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FROM DETERMINISTIC TO STOCHASTIC EQUILIBRIUM PROBLEMS: EXISTENCE OF SOLUTIONS AND APPLICATION TO CONVEX PROGRAMMING

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In this paper, by means of Castaing representation Theorem, we extend the results by M. Ait Mansour, R-.A. Elakri and M. Laghdir [*Equilibrium and quasi-equilibrium problems under φ-quasimonotonicity and φ-quasiconvexity, Existence, stability and applications*, Minimax Theory and its Applications, **2** (2) (2017), 175–229] to random equilibrium problems wherein the objective data are subject to a random perturbation. We obtain deterministic as well as random (or measurable) and integral solutions to random equilibrium problems. The case when the random parameter has a probability of realization suggests us to introduce stochastic formulations of equilibria covering the expected value approach and the almost sure method. Thus, we prove the existence of two further kinds of equilibria under uncertainty: almost sure solutions and expected stochastic equilibrium points. Finally, we present an application to stochastic convex and quasiconvex programming for which we establish the existence of almost sure minimizers.

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1. Introduction

Given a real-valued bifunction $f : K \times K \longrightarrow \mathbb{R}$ and a closed convex subset K of a normed space X by $EP(f,K)$ we understand: Find $\bar{x} \in K$ such that

$$EP(f,K) \qquad f(\bar{x},y) \geq 0, \ \forall y \in K.$$

It is well known that optimization problems , variational inequalities, fixed points and many other related problems can be converted into the equilibrium problem in the sense of Blum and Oetlli [5], which thereby turns out to be a convenient and unified framework for the study of all these problems. We refer to [1–8, 16] and references therein for different studies of the problem $EP(f,K)$ concerning existence of solutions, stability and proximal algorithms.

Parametric equilibrium problems have been investigated in many papers as in [2, 3] for example wherein the perturbing parameters are determined in normed spaces. Yet, the deterministic perturbation in the problem $EP(f,K)$ proves to be inadequate for many interesting real applications such as in finance problems or market equilibrium models, traffic networks and operation research where the corresponding preference and/or objective bifunction may be subject to a randomness and uncertainty. Actually, in these applied models, the bifunction f and/or on the feasibility sets involve in reality random components. These considerations suggest to consider, naturally, stochastic and random generalizations of the deterministic problem $EP(f,K)$, which are evidently of a particular relevance although they require a quite different material in addition to the problematic of defining adequate random and stochastic solutions for adequate formulations. By considering a continuous probability space for the random variable, we shall introduce random, almost-sure, integral or else stochastic solutions in terms of an expected-value formulation. Such models with random or uncertain data have been treated sufficiently in the literature for optimization problems and variational inequalities, see for instance the survey by G-H. Lin and M. Fukushima [20], see also the very recent paper by B. Jadamba, A. Khan and F. Raciti [17] wherein a detailed comparison between the measure theoretic and integral approach and the more OR-oriented sample-path approach is performed. However, as far as we know the stochastic abstract formulation for the problem $EP(f,K)$ has not yet been considered. One of the usual methods to deal with stochastic problems is to define a convenient pointwise deterministic equivalent model. In this way, motivated by E. Kalmoun [18], K.-K. Tan and X.-Z Yuan [22] and the recent works by Gwinner and Raciti [11–14] on random variational inequalities and inspired by the deterministic case of $EP(f,K)$ we first introduce the random equilibrium problem as follows: Find a function $\gamma : \Omega \longrightarrow K$ such that:

$$REP(f,K) \qquad f(\omega,\gamma(\omega),y) \geq 0 \;\; \forall y \in K, \text{a. e.} \omega \in \Omega.$$

Here, X and K are still as before, Ω is a space of random events and $f : \Omega \times K \times K \longrightarrow \mathbb{R}$ is a given (random) function. For every $\omega \in \Omega$, whenever it exists, $\gamma(\omega)$ satisfying such a random inequality, it is called a *deterministic solution* of the $REP(f,K)$. If in addition γ is measurable then it is called a *random solution* to $REP(f,K)$. On the basis of the famous Castaing representation Theorem [19], we first prove for a variety of assumptions two existence results of random solutions obtained, under measurability assumptions, from those corresponding to the deterministic problem $EP(f,K)$: Theorem 3.8 and Theorem 3.9. Integral solutions are then derived in Corollary 3.11. Further, we introduce new stochastic formulations for equilibrium problems for which we establish almost-sure solutions as well as expected-value stochastic equilibria in Theorem 4.3. The latter enables us to obtain almost-sure minimizers of stochastic programming for Carathéodory real-valued functions, see Corollary 4.8 which covers the case of convexity assumption while in the quasiconvex setting we reduce the problem to the one of a finite system of equilibrium problems having a common stochastic solution. Here, as a first step, we restrict ourselves to the randomness at the level of the bifunction.

The reminder of the paper is organized as follows. Section 2 summarizes the preliminaries and the background we need for our treatment. In Section 3 we concentrate our attention on the random equilibrium problems for which we prove the existence of deterministic solutions, random solutions as well as integral solutions. In Section 4, we introduce stochastic equilibrium problems wherein both of the expected-value and the almost sure formulations are considered, and discuss an application to stochastic quasiconvex programming.

