilpotent Lie algebra equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$. We show that if $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is an H-type Lie algebra in the sense of Kaplan, then $\mathfrak{g}$ must be isomorphic to a complex Heisenberg Lie algebra $\mathfrak{h}_{\mathbb{C}}^{2n+1}$. This shows that the class of complex H-type Lie algebras is very small.

1. Introduction

Since their introduction by Kaplan [8], H-type Lie algebras, and their corresponding nilpotent Lie groups, have attracted interest as a natural generalization of the classical real Heisenberg Lie algebra $\mathfrak{h}^3$ of dimension 3 and the corresponding real Heisenberg group $\mathbb{H}^3$. The Heisenberg group is a motivating example in many areas of mathematics, and in many cases, facts about the Heisenberg group carry over into the H-type setting. For instance, H-type groups carry a natural structure as sub-Riemannian manifolds, and the analysis of their sub-Laplacians has attracted considerable interest. As a sampling, we mention [1, 3, 5–7, 9].

The H-type condition for a (real) Lie algebra $\mathfrak{g}$ is dependent on a choice of inner product $\langle \cdot, \cdot \rangle$ (i.e. a positive definite, symmetric, bilinear form) on $\mathfrak{g}$, so it is really a property of the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. For example, in $\mathfrak{h}^3$, the natural Euclidean inner product will do.

Entrato in redazione: 3 agosto 2017

AMS 2010 Subject Classification: 17B30, 22E30, 22E25, 32M05, 32Q99
Keywords: complex Lie algebra, H-type group, complex Heisenberg group

Among the H-type Lie algebras are the complex Heisenberg (or Heisenberg–Weyl) Lie algebras $\mathfrak{h}_{\mathbb{C}}^{2n+1}$, equipped with their natural Euclidean inner products. The Euclidean inner product on $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ is Hermitian with respect to the complex structure, which is a natural compatibility condition. As such, analysis on the complex Heisenberg groups $\mathbb{H}_{\mathbb{C}}^{2n+1}$ can take advantage of all the tools of complex geometry, together with the many results for H-type groups mentioned above. The purpose of this note is to show that there are no other complex Lie algebras with this property.

Another way to state this result is that if a Lie algebra is required to carry a complex structure and simultaneously be H-type, in a compatible way, then it is forced to be a complex Heisenberg Lie algebra. This is somewhat similar in spirit to a result in [10], where it is shown that a real Carnot group of corank 1 is forced to be "almost" a Heisenberg group; it splits into an (anisotropic) Heisenberg part and a Euclidean part. (Here, corank 1 means that the Lie algebra $\mathfrak{g}$ can be written $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$, $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$, and $\dim \mathfrak{g}_2 = 1$.) However, we stress that in the present paper, we make no *a priori* assumptions about the dimension or rank of the Lie algebra.

As an application, we refer to [4], in which we studied a property known as strong hypercontractivity for the hypoelliptic heat kernel on a stratified complex Lie group. An essential hypothesis for this result was that the heat kernel should satisfy a logarithmic Sobolev inequality. For most Lie groups, it remains an open problem to determine whether this inequality holds, but it follows from the results of [3, 6] that the inequality holds in every H-type Lie group. Thus, the strong hypercontractivity theorem proved in [4] holds in particular for any complex Lie group which, when considered as a real Lie group, is also H-type. The result of the present note implies that these Lie groups are precisely the family $\mathbb{H}_{\mathbb{C}}^{2n+1}$. As this is a relatively limited class of examples, we see this as further motivation to try to extend the logarithmic Sobolev inequality beyond the H-type case.

2. Definitions and examples

We begin by recalling the definition of an H-type Lie algebra, as formulated in [2, Definition 18.1.1]. (Kaplan's original definition [8] is equivalent, but slightly less convenient for our purposes.) Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra equipped with an inner product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and let $\mathfrak{v} = \mathfrak{z}^{\perp}$. For $z \in \mathfrak{z}$ and $u \in \mathfrak{v}$, define $J_z u$ as the unique element of $\mathfrak{v}$ satisfying

$$\langle J_z u, v \rangle = \langle z, [u, v] \rangle \quad \text{for all } v \in \mathfrak{v}. \tag{1}$$

It is clear that each $J_z : \mathfrak{v} \to \mathfrak{v}$ is a linear map, and moreover $z \mapsto J_z$ is linear in z.

Definition 2.1. We say that $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is **H-type** if the following two conditions hold:

1. $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$
2. For each $z \in \mathfrak{z}$ with $\|z\| = 1$, $J_z : \mathfrak{v} \to \mathfrak{v}$ is an isometry with respect to $\langle\cdot,\cdot\rangle$.

We observe that an H-type Lie algebra is necessarily nilpotent of step 2. A simply-connected Lie group is said to be H-type if its Lie algebra is H-type in the above sense.

