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WEAK WELL-POSEDNESS OF THE DIRICHLET PROBLEM FOR EQUATIONS OF MIXED ELLIPTIC-HYPERBOLIC TYPE

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Equations of mixed elliptic-hyperbolic type with a homogeneous Dirichlet condition imposed on the entire boundary will be discussed. Such *closed* problems are typically overdetermined in spaces of classical solutions in contrast to the well-posedness for classical solutions that can result from *opening the boundary* by prescribing the boundary condition only on a proper subset of the boundary. Closed problems arise, for example, in models of transonic fluid flow about a given profile, but very little is known on the well-posedness in spaces of weak solutions. We present recent progress, obtained in collaboration with D. Lupo and C.S. Morawetz, on the well-posedness in weighted Sobolev spaces as well as the beginnings of a regularity theory.

1. Introduction.

The purpose of this note is to examine the question of well-posedness for the Dirichlet problem for a second order linear partial differential equation of mixed elliptic-hyperbolic type. That is, given $f \in \mathcal{H}_0$, we ask if it is possible to show the existence of a unique $u \in \mathcal{H}_1$ which solves in some reasonable sense the problem

$$Lu = K(y)u_{xx} + u_{yy} = f \text{ in } \Omega \tag{1.1}$$

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$$u = 0 \text{ on } \partial\Omega \tag{1.2}$$

where $\mathcal{H}_0$, $\mathcal{H}_1$ are functions spaces to be determined, $K \in C^1(\mathbb{R})$ satisfies

$$K(0) = 0 \text{ and } yK(y) > 0 \text{ for } y \neq 0, \tag{1.3}$$

Ω is a bounded open and connected subset of $\mathbb{R}^2$ with piecewise C^1 boundary. We will assume throughout that

$$\Omega^{\pm} := \Omega \cap \mathbb{R}^2_{\pm} \neq \emptyset, \tag{1.4}$$

so that (1.1) is of mixed elliptic-hyperbolic type. We will call Ω a *mixed domain* if (1.4) holds.

While the Dirichlet problem (1.1)–(1.2) is classically well-posed for an elliptic operator L, the presence of a hyperbolic subregion Ω^- leads to an over-determined problem in spaces of classical regularity. This phenomenon is well known for purely hyperbolic equations, as first noted by Picone [13]. In the mixed type case, under mild assumptions on the function K and the geometry of the boundary one has a uniqueness result of the following form: *Assume that $\tilde{u}$ is a sufficiently smooth solution to $L\tilde{u} = 0$ in $\Omega \setminus \Sigma$ such that $\tilde{u} = 0$ on $\partial\Omega \setminus \Sigma$. Then $\tilde{u} \equiv 0$ on $\overline{\Omega \setminus \Sigma}$.* Here Σ is a backward light cone with vertex at an interior point $C = (x_0, 0)$ on the parabolic line (see Figure 1). Such uniqueness theorems have been proven by a variety of methods, including energy integrals as in [15] and maximum principles as in [1] and [8]. From this uniqueness theorem, one can show that if u is a sufficiently smooth solution to (1.1) which vanishes only on the proper subset of the boundary $\Gamma = \partial\Omega \setminus \Sigma$, it must vanish on $\overline{\Omega}$. Hence, if one wants to impose the boundary condition on all of the boundary, one must expect in general that some real singularity must be present. Moreover, in order to prove well-posedness, one must make a good guess about where to look for the solution; that is, one must choose some reasonable function space which admits a singularity strong enough to allow for existence but not so strong as to lose uniqueness. This, in practice, has proven to be the main difficulty of the problem.

A boundary value problem such as (1.1)–(1.2) in which the boundary condition is imposed on the entire boundary will se called *closed*, while a problem with the boundary condition imposed on a proper subset will be called *open*. Both open and closed boundary value problems associated to (1.1) are of interest in the study of transonic fluid flow. More precisely, equations of the form (1.1) describe the flow in the hodograph plane where u represents either a stream function (with Dirichlet boundary conditions) or a perturbation of the