2. Preliminaries

Through this paper unless otherwise is specified, X is a normed space and K is closed convex subset of X. The topological dual of X will be denoted by X^* with duality paring $\langle .,. \rangle$. X^* will always be endowed with its weak-* topology. B_X will stand for the closed unit ball of X, and for any $x_0 \in K$ and $r > 0$ we denote by $B(x_0, r)$ the open ball centered in x_0 with radius r. For an arbitrary subset A of X, we denote by $\overline{A}$ (or $cl(A)$) the closure in X, by $\overline{A}^K$ the closure in K, $Conv(A)$ its convex hull. If $A \subseteq K$ we denote by $int(A)$ the topological interior of A, by A^c the complementary of A in K that is the set $\{x \in K : x \notin A\}$. For a multifunction $F : K \longrightarrow 2^K$, the notation $Gr(F)$ stands for the graph of F that is the set of elements (x,y) in $K \times K$ satisfying $y \in F(x)$.

For the classical concepts of semicontinuity and closedness for set-valued maps we refer to [9].

2.1. Generalized convexity and generalized quasimonotonicity

Given a convex subset K of X, a function $g : K \longrightarrow \mathbb{R}$ is said to be:

- Convex if for all $x, y \in K$,

$$g(tx + (1-t)y) \leqslant tg(x) + (1-t)g(y) \qquad \forall t \in [0,1].$$

- Concave if $-g$ is convex.

- Quasiconvex if for all $x, y \in K$

$$g(tx + (1-t)y) \leqslant \max\{g(x), g(y)\} \qquad \forall t \in [0,1].$$

- Quasiconcave if for all $x, y \in K$ such that $x \neq y$,

$$g(tx + (1-t)y) \geq \min\{g(x), g(y)\} \qquad \forall t \in [0,1].$$

- Strictly quasiconvex if for all $x, y \in K$ such that $x \neq y$,

$$g(tx + (1-t)y) < \max\{g(x), g(y)\} \qquad \forall t \in]0,1[.$$

- Semistrictly quasiconvex if for all $x, y \in K$ such that $x \neq y$,

$$g(x) < g(y) \Longrightarrow g(tx + (1-t)y) < g(y), \qquad \forall t \in]0,1[. \tag{1}$$

- *Second type semistrictly quasiconvex* if g is quasiconvex and (1) holds.
- Semistrictly quasiconcave if $-f$ is semistrictly quasiconvex;
- Upper semicontinuous at $x_0 \in K$ means that:

$$\limsup_{x \longrightarrow x_0} g(x) \leqslant g(x_0),$$

  where lim sup is the upper limit of the function g at point x_0.

- Upper hemicontinuous if it is upper semicontinuous on every segment of K.

- Lower hemicontinuous if $-f$ is upper Hemicontinuous.

The next definition present a mode of generalized convexity that has some links with φ-quasimonotonicity, see [1].

Definition 2.1. A function $g : K \longrightarrow \mathbb{R}$ is said to be φ-quasiconvex if for any $x, y \in K$ and any $t \in [0,1]$,

$$g(tx+(1-t)y) \leqslant \max\Big(g(x), g(y)\Big) - t(1-t)\varphi(x,y).$$

We recall now the following generalizations of monotonicity introduced in [1]:

Definition 2.2. Let f, φ be two real-valued bifunctions defined on $K \times K$. Assume that $\varphi(x,y) = \varphi(y,x)$ for all $x, y \in K$. f is said to be

- φ-*monotone* if for all $x, y \in K$,

$$f(x,y) + f(y,x) \leqslant \varphi(y,x). \tag{2}$$

- φ-*pseudomonotone* if for all $x, y \in K$,

$$f(x,y) \geqslant 0 \Longrightarrow f(y,x) \leqslant \varphi(y,x) \tag{3}$$

  or equivalently

$$f(x,y) > \varphi(x,y) \Longrightarrow f(y,x) < 0. \tag{4}$$

- φ-*quasimonotone* if, and only if for all $x, y \in K$,

$$f(x,y) > 0 \Longrightarrow f(y,x) \leqslant \varphi(y,x) \tag{5}$$

  equivalently,

$$f(x,y) > \varphi(x,y) \Longrightarrow f(y,x) \leqslant 0 \quad \forall x, y \in K. \tag{6}$$

- φ-*Properly quasimonotone* if, and only if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in K$ and $\overline{x} \in Conv\{x_1, \ldots, x_n\}$ there exists $i \in \{1, \ldots, n\}$ such that

$$f(x_i, \overline{x}) \leqslant \varphi(x_i, \overline{x}).$$

Remark 2.3. According to [1] we define similarly the following:

- φ-*relaxed monotone* if $\varphi \geqslant 0$ and for all $x, y \in K$,

$$f(x,y) + f(y,x) \leqslant \varphi(y,x). \tag{7}$$

- φ-*strongly monotone* if $\varphi \geqslant 0$ and for all $x, y \in K$,

$$f(x,y) + f(y,x) \leqslant -\varphi(y,x). \tag{8}$$

- φ-*relaxed pseudomonotone* if, and only if $\varphi \geqslant 0$ and for all $x, y \in K$,

$$f(x,y) \geqslant 0 \Longrightarrow f(y,x) \leqslant \varphi(y,x) \tag{9}$$

  or equivalently

- φ-*strongly pseudomonotone* if, and only if $\varphi \geqslant 0$ and for all $x, y \in K$,

$$f(x,y) \geqslant 0 \Longrightarrow f(y,x) \leqslant \varphi(y,x) \tag{10}$$

  or equivalently

- φ-*relaxed quasimonotone* if f is φ-quasimonotone and $\varphi \geqslant 0$;
- φ-*strongly quasimonotone* if, and only if $\varphi \geqslant 0$ and for all $x, y \in K$,

$$f(x,y) > 0 \Longrightarrow f(y,x) \leqslant -\varphi(y,x) \tag{11}$$

2.2. Minty solutions and upper sign properties

Definition 2.4. Let f be a real-valued bifunction defined on $K \times K$ and $\mu \geqslant 0$. A point $\bar{x} \in K$ is said to be a μ-*global weak Minty* equilibrium or μ-*global relaxed Minty equilibrium* for f if

$$f(y,\bar{x}) \leqslant \mu ||y - \bar{x}||^2, \ \forall y \in K. \tag{12}$$

The set of μ-global weak Minty equilibrium points for f will be denoted by $M_w^{\mu}(f,K)$.

If (12) is satisfied only in a neighborhood of $\bar{x}$, then $\bar{x}$ coincides with the local μ-relaxed Minty equilibrium for f introduced in [6] in the sense of the following:

Definition 2.5. Let $\mu \geq 0$, $K \subset X$ and $f : K \times K \to \mathbb{R}$. A point $x \in K$ will be said a *local relaxed μ-Minty equilibrium* for f if, and only if there exists a neighborhood $\mathcal{V}_x$ of x such that

$$f(y,x) \leqslant \mu\|y-x\|^2, \quad \forall y \in K \cap \mathcal{V}_x.$$

For any $\mu \geqslant 0$, $M_L^{\mu}(f,K)$ will stand for the local Minty equilibrium points of f over K.

The following concept plays a fundamental role in the connection of Minty equilibrium points with standard ones.

Definition 2.6. Let $\mu \geq 0$ and K be a convex subset of X. A bifunction $f : K \times K \to \mathbb{R}$ will be said to have the *local μ-upper sign property* at $x \in K$ if there exists a convex neighborhood $\mathcal{V}_x$ of x such that for all $y \in \mathcal{V}_x \cap K$,

$$f(z_t,x) \leqslant \mu\|z_t - x\|^2, \forall t \in]0,1[\Longrightarrow f(x,y) \geqslant 0, \tag{13}$$

where $z_t = (1-t)x + ty$. If (13) is satisfied for all $y \in K$ then f will be said to have the *global μ-upper sign property* in x

We need equally to recall the following property which is essential together with (13) in the next Proposition.

$$f(x,y) < 0 \Longrightarrow f(x,(1-t)x+ty) < 0, \ \forall t \in]0,1[. \tag{14}$$

From now on we denote by $S(f,K)$ the solution set to $EP(f,K)$.

Proposition 2.7. *Let $\mu > 0$. Assume that f has the local μ-upper sign property in x and (14) is satisfied. Then we have*

1) $M_L^{\mu}(f,K) \subset S(f,K)$.

2) If, in addition, $f(x,x) = 0$ for all $x \in K$ and f is μ-relaxed quasimonotone and strictly quasiconvex in y then

$$M_w^{\mu}(f,K) = M_L^{\mu}(f,K) = S(f,K). \tag{15}$$

Proof. See [1, Proposition 5.32].

□

2.3. Existence of equilibria under generalized quasiconvexity

Now, we recall the assumptions and existence results established in [1] on which we will base our random and stochastic treatment.

($\mathbb{H}_0$) $f(x,x) = 0$ for all $x \in K$;

($\mathbb{H}_1$) $f(x,x) \geqslant 0$ for all $x \in K$;

($\mathbb{H}_2$) $F_\varphi(x) = \{y \in K|\ f(x,y) \leqslant \varphi(x,y)\}$ is compactly closed for all $x \in K$ i.e., $F_\varphi(x) \cap B$ is closed for every compact $B \subset K$;

($\mathbb{H}_3$) f is properly φ-quasimonotone;

($\mathbb{H}_4$) f is φ-quasimonotone and $f(x,.)$ is strictly quasiconvex for all $x \in K$;

($\mathbb{H}_5$) $f(.,y)$ is diagonally quasiconcave for every $y \in K$;

($\mathbb{H}_6$) f is φ-quasimonotone and not properly φ-quasimonotone, and $f(x,.)$ is quasiconvex for all $x \in K$;

($\mathbb{H}_7$) f is φ-pseudomonotone and $f(x,.)$ is quasiconvex for all $x \in K$.