Now suppose that $\mathfrak{g}$ is a complex Lie algebra, whose complex structure we denote by i. If we wish to equip $\mathfrak{g}$ with a real inner product, it is natural to demand some compatibility with the complex structure. Specifically, we would like the inner product to be **Hermitian**, i.e., for $x, y \in \mathfrak{g}$ we have $\langle ix, iy\rangle = \langle x, y\rangle$. We may then define J in terms of this inner product by (1). We observe for later use that, as a consequence of the Hermitian property of the inner product, we have for $\alpha, \beta \in \mathbb{C}$ and $u, z \in \mathfrak{g}$,

$$J_{\alpha z}(\beta u) = \alpha\bar{\beta} J_z u. \tag{2}$$

That is, $J_z u$ is complex linear in z and conjugate linear in u.

The question of interest in this note is when both of the above properties hold, motivating the following definition.

Definition 2.2. A **complex H-type Lie algebra** is a pair $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, where $\mathfrak{g}$ is a complex Lie algebra and $\langle\cdot,\cdot\rangle$ is an inner product on $\mathfrak{g}$, such that the following two conditions hold:

- The inner product $\langle\cdot,\cdot\rangle$ is Hermitian with respect to the complex structure of $\mathfrak{g}$.
- Forgetting the complex structure on $\mathfrak{g}$, the pair $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is H-type in the sense of Definition 2.1.

We can likewise define a **complex H-type Lie group** as a connected and simply connected complex Lie group G equipped with a Hermitian left-invariant Riemannian metric g which, when viewed as an inner product on the Lie algebra of G, satisfies the above conditions.

Example 2.3. The **complex Heisenberg Lie algebra** of complex dimension $2n+1$ is the complex Lie algebra $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ generated (over $\mathbb{C}$) by the basis of the $2n+1$ vectors $\{x_1, y_1, \ldots, x_n, y_n, z\}$, with the bracket defined by $[x_k, y_k] = z$, and for $j \neq k$, $[x_j, y_k] = [x_j, z] = [y_j, z] = 0$. We may equip $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ with the real inner product $\langle\cdot,\cdot\rangle$ that makes all of $x_k, ix_k, y_k, iy_k, z, iz$ orthonormal; it is clear that this inner product is Hermitian. The center $\mathfrak{z}$ of $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ is spanned (over $\mathbb{C}$) by z, so we clearly have $[\mathfrak{v},\mathfrak{v}] = \mathfrak{z}$. Defining J as above, it is easy to compute

$$J_z x_k = y_k \quad J_z y_k = -x_k \quad J_z i x_k = -i y_k \quad J_z i y_k = i x_k$$

so that J_z is an isometry. Moreover, every element $w \in \mathfrak{z}$ is of the form $w = \alpha z$ for some $\alpha \in \mathbb{C}$, and $\|w\| = |\alpha|$, so using (2) we see that J_w is an isometry whenever $\|w\| = 1$. Thus $(\mathfrak{h}_{\mathbb{C}}^{2n+1}, \langle\cdot,\cdot\rangle)$ is a complex H-type Lie algebra.

Of course, the complex Heisenberg Lie algebras are a very special family within the far larger class of all complex Lie algebras. Likewise, the class of H-type Lie algebras, although fairly restrictive, is still much broader than this specific family. For instance, there exist H-type Lie algebras having centers of any given real dimension [8], while the complex Heisenberg Lie algebras all have centers of real dimension 2.

Nevertheless, we shall now prove that the complex Heisenberg Lie algebras are, up to isometric isomorphism, the only complex H-type Lie algebras.

3. Main result

Theorem 3.1. *Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a complex H-type Lie algebra as defined above. Then for some n, $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is isometrically isomorphic to $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ with its standard Hermitian inner product.*

In particular, complex H-type Lie algebras are completely classified by their dimension. We also immediately obtain the analogous classification of complex H-type Lie groups.

Proof. Suppose $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is complex H-type, and let $\mathfrak{v}$, $\mathfrak{z}$ and J be defined as above.

We recall the well-known Clifford algebra identity for H-type Lie algebras:

$$J_z J_w + J_w J_z = -2\langle z, w\rangle I, \quad z, w \in \mathfrak{z}. \tag{3}$$

To prove this, first consider the case when $w = z$ and $\|z\| = 1$. Then for any $u, v \in \mathfrak{v}$, we have

$$\langle J_z^2 u, v\rangle = \langle z, [J_z u, v]\rangle = -\langle z, [v, J_z u]\rangle = -\langle J_z v, J_z u\rangle = -\langle v, u\rangle$$

since J_z is an isometry. So $J_z^2 = -I$. The general case follows by scaling and polarization.

We begin by showing that $\mathfrak{z}$ must have complex dimension 1. If not, then we can find $z, w \in \mathfrak{z}$ with $\|z\| = \|w\| = 1$ and $\langle z, w\rangle = \langle iz, w\rangle = 0$. Then by (3) and (2) we have

$$\begin{aligned}
0 &= -2\langle z, w\rangle I = J_z J_w + J_w J_z \\
0 &= -2\langle iz, w\rangle I = J_{iz} J_w + J_w J_{iz} = iJ_z J_w + J_w iJ_z = i(J_z J_w - J_w J_z).
\end{aligned}$$

Thus $J_w J_z = J_z J_w = 0$, contradicting the requirement that J_z, J_w be isometries.