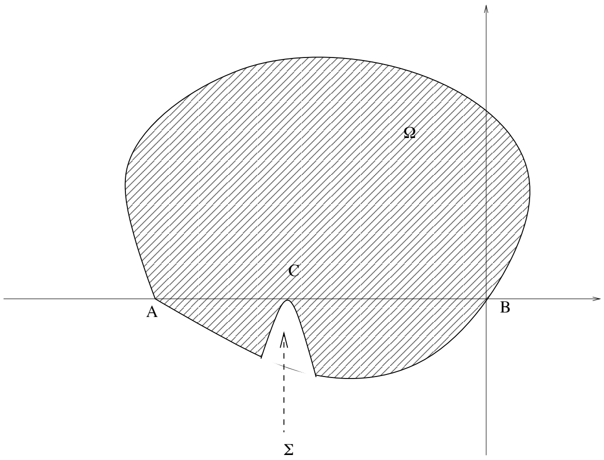

Figure 1: Shaded region $\Omega \setminus \Sigma$

velocity potential (with conormal boundary conditions) and the variables x, y represent the flow angle and a rescaled flow speed respectively (see the classic monograph [2] or the modern survey [11], for example). Open problems arise in flows in nozzles and in proving non-existence of smooth flows past airfoils and closed problems arise in constructing smooth transonic flows about airfoils, see [9]. This connection to transonic flows is our principal motivation.

Despite the interest in closed problems for mixed type equations, the literature essentially contains only two results on well-posedness. The first, due to Morawetz [10] concerns the Dirichlet problem for the Tricomi equation ($K(y) = y$) and the second due to Pilant [14] concerns conormal boundary conditions for the Lavrentiev-Bitsadze equation ($K(y) = \text{sgn}(y)$). In both cases, the restrictions on the boundary geometry are quite severe in that the domains must be lens-like and thin in some sense. Such restrictions on boundary geometry and the type change function are not particularly welcome in the transonic flow applications since the boundary geometry reflects profile or nozzle shape and the approximation $K(y) \sim y$ is valid only for nearly sonic speeds. The main purpose of our investigation is to show well-posedness continues to hold for classes of type change functions and more general domains. Herein we will describe a part of the recent progress made on the Dirichlet problem. The results are contained in [6] which also contains additional results, including work on problems with mixed boundary conditions.

2. Background notions.

In order to present the well-posedness results for (1.1)–(1.2), we would like to first make precise the setting in which we will work. The function $K \in C^1(\mathbb{R})$ will be taken to satisfy (1.3) and additional assumptions as necessary.

In all that follows, Ω will be a bounded mixed domain (open, connected, satisfying (1.4)) in $\mathbb{R}^2$ with piecewise C^1 boundary so that we may apply the divergence theorem and ν will denote the external normal field. Since the differential operator (1.1) is invariant with respect to translations in x, we may assume that the origin is the point on the *parabolic line* $AB := \{(x, y) \in \overline{\Omega} : y = 0\}$ with maximal x coordinate; that is, $B = (0, 0)$. This will simplify certain formulas without reducing the generality of the results. We will sometimes require that Ω is *star-shaped with respect to the flow of a given (Lipschitz) continuous vector field* $V = (V_1(x, y), V_2(x, y))$; that is, for every $(x_0, y_0) \in \overline{\Omega}$ one has $\mathcal{F}_t(x_0, y_0) \in \overline{\Omega}$ for each $t \in [0, +\infty]$ where $\mathcal{F}_t(x_0, y_0)$ represents the time-t flow of (x_0, y_0) in the direction of V.

We will make use of several natural spaces of functions and distributions. We define $H_0^1(\Omega; K)$ as the closure of $C_0^\infty(\Omega)$ (smooth functions with compact support) with respect to the weighted Sobolev norm

$$||u||_{H^1(\Omega;K)} := \left[\int_\Omega \left(|K|u_x^2 + u_y^2 + u^2\right)\, dxdy\right]^{1/2}.$$

Since $u \in H_0^1(\Omega; K)$ vanishes weakly on the entire boundary, one has a *Poincarè inequality*: there exists $C_P = C_P(\Omega, K)$

$$(2,1) \qquad ||u||^2_{L^2(\Omega)} \leq C_P \int_\Omega \left(|K|u_x^2 + u_y^2\right)\, dxdy, \quad u \in H_0^1(\Omega; K).$$

The inequality (2.1) is proven in the standard way by integrating along segments parallel to the coordinate axes for $u \in C_0^1(\Omega)$ and then using continuity. An equivalent norm on $H_0^1(\Omega; K)$ is thus given by

$$(2.2) \qquad ||u||_{H_0^1(\Omega;K)} := \left[\int_\Omega \left(|K|u_x^2 + u_y^2\right)\, dxdy\right]^{1/2}.$$