Next, we recall a suitable existence result for our purpose on the following generalized Minty equilibrium:

$$\varphi - MEP(f,K)) : \quad f(y,\bar{x}) \leqslant \varphi(y,\bar{x}),\ \forall y \in K.$$

Proposition 2.8. *([1]). Let $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction and let K be a convex and compact subset of X. Assume that ($\mathbb{H}_i$) is fulfilled for $i \in \{1,2\}$ and either ($\mathbb{H}_4$) or ($\mathbb{H}_7$) is satisfied. Then, $\varphi - (MEP)$ admits at least one solution.*

The following coercivity conditions will be needed in the next result.
(C_1) There exists a compact and convex subset B of K, for all $x \in K \setminus B$, there exists $y \in B$ such that

$$f(y,x) > \varphi(y,x).$$

(C_2) There exists a compact and convex subset B of K, for all $x \in K \setminus B$, there exists $y \in B$ such that

$$f(x,y) < \varphi(x,y).$$

Theorem 2.9. *([1]). Let $\mu \geq 0$ and φ the bivariate function defined by*

$$\varphi(x,y) = \mu\|x-y\|^2.$$

Assume that K is a convex subset of X and $f : K \times K \longrightarrow \mathbb{R}$ a bifunction such that ($\mathbb{H}_2$) is satisfied. Then the following assertions are satisfied:

1) *$M_w^\mu(f,K)$ is nonempty and $M_w^\mu(f,K) \subset B$ if (C_1) and one of the following conditions are satisfied:*

   i) $(\mathbb{H}_3)$*;*

   ii) $(\mathbb{H}_i)$ *for* $i \in \{1,4\}$;

   iii) $(\mathbb{H}_i)$ *for* $i \in \{0,5\}$.

   iv) $(\mathbb{H}_i)$ *for* $i \in \{1,7\}$;

   If, in addition, f has the global μ-upper sign property in x, then $\emptyset \neq M_w^\mu(f,K) \subset S(f,K)$.

2) *If $(\mathbb{H}_6)$ holds true, the condition (14) is verified, f has the local μ-upper sign property and non-continuous in x. then $\emptyset \neq M_L^\mu(f,K) \subset S(f,K)$. If in addition, f is strictly quasiconvex and satisfies $(\mathbb{H}_0)$ then $M^\mu(f,K) = S(f,K) = M_L^\mu(f,K)$. If moreover f satisfies (C_1) or (C_2) then $S(f,K) \subset B$.*

Concerning examples of classes of functions which satisfy our definitions in Remark 2.3 and Definition 2.4, we refer to [1], wherein Proposition 2.8 and Theorem 2.9 are also discussed in the context of variational and quasi-variational inequalities as well as quasiconvex programming, see also [6] for illustrations of the related assumptions.

3. Random equilibrium problems

In this section, we deal with random equilibrium problems where a random data, say ω, arises in the bifunction. We need to recall the following notations and basic facts on measurability. X is still a normed space as before, till otherwise is specified, and K a closed convex of X. Let $(\Omega, \mathcal{F})$ be a measurable space with $\mathcal{F}$ is a σ-algebra of subsets of Ω, that is, a collection of subsets of Ω closed under countably many set operations (complement, union and intersection) and $\mathcal{B}(X)$ is the Borel σ -algebra, that is, the smallest σ-algebra containing all open subsets of X. We denote by $\mathcal{F} \otimes \mathcal{B}(X)$ the σ-algebra generated by $\{A \times B : A \in \mathcal{F} \text{ and } B \in \mathcal{B}(X)\}$.

- A function $g : \Omega \longrightarrow X$ is said to be *$\mathcal{F}$-measurable* (or simply measurable or else *random*) if $g^{-1}(A) \in \mathcal{F}$ for every Borel subset A in X.
- A map $g : \Omega \times K \longrightarrow \mathbb{R}$ is said to be *Carathéodory function* if for each fixed $x \in K$, $g(.,x)$ is measurable with respect to $\mathcal{F}$, and for every $\omega \in \Omega$, $g(\omega,.)$ is continuous.

- A multifunction $F:\Omega\longrightarrow 2^X$ is said to have a *measurable* selection if there exists a measurable function $f:\Omega\longrightarrow X$ such that $f(\omega)\in F(\omega)$ for all $\omega\in\Omega$.

- A multifunctions $F:\Omega\longrightarrow 2^X$ has a *Castaing representation* ([19]) whenever there is a countable family of measurable selections $(f_i)_{i\geq 1}$ such that $(f_i(\omega))_{i\geq 1}$ is dense in $F(\omega)$ (with respect to the topology of X) for each $\omega\in\Omega$.

Remark 3.1. If we replace the set Ω by $\Omega\backslash\tilde{\Omega}$, where $\tilde{\Omega}$ is a subset of Ω with null measure, the conclusion of Castaing representation may be stated for almost every $\omega\in\Omega$. The same remark is valid for measurable selections.

Definition 3.2. (Suslin operation and Suslin families, [19])
Let Λ and $\Lambda^{'}$ be the sets of infinite and finite sequences of positive integers respectively and $\mathfrak{S}$ a family of sets. For $\sigma=(\sigma)_{i\geqslant 1}\in\Lambda$, we denote by $\sigma|_n$ the finite sequences $(\sigma_1,...,\sigma_n)$. Let $A:\Lambda^{'}\longrightarrow\mathfrak{S}$. Under these notations:

- the element (of $\mathfrak{S}$) $\bigcup_{\sigma\in\Lambda}\bigcap_{n=1}^{\infty}A(\sigma|_n)$ is said to be obtained from $\mathfrak{S}$ by the *Suslin operation*.