Therefore, $\mathfrak{z}$ is the complex span of a single unit vector z. We recursively construct an orthonormal basis for $\mathfrak{v}$ over $\mathbb{R}$, of the form $\{x_k, ix_k, y_k, iy_k : k = 1, \dots, n\}$. Suppose $\{x_k, ix_k, y_k, iy_k : k = 1, \dots, m-1\}$ have been constructed and do not span $\mathfrak{v}$. Let x_m be any unit vector orthogonal to all of x_k, ix_k, y_k, iy_k for $k = 1, \dots, m$. Then set $y_m = J_z x_m$. We have $\|y_m\| = 1$, and a few straightforward computations verify that $\{x_k, ix_k, y_k, iy_k : k = 1, \dots, m\}$ are now orthogonal. When the process terminates, we have the desired orthonormal basis.

To compute brackets, for $j \neq k$ we have

$$\begin{aligned}
\langle z, [x_k, y_k]\rangle &= \langle J_z x_k, y_k\rangle = \langle y_k, y_k\rangle = 1 \\
\left\langle z, [x_k, x_j]\right\rangle &= \left\langle J_z x_k, x_j\right\rangle = \left\langle y_k, x_j\right\rangle = 0 \\
\left\langle z, [y_k, y_j]\right\rangle &= \left\langle J_z y_k, y_j\right\rangle = \left\langle J_z y_k, J_z x_j\right\rangle = \left\langle y_k, x_j\right\rangle = 0 \\
\left\langle z, [x_k, y_j]\right\rangle &= \left\langle J_z x_k, y_j\right\rangle = \left\langle y_k, y_j\right\rangle = 0.
\end{aligned}$$

Similar computations show that if z is replaced by iz, all of the above expressions vanish. Each bracket is in $\mathfrak{z}$ and hence a complex scalar multiple of z, so we have

$$[x_k, y_k] = z, \quad [x_k, x_j] = [y_k, y_j] = [x_k, y_j] = 0.$$

The corresponding brackets for ix_k, iy_k, etc, follow from the complex bilinearity of the bracket. These are precisely the same relations as for the complex Heisenberg Lie algebra $\mathfrak{h}_{\mathbb{C}}^{2n+1}$, and the basis is orthonormal, just as for the standard inner product on $\mathfrak{h}_{\mathbb{C}}^{2n+1}$. Therefore, the unique complex linear map $\mathfrak{g} \to \mathfrak{h}_{\mathbb{C}}^{2n+1}$ sending $x_1, y_1, \dots, x_n, y_n, z \in \mathfrak{g}$ to the standard basis for $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ (described in Example 2.3) is an isometric isomorphism of complex Lie algebras. □

Acknowledgements

The author thanks Maria Gordina, Leonard Gross, Martin Moskowitz, Laurent Saloff-Coste, and the anonymous referee for helpful suggestions.

This work was supported by a grant from the Simons Foundation (#355659, Nathaniel Eldredge).

REFERENCES

[1] Zoltán M. Balogh - Jeremy T. Tyson, *Polar coordinates in Carnot groups*, Math. Z. 241 no. 4 (2002), 697–730.

[2] A. Bonfiglioli - E. Lanconelli - F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[3] Nathaniel Eldredge, *Gradient estimates for the subelliptic heat kernel on H-type groups*, J. Funct. Anal. 258 no. 2 (2010), 504–533, arXiv:0904.1781.

[4] Nathaniel Eldredge - Leonard Gross - Laurent Saloff-Coste, *Strong hypercontractivity and logarithmic Sobolev inequalities on stratified complex Lie groups*, To appear in Transactions of the American Mathematical Society. arXiv:1510.05151, 2015.

[5] Jianxun He - Mingkai Yin, *L^p Hardy type inequality in the half-space on the H-type group*, J. Inequal. Appl. (2016), 2016:129, 10.

[6] Jun-Qi Hu - Hong-Quan Li, *Gradient estimates for the heat semigroup on H-type groups*, Potential Anal. 33 no. 4 (2010), 355–386.

[7] J. Inglis - V. Kontis - B. Zegarliński, *From U-bounds to isoperimetry with applications to H-type groups*, J. Funct. Anal. 260 no. 1 (2011), 76–116.

[8] Aroldo Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. 258 no. 1 (1980), 147–153.

[9] Hans Martin Reimann, *Rigidity of H-type groups*, Math. Z. 237 no. 4 (2001), 697–725.

[10] Luca Rizzi, *Measure contraction properties of Carnot groups*, Calc. Var. Partial Differential Equations 55 no. 3 (2016), Art. 60, 20.

NATHANIEL ELDREDGE
School of Mathematical Sciences
University of Northern Colorado
e-mail: `neldredge@unco.edu`