We denote by $H^{-1}(\Omega; K)$ the dual space to $H_0^1(\Omega; K)$ equipped with its negative norm in the sense of Lax

$$(2.3) \qquad ||w||_{H^{-1}(\Omega;K)} := \sup_{0\neq\varphi\in C_0^\infty(\Omega)} \frac{|\langle w, \varphi\rangle|}{||\varphi||_{H_0^1(\Omega;K)}},$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket and one has the *generalized Schwartz inequality*

$$|\langle w, \varphi \rangle| \leq ||w||_{H^{-1}(\Omega;K)} ||\varphi||_{H_0^1(\Omega;K)}, \quad w \in H^{-1}(\Omega; K), \ \varphi \in H_0^1(\Omega; K). \tag{2.4}$$

One clearly has a rigged triple of Hilbert spaces

$$H_0^1(\Omega; K) \subset L^2(\Omega) \subset H^{-1}(\Omega; K),$$

where the scalar product (on L^2 for example) will be denoted by $(\cdot, \cdot)_{L^2(\Omega)}$.

It is routine to check that the second order operator L in (1.1) is formally self-adjoint when acting on distributions $\mathcal{D}'(\Omega)$ and gives rise to a unique continuous and self-adjoint extension

$$L : H_0^1(\Omega; K) \to H^{-1}(\Omega; K) \tag{2.5}$$

We will also make use of suitably weighted versions of $L^2(\Omega)$ and their properties. In particular, for $K \in C^1(\mathbb{R})$ satisfying (1.3) we define

$$L^2(\Omega; |K|^{-1}) := \{ f \in L^2(\Omega) : \ |K|^{-1/2} f \in L^2(\Omega) \},$$

equipped with its natural norm

$$||f||_{L^2(\Omega;|K|^{-1})} = \left[\int_\Omega |K|^{-1} f^2 \, dxdy \right]^{1/2},$$

which is the dual space to the weighted space $L^2(\Omega; |K|)$ defined as the equivalence classes of square integrable functions with respect to the measure $|K| \, dxdy$; that is, with finite norm

$$||f||_{L^2(\Omega;|K|)} = \left[\int_\Omega |K| f^2 \, dxdy \right]^{1/2}.$$

One has the obvious chain of inclusions

$$L^2(\Omega; |K|^{-1}) \subset L^2(\Omega) \subset L^2(\Omega; |K|),$$

where the inclusion maps are continuous and injective (since K vanishes only on the parabolic line, which has zero measure).

3. Results on existence and uniqueness..

As a first step, using standard functional analytic techniques, one can obtain results on weak existence and strong uniqueness for solutions to the Dirichlet problem (1.1)–(1.2). The key point is to obtain a suitable *a priori* estimate by performing an energy integral argument with a well chosen multiplier. One obtains the following result

Theorem 3.1. *Let Ω be any bounded region in $\mathbb{R}^2$ with piecewise C^1 boundary. Let $K \in C^1(\mathbb{R})$ be a type change function satisfying* (1.3) *and*

$$K' > 0 \tag{3.1}$$

$$\exists \delta > 0 : \ 1 + \left(\frac{2K}{K'}\right)' \geq \delta \tag{3.2}$$

a) *There exists a constant $C_1(\Omega, K)$ such that*

$$||u||_{H^1_0(\Omega;K)} \leq C_1 ||Lu||_{L^2(\Omega)}, \quad u \in C^2_0(\Omega). \tag{3.3}$$

b) *For each $f \in H^{-1}(\Omega; K)$ there exists $u \in L^2(\Omega)$ which weakly solves* (1.1)–(1.2) *in the sense that*

$$(u, L\varphi)_{L^2} = \langle f, \varphi \rangle, \quad \varphi \in H^1_0(\Omega; K) : \ L\varphi \in L^2(\Omega). \tag{3.4}$$

This theorem is the union of Lemma 2.1 and Theorem 2.2 of [6], where the complete proof is given. It should be noted that, since L is formally self-adjoint, the estimate (3.3) also holds for $L^t = L$ and that, in (3.4), L is the self-adjoint extension (2.5). The estimate (3.3) follows from an energy integral argument (the method of multipliers) in which one considers an arbitrary $u \in C^2_0(\Omega)$ and and seeks to estimate the expression $(Mu, Lu)_{L^2}$ from above and below where $Mu = au + bu_x + cu_y$ is the multiplier to be determined. Using

$$a \equiv -1, \quad b \equiv 0, \quad c = c(y) = \max\{0, -4K/K'\} \tag{3.5}$$

one has the needed positive lower bound, while the Cauchy-Scwartz inequality is used for the upper bound. The two estimates are combined with the Poincarè inequality (2.1) to complete the estimate (3.3). The proof of the existence in part **b** is a standard argument using the Hahn-Banach theroem and the Riesz representation theorem.