- $\mathfrak{S}$ is said to be *Suslin family* if it is stable by the Suslin operation.

- A topological space is a *Polish* space if it is a separable complete metrizable space.

- A topological space is *Suslin* if it is a topological space that is the continuous image of a Polish space.

- A subset of X is *Suslin* if it is a Suslin (topological) space.

Motivated by the recent works by Gwinner and Raciti on random variational inequalities [11, 13] and inspired by the deterministic case, given a space of random events Ω and a real-valued function $f:\Omega\times K\times K\longrightarrow\mathbb{R}$ we introduce the random equilibrium problem as follows: Find a function $\gamma:\Omega\longrightarrow K$ such that:

$$REP(f,K)\qquad f(\omega,\gamma(\omega),y)\geqslant 0\quad\forall y\in K,\,for\text{ a. e.}\omega\in\Omega.$$

Notation: For each $\omega\in\Omega$, we write $f_\omega=f(\omega,.,.)$.
For each $\omega\in\Omega$, we consider the problem: Find $x(\omega)\in K$ such that

$$EP(f_\omega,K)\qquad f_\omega(x(\omega),y)\geqslant 0,\quad\forall y\in K.$$

Definition 3.3. Whenever the solution $\gamma : \Omega \to K$ to $REP(f,K)$ exists it is called a deterministic solution to $REP(f,K)$. If in addition γ is measurable map then it is called a random solution to $REP(f,K)$.

The solution set to $EP(f_\omega,K)$ will be denoted by $S(\omega)$ i.e., the solution map $S : \Omega \rightrightarrows K$ is the multifunction defined by

$$S(\omega) =: \bigcap_{y\in K} \{x \in K : f_\omega(x,y) \geqslant 0\}. \tag{16}$$

If $(\Omega,\mathcal{F},\rho)$ is a measure space (where ρ is measure on $\mathcal{F}$) and f is integrable in ω then any random solution $\gamma : \Omega \longrightarrow K$ to $REP(f,K)$ is also a solution to the corresponding integral problem: Find a measurable function $U : \Omega \longrightarrow K$ such that:

$$IREP(f,K) \qquad \int_\Omega f(\omega,U(\omega),y)d\rho(\omega) \geqslant 0 \quad \forall y \in K.$$

Let $\mu \geqslant 0$. The corresponding random relaxed Minty problem is as follows: Find $\gamma : \Omega \longrightarrow K$ such that:

$$\mu - RMEP(f,K) \qquad f(\omega,y,\gamma(\omega)) \leqslant \mu\|y-\gamma(\omega)\|^2, \quad \forall y \in K, \text{ a. e. } \omega \in \Omega.$$

We also consider for each $\omega \in \Omega$ and $f_\omega = f(\omega,.,.)$ the problem: Find $x(\omega) \in K$ such that

$$\mu - MEP(f_\omega,K) \qquad f_\omega(y,x(\omega)) \leqslant \mu\|y-x(\omega)\|^2, \quad \forall y \in K. \tag{17}$$

The solution set to $\mu - MEP(f_\omega,K)$ will be denoted by $M^\mu(\omega)$ i.e., the solution map $M : \Omega \rightrightarrows K$ is the multifunction defined by

$$M^\mu(\omega) =: \bigcap_{x\in K} \{y \in K : f_\omega(x,y) \leqslant \mu\|y-x\|^2\}.$$

If (17) is true only for y in a neighborhood of $x(\omega)$, for a given value of $\omega \in \Omega$, we rather speak about local deterministic solutions whose set will be denoted by $M_L^\mu(f_\omega,K)$ or simply by $M_L^\mu(\omega)$.
The random existence theorem we propose is based on the following lemma, which is due to Less (see [19]).

Lemma 3.4. *Let $\mu \geqslant 0$. If $(\Omega,\mathcal{F})$ is measurable space with $\mathcal{F}$ is a Suslin family, X is a Suslin space, and $F : \Omega \longrightarrow 2^X$ is a map with nonempty values such that $GrF \in \mathcal{F} \otimes \mathcal{B}(X)$. Then F has a Castaing representation.*

Lemma 3.5. *Assume that K is a separable, closed and convex subset of X and that the solution map M^{μ} has nonempty values. If $f(\omega,.,y)$ is lower semicontinuous (in x) for all $\omega \in \Omega$ and all $y \in K$, then, there exists a countable dense sequence $(x_n)_{n\geqslant 1}$ in K such that*

$$Gr(M^{\mu}) = \bigcap_{n\geq 1}\{(\omega,y) \in \Omega\times K : f(\omega,x_n,y) \leqslant \mu\|x_n-y\|^2\}. \tag{18}$$

If in addition for all $x \in K$, $f(.,x,.)$ is $\mathcal{F}\otimes\mathcal{B}(K)-\mathcal{B}(\mathbb{R})$-measurable, then $Gr(M^{\mu}) \in \mathcal{F}\otimes\mathcal{B}(K)$.