The estimate (3.3) also shows that sufficiently strong solutions must be unique. We say that $u \in H^1_0(\Omega; K)$ is a *strong solution* of the Dirichlet problem (1.1)–(1.2) if there exists an approximating sequence $u_n \in C^2_0(\Omega)$ such that

$$||u_n - u||_{H^1(\Omega;K)} \to 0 \ \text{ and } \ ||Lu_n - f||_{L^2(\Omega)} \to 0 \ \text{ as } \ n \to +\infty$$

The following theorem is an immediate consequence of the definition.

Theorem 3.2. *Let Ω be any bounded region in $\mathbb{R}^2$ with piecewise C^1 boundary. Let $K \in C^1(\mathbb{R})$ be a type change function satisfying* (1.3), (3.1) *and* (3.2). *Then any strong solution of the Dirichlet problem* (1.1)–(1.2) *must be unique.*

Remarks.

1. The class of admissible K is very large and includes the standard models for transonic flow problems such as the Tricomi equation with $K(y) = y$ and the Tomatika-Tamada equation $K(y) = A(1 - e^{-2By})$ with $A, B > 0$ constants.
2. The result also holds for non strictly monotone functions such as the Gellerstedt equation with $K(y) = y|y|^{m-1}$ where $m > 0$. In this case, one can check that in place of (3.5) it is enough to choose the *dilation multiplier* introduced in [7]

$$a \equiv 0, \quad b = (m+2)x, \quad c = 2y.$$

3. No boundary geometry hypotheses have yet been made; in particular, there are no star-like hypotheses on the elliptic part and no sub-characteristic hypotheses on the hyperbolic part. These kinds of hypotheses will enter when we look for solutions in a stronger sense.

It is clear that the existence is in a very weak sense; too weak, in fact, to be very useful. In particular, the sense in which the solution vanishes at the boundary is only by duality and one may not have uniqueness. Example 2.4 of [6] gives one way in which things may go wrong. In any case, the existence result is a first general indication that while the closed Dirichlet problem is generically over-determined for regular solutions, it is generically not over-determined if one looks for a solution which is taken in a sufficiently weak sense. Moreover, while uniqueness generically holds for strong solutions, one must show that such strong solutions exist.

We are now ready for the well-posedness result, which shows that there is a way to steer a course between the weak existence and the strong uniqueness result for the Dirichlet problem by following the path laid out by Didenko [3] for open boundary value problems. The suitable notion of solutions is contained in the following definition.

Definition 3.3. We say that $u \in H_0^1(\Omega; K)$ is a generalized solution of the Dirichlet problem (2.1)–(2.2) if there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$||u_n - u||_{H_0^1(\Omega;K)} \to 0 \text{ and } ||Lu_n - f||_{H^{-1}(\Omega;K)} \to 0, \text{ for } n \to +\infty$$

or equivalently

$$\langle Lu, \varphi\rangle = -\int_\Omega \left(Ku_x\varphi_x + u_y\varphi_y\right)\, dxdy = \langle f, \varphi\rangle, \quad \varphi \in H_0^1(\Omega, K),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $H_0^1(\Omega; K)$ and $H^{-1}(\Omega; K)$, L is the continuous extension defined in (2.5), and the relevant norms are defined in (2.2) and (2.3).