Proof. Since K is separable, there exists a countable dense sequence $(x_n)_{n\geqslant 1}$ in K. Then, for all $x \in K$, there exists a subsequence $(n_k)_k$ such that

$$(x_{n_k})_{k\geqslant 1} \to x. \tag{19}$$

Clearly, for all $\omega \in \Omega$, $M^{\mu}(\omega) \subset \bigcap_{n\geqslant 1}\{y \in K : f(\omega,x_n,y) \leqslant \mu\|x_n-y\|^2\}$. Fix $\omega \in \Omega$ and let $y \in K$ such that

$$f(\omega,x_n,y) \leqslant \mu\|x_n-y\|^2, \;\; \forall n \geqslant 1. \tag{20}$$

In particular, for any subsequence $(n_k)_k$,

$$f(\omega,x_{n_k},y) \leqslant \mu\|x_{n_k}-y\|^2, \;\; \forall k \geqslant 1. \tag{21}$$

Hence, from the lower semicontinuity of f in x and (19), it follows that

$$f(\omega,x,y) \leqslant \mu\|x-y\|^2, \forall x \in K.$$

This shows that $y \in M^{\mu}(\omega)$. Hence,

$$M^{\mu}(\omega) = \bigcap_{n\geqslant 1}\{y \in K : f(\omega,x_n,y) \leqslant \mu\|x_n-y\|^2\}. \tag{22}$$

Consequently,

$$Gr(M^{\mu}) = \bigcap_{n\geq 1}\{(\omega,y) \in \Omega\times K : f(\omega,x_n,y) \leqslant \mu\|x_n-y\|^2\}.$$

Now, if for all $x \in K$, $f(.,x,.)$ is $\mathcal{F}\otimes\mathcal{B}(K)-\mathcal{B}(\mathbb{R})$-measurable, then for all $n \geqslant 1$, we have $\{(\omega,y) \in \Omega\times K : f(\omega,x_n,y) \leqslant \mu\|x_n-y\|^2\} \in \mathcal{F}\otimes\mathcal{B}(K)$. Therefore $Gr(M^{\mu})$, as a countable intersection of elements of $\mathcal{F}\otimes\mathcal{B}(K)$, is an element of $\mathcal{F}\otimes\mathcal{B}(K)$.

□

Following lines of the proof of Lemma 3.5, measurability property for the solution map $S(\omega) =: \bigcap_{y\in K}\{x \in K : f(\omega,x,y) \geqslant 0\}$ is established in the following:

Lemma 3.6. *Assume that K is a separable, closed and convex subset of X and that the solution map S has nonempty values. If $f(\omega,x,.)$ is upper semicontinuous (in y) for all $\omega \in \Omega$ and all $x \in K$, then, there exists a countable dense sequence $(y_n)_{n\geqslant 1}$ in K such that*

$$Gr(S) = \bigcap_{n\geq 1}\{(\omega,x) \in \Omega\times K : f(\omega,x,y_n) \geqslant 0\}. \tag{23}$$

If in addition for all $y \in K$, $f(.,.,y)$ is $\mathcal{F}\otimes\mathcal{B}(K) - \mathcal{B}(\mathbb{R})$-measurable, then $Gr(S) \in \mathcal{F}\otimes\mathcal{B}(K)$.

Remark 3.7. In the next results, we adopt the Castaing representation for almost every $\omega \in \Omega$.

Theorem 3.8. *Let $\mu \geqslant 0$, K be a separable, closed, convex and Suslin subset of Suslin normed space X and $f : \Omega\times K\times K \longrightarrow \mathbb{R}$ a real-valued function such that for almost every $\omega \in \Omega$ $f(\omega,.,.)$ satisfies (C_1), $(\mathbb{H}_2)$ and one of the conditions :*

i) $(\mathbb{H}_3)$;

ii) $(\mathbb{H}_i)$ *for* $i \in \{1,4\}$;

iii) $(\mathbb{H}_i)$ *for* $i \in \{0,5\}$.

iv) $(\mathbb{H}_i)$ *for* $i \in \{1,7\}$.

Then:

1) *The problem $\mu - RMEP(f,K)$ admits a deterministic solution i.e., for almost every $\omega \in \Omega$, the problem $\mu - RMEP(f_\omega,K)$ has a solution i.e., $M^\mu(\omega)$ is nonempty and $M^\mu(\omega) \subset B$ for a.e. $\omega \in \Omega$.*

2) *if in addition $f(\omega,.,y)$ is lower semicontinuous (in x) for all $\omega \in \Omega$ and all $y \in K$, and $f(.,x,.)$ is $\mathcal{F}\otimes\mathcal{B}(K) - \mathcal{B}(\mathbb{R})$-measurable for each $x \in K$ then the problem $\mu - RMEP(f,K)$ admits a countable family of random solutions $\gamma_i : \Omega \to K$, $i \geq 1$, such that $\{\gamma_i(\omega)|\ i \geq 1\}$ is dense in $M^\mu(\omega)$ for almost every $\omega \in \Omega$.*

3) *If, more than the measurability assumption in 2), for almost every* $\omega \in \Omega$, $f(\omega,.,.)$ *has the global* μ*-upper sign property in x, then* $\emptyset \neq M^{\mu}(\omega) \subset S(\omega)$ *and* $REP(f,K)$ *admits a countable family of random solutions* $\gamma_i : \Omega \to K$, $i \geq 1$.

Proof. The assertion 1) is a direct consequence of assertion 1) of Theorem 2.9. Indeed, by this Theorem, for almost every $\omega \in \Omega$, the problem $\mu - MEP(f_{\omega},K)$ has at least a solution denoted by $\gamma(\omega)$ and moreover $M^{\mu}(\omega) \subset B$ for a.e. $\omega \in \Omega$. Then the map $\gamma : \Omega \to K$, $\omega \mapsto \gamma(\omega)$ is a deterministic solution to $\mu - RMEP(f,K)$. For the second assertion 2), M^{μ} has a nonempty image in view of 1). Therefore, thanks to Lemma 3.5, $Gr(M^{\mu})\mathcal{F} \otimes \mathcal{B}(K)$. Thus by Lemma 3.4, the map M^{μ} has Castaing representation. Accordingly, there exists $(\gamma_i)_{i\geq 1}$ such that for each $i \in \{1,......,n\}$:

- $\gamma_i : \Omega \longrightarrow K$ is measurable.
- $(\gamma_i(\omega))_{i\geq 1}$ is dense in $M^{\mu}(\omega)$ for almost every $\omega \in \Omega$.
- $f(\omega,x,\gamma_i(\omega)) \leqslant \mu\|x-\gamma_i(\omega)\|^2$, for almost every $\omega \in \Omega$ and all $x \in K$, for each $i \in \{1,......,n\}$.

Finally, the assertion in 3) descends immediately from 2) and the μ-(global) upper sign property of f in x. □

Theorem 3.9. *Let* $\mu \geq 0$ *and K be a separable, closed, convex and Suslin subset of Suslin normed space X and* $f : \Omega \times K \times K \longrightarrow \mathbb{R}$ *a real-valued function such that for almost every* $\omega \in \Omega$, $f(\omega,.,.)$ *satisfies* ($\mathbb{H}_2$) *and* ($\mathbb{H}_6$)*. Then,*

1) *for almost every* $\omega \in \Omega$, *the problem* $\mu - MEP(f_{\omega},K)$ *admits a local deterministic solution.*

2) *If in addition, for almost every* $f(\omega,.,.)$ *has the local* μ*-upper sign property but non-continuous in x and satisfies the condition (14), then the problem* $EP(f_{\omega},K)$ *has a deterministic solution i.e.,* $S(\omega)$ *is nonempty-set for a.e.* $\omega \in \Omega$

3) *If more than the assumption in 2), for almost every* $\omega \in \Omega$, ***either***

   a) $f(\omega,.,.)$ *is strictly quasiconvex in y, satisfies* ($\mathbb{H}_0$)*; and* $f(.,x,.)$ *is* $\mathcal{F} \otimes \mathcal{B}(K) - \mathcal{B}(\mathbb{R})$*-measurable for each* $x \in K$ *and* $f(\omega,.,y)$ *is lower semicontinuous (in x) for a.e.* $\omega \in \Omega$ *and all* $y \in K$, ***or***

   b) $f(\omega,x,.)$ *is upper semicontinuous (in y) for a.e* $\omega \in \Omega$, *for all* $x \in K$, *and* $f(.,.,y)$ *is* $\mathcal{F} \otimes \mathcal{B}(K) - \mathcal{B}(\mathbb{R})$*-measurable for each* $y \in K$, *then $REP(f,K)$ admits a countable family of random solutions $\gamma_i : \Omega \to K$, $i \geqslant 1$. Moreover, in the case b), $\{\gamma_i(\omega) | \, i \geq 1\}$ is dense in $S(\omega)$ for a.e. $\omega \in \Omega$.*

Proof. By the assertion 2) of Theorem 2.9, for almost $\omega \in \Omega$, the Minty problem $MEP(f_\omega, K)$ admits at least a local solution i.e., $M_L^\mu(\omega) \neq \emptyset$ for a.e. $\omega \in \Omega$, which shows the required conclusion in 1). The second point 2) is straight from assertion 1) of Proposition 2.7 i.e., for almost every $\omega \in \Omega$, $\emptyset \neq M_L^\mu(\omega) \subset S(\omega)$. Let us now prove 3). In the case a), from the assertion 2) of Proposition 2.7, $M^\mu(\omega) = M_L^\mu(\omega) = S(\omega)$ a. e. $\omega \in \Omega$. Then the required result descends from Theorem 3.8. The conclusion in case b) can be easily obtained in a similar way by Lemma 3.6 and Castaing representation Lemma (Lemma 3.4).

□

Remark 3.10. In the assertion 3) of Theorem 3.9 if f satisfies moreover (C_1) or (C_2) then $S(\omega) \subset B$ for almost every $\omega \in \Omega$.

Corollary 3.11. *Let $(\Omega, \mathcal{F}, \rho)$ be a measure space. Assume that f is integrable in ω and that the conditions of Theorem 3.8 or Theorem 3.9 are satisfied. Then, the integral random problem $IREP(f,K)$ has a solution.*

Remark 3.12. In the next Section we obtain better than the integral solutions. Precisely, we establish almost sure or expected-value solutions under appropriate conditions.

4. Stochastic equilibria

Now we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $\mathbb{E}[.]$ the expectation operator with respect to the probability measure $\mathbb{P}$. Given a closed and convex subset of X and a random bifunction $f : \Omega \times K \times K$, inspired by stochastic variational inequalities, see for example [20], let us agree to introduce the stochastic equilibrium problem : find $x \in K$ such that the following holds:

$$SEP(f,K) \qquad \mathbb{E}[f(\omega,x,y)] \geqslant 0, \ \ \forall y \in K. \tag{24}$$

Of course, the stochastic equilibrium problem defined in (24) includes as particular cases stochastic optimization problems, stochastic variational inequalities, stochastic complementarity problems, see for instance the survey [20]. As first step, we have restricted the abstract stochastic formulation to randomness of the data at the level of the objective bivariate function.

In addition to this expected-value formulation we can also define the almost-sure formulation as follows. Find $x \in K$ such that

$$ASEP(f,K) \qquad f(\omega,x,y) \geqslant 0, \ \ \forall y \in K, \ \ a.e. \ \omega \in \Omega. \tag{25}$$

Remark 4.1. Observe that whenever f is integrable in ω for every $x,y \in K$, then any solution to (25) is a solution to (24).

Remark 4.2. To apply probability concepts in an easiest way, one usually define a measurable function $\mathcal{X}$, called a random variable, from $(\Omega,\mathcal{F})$ to $(\mathbb{R}^d,\mathcal{B}_d)$, where $\mathcal{B}_d$ is the Borel σ-algebra, that is, the smallest σ-algebra containing all open subsets of $\mathbb{R}^d$, d being an integer such that $d \geqslant 1$.: $\mathcal{X} : (\Omega,\mathcal{F}) \to (\mathbb{R}^d,\mathcal{B}_d)$. Then, with the probability $\mathbb{P}$, the random variable $\mathcal{X}$ defines a new probability space $(\mathbb{R}^d,\mathcal{B}_d,\mathbb{P}\circ\mathcal{X}^{-1})$. Thus, the properties of probability space $(\Omega,\mathcal{F},\mathbb{P})$ can be exploited trough the new one $(\mathbb{R}^d,\mathcal{B}_d,\mathbb{P}\circ\mathcal{X}^{-1})$. Accordingly, the obtained stochastic solutions will be applicable in an easiest way rather than the dependance on the abstract set Ω. Therefore, we may suppose that Ω is a subset of $\mathbb{R}^d$, which, via the random variable $\mathcal{X}$, allows to assign meaning to the realizations ($\omega \in \Omega$) consistent with the application of interest.

Theorem 4.3. *Let K be a convex and compact of the normed space X. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $f : \Omega \times K \times K \longrightarrow \mathbb{R}$ a (random) real-valued function such that for* ***almost*** *all $\omega \in \Omega$, $f(\omega,.,.)$ is continuous and moreover satisfies the assumption $(\mathbb{H}_1)$, $f(\omega,.,.)$ is μ-pseudomonotone for* ***almost*** *all $\omega \in \Omega$, and that $f : \Omega \times K \to \mathbb{R}$, $(\omega,y) \mapsto f(\omega,x,y)$ is convex for all $x \in K$ and almost every $\omega \in \Omega$. Then the almost sure stochastic problem (25) admits a solution. If in addition f is integrable in ω then the expected-value stochastic problem (24) has also a solution.*

Remark 4.4. Notice that the μ-pseudomonotonicity condition on $f(\omega,.,.)$ together with the convexity of $(\omega,y) \mapsto f(\omega,x,y)$ ensure that $f(\omega,.,.)$ fulfills the assumption $(\mathbb{H}_7)$ for **almost** every $\omega \in \Omega$.

Proof. Let $\mu \geqslant 0$. Set Ω_0 the subset of Ω for which the assumptions of the Theorem are not satisfied. Let $\omega_1,\ldots,\omega_k \in \Omega\backslash\Omega_0$ and consider the bifunction $\tilde{f} : K \times K$ defined by

$$\tilde{f}(x,y) = \inf_{i\in\{1,\ldots,k\}} f_{\omega_i}(x,y).$$

Observe that, by [21, b) of Proposition 1.26], $\tilde{f}$ is lower semicontinuous in y as a pointwise infimum of lower semicontinuous functions in y, (in our case $\tilde{f}$ is even continuous in (x,y)) which means that $\tilde{f}$ verifies $(\mathbb{H}_2)$. Furthermore, by [21, a) of Proposition 2.22] $\tilde{f}$ is convex in y as a pointwise infimum of parameterized convex functions in y. In addition, it is a simple matter to check (for bifunctions) the stability of relaxed μ-pseudomonotonicity and positivity of values on the diagonal under any finite pointwise infimum. Then, $\tilde{f}$ satisfies also $(\mathbb{H}_i)$ for $i \in \{1,7\}$. Hence, from Proposition 2.8, $\mu - MEP(\tilde{f},K)$ admits a solution $x^\star \in K$ which is, thanks to continuity of $\tilde{f}$, a solution to $EP(\tilde{f},K)$.

This implies that $x^\star$ is a solution to both $EP(f_{\omega_i}, K)$ for all $i \in \{1,\dots,k\}$ i.e., $\bigcap_{i\in\{1,\dots,k\}} S(\omega_i) \neq \emptyset$. Accordingly, the family $(S(\omega))