Our first result concerns the Gellerstedt operator; that is, with K of pure power type

$$K(y) = y|y|^{m-1}, \quad m > 0. \tag{3.6}$$

Theorem 3.4. *Let Ω be a bounded mixed domain with piecewise C^1 boundary and parabolic segment AB with $B = 0$. Let K be of pure power form* (3.6). *Assume that Ω is star-shaped with respect to the vector field $V = (-(m+2)x, -\mu y)$ where $\mu = 2$ for $y > 0$ and $\mu = 1$ for $y < 0$. Then*

a) *there exists $C_1 > 0$ such that*

$$||u||_{L^2(\Omega;|K|)} \leq C_1 ||Lu||_{H^{-1}(\Omega;K)}, \quad u \in C_0^\infty(\Omega). \tag{3.7}$$

b) *for each $f \in L^2(\Omega; |K|^{-1})$ there exists a unique generalized solution $u \in H_0^1(\Omega; K)$ in the sense of Definition 3.3 to the Dirichlet problem* (1.1)–(1.2).

Note that the restriction on the boundary geometry allows for both non lens-like and lens-like domains. Moreover, the star-shaped assumption implies that the hyperbolic boundary is non-characteristic.

This theorem is the union of Lemma 3.3 and Theorem 3.2 of [6]. The existence in part **b)** follows from the a priori estimate of in much the same way as in Theorem 2.1. The uniqueness follows also from the estimate and the Definition 2.3. To obtain the estimate (3.7), the basic idea is to estimate from above and below the expression $(Iu, Lu)_{L^2(\Omega)}$ for each $u \in C_0^\infty(\Omega)$ where $v = Iu$ is the solution to the following auxiliary Cauchy problem

$$\begin{cases} Mv := av + bv_x + cv_y = u & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \setminus B \end{cases}$$

where $B = (0, 0)$ is the righthand endpoint of the parabolic line and

$$a \equiv -1/4, \quad (b, c) = -V = ((m+2)x, \mu y). \tag{3.8}$$

One analyzes first the properties of the solution v, which is shown to lie in $C^\infty(\Omega^\pm) \cap C^0(\overline{\Omega}) \cap H_0^1(\Omega; K)$. Then one estimates from below where the choice (3.8) ensures the positivity of the quadratic form (Iu, Lu). The rest of the estimate proceeds as before, using the generalized Schwartz inequality (2.4), continuity properties of the differential operator M, and the Poincarè inequality (2.1).

Remarks.

1. One can eliminate almost entirely the boundary geometry restrictions in the elliptic part of the domain by patching together the a priori estimate (3.7) with an easy estimate on elliptic subdomans using the multiplier $Mu = u$ (see Theorem 3.4 of [6].

2. One can replace the type change functions K of pure power type with more general forms in which there is a bound on the variation of $(K/K')'$ for y small (see Proposition 3.5 of [6]).

3. With respect to the original result of Morawetz, our improvements are due in part from the fact that we work directly on the second order equation instead of reducing to a first order system, as is done in [10]. Working with the equation allows for a greater freedom in choosing the multipliers (a, b, c); for the first order system there is no coefficient corresponding to a.

4. The norm $H^1(\Omega; K)$ employed here has a weight $|K|$ which vanishes on the entire parabolic line and hence one might worry that the solution is not locally H^1 due to the term $|K|u_x^2$. In fact, the solution does lie in $H^1_{\text{loc}}(\Omega)$ as will be discussed below. On the other hand, the norms used in [10] for treating the equation by way of a first order system were carefully constructed so as not to have weights vanishing on the interior.

4. Local regularity results.

We now discuss the beginnings of a regularity theory for the Dirichlet problem (1.1)–(1.2). in particular, for the applications to transonic flow, it would be important to know that the solutions were at least continuous in the interior. As noted in the Remark 4 above, a generalized solution in the sense of Definition 3.3 with $f \in L^2(\Omega; |K|^{-1})$ will be locally in H^1 as follows from a microlocal analysis argument, if K is smooth enough, using Hörmander's theorem on the propagation of singularities (cf. [12]). Moreover, one can show that u is locally H^2 and hence continuous by the Sobolev embedding theorem, provided that one assumes more regularity on f than is required to the existence and uniqueness.

We will restrict our discussion to the important special case of the Tricomi equation ($K(y) = y$); that is

$$Tu = yu_{xx} + u_{yy} = f \text{ in } \Omega \tag{4.1}$$

$$u = 0 \text{ on } \partial\Omega \tag{4.2}$$

where Ω an admissible mixed domain for the existence result of Theorem 3.4 in this special case. This will allows us to give a direct argument to obtain the local H^1 regularity which avoids the use of microlocal analysis as was suggested to us by Louis Nirenberg. His argument involves a formal estimate which states that locally one can remove the presence of the weight $|K| = |y|$ in the norm (2.2) for sufficiently smooth solutions where mollifying in the x direction yields the needed regularity. The formal estimate is the following (see Lemma 5.1 of [6]).

Lemma 4.1. (Nirenberg) *Let u be a weak solution to the inhomogeneous Tricomi equation* (4.1). *Assume that*

$$F := \int_\Omega f^2\, dxdy < +\infty$$
$$E := \int_\Omega \left(|y|u_x^2 + u_y^2\right)\, dxdy < +\infty.$$

If u is smooth enough, then for each compact subdomain $G \subset \Omega$ there exists a constant $C = C(\Omega, G)$ such that

$$\int_G u_x^2\, dxdy \le C(E + F).$$

The proof involves a sequence of integration by parts starting from the integral of ζu_x^2 where $\zeta \in C_0^\infty(\Omega)$ is a cutoff function with $\zeta \equiv 1$ on G. One checks that the $u_x^2 \in L^1_{\text{loc}}(\Omega)$ is needed to start the estimate and then $u \in H^2_{\text{loc}}(\Omega)$ would be enough to justify the various integration by parts performed.

In order to extend the estimate to a generalized solution u of (4.1)–(4.2), one mollifies in the x-direction. Using the standard properties of this smoothing process (cf. [5]) one finds that the mollified function u_ϵ solves (4.1) with f replaced by its mollification f_ϵ and that $u_\epsilon \in H^2_{\text{loc}}(\Omega)$. Thus one can apply Lemma 4.1 to u_ϵ and pass to a limit with the aid of Fatou's lemma (see Lemma 5.3 and Theorem 5.4 of [6]). This yields the desired result.

Theorem 4.2. *Let Ω be a mixed domain and $f \in L^2(\Omega)$. If $u \in H^1(\Omega, y)$ is a weak solution of the inhomogeneous Tricomi equation $Tu = f$, then $u \in H^1_{\text{loc}}(\Omega)$.*

Finally, higher order local regularity can be achieved by assuming more regularity on the forcing term f.

Theorem 4.3. *Let $u \in H_0^1(\Omega, y)$ be the unique generalized solution to the Dirichlet problem for the inhomogeneous Tricomi equation $Tu = f$ with $f \in L^2(\Omega, |y|^{-1})$. If, in addition, $f_x \in L^2(\Omega, |y|^{-1})$, then $u \in H^2_{\rm loc}(\Omega)$.*

This is Theorem 5.5 of [6]. The idea of the proof is to consider the weak derivative $u_x \in L^2(\Omega, |y|) \cap L^2_{\rm loc}(\Omega)$ which is a weak solution of the equation $Tu_x = f_x$ although no claim is made about its boundary values. One could call this an *interior weak solution* in analogy with the definition of Sarason [16] for first order systems. Comparing u_x with $w \in \in H_0^1(\Omega; y)$ the unique generalized solution to (4.1)–(4.2) with right hand side $f_x \in L^2(\Omega, |y|^{-1})$ one has

$$(w, Tv)_{L^2(\Omega)} = (f_x, v)_{L^2(\Omega)} = (u_x, Tv)_{L^2(\Omega)}, \quad \forall v \in C_0^\infty(\Omega).$$

Now, if one knew that for each compact subdomain G of Ω one has

$$\{g \in L^2(G) : \quad g = Tv, \quad v \in C_0^\infty(\Omega)\} \text{ is dense in } L^2(G), \tag{4.3}$$

then the result would follow. The verification of this density claim depends on the fact that: every interior weak solution is an *interior strong solution* in the sense that there exists a sequence of functions $U_n \in C_0^\infty(\Omega)$ such that

$$\lim_{n \to +\infty} \left[||U_u - U||_{L^2(G)} + ||TU_n - g||_{L^2(G)} \right] = 0.$$

That interior weak solutions are interior strong solutions follows for the Tricomi equation because T is *principally normal* and Ω admits a function $\psi \in C^2(\overline{\Omega})$ such that the level sets of ψ are *pseudo-convex with respect to T*, facts noted by Hörmander (cf. Theorem 8.7.4 of [4] and Section 8.6 of [4]).

Remark. One can generalize Theorems 4.2 and 4.3 to include more general type change functions $K(y)$. One only needs that K is sufficiently smooth in order to apply Hörmander's results on singularity propagation as well as the weak equals strong result used to obtain the density (4.3).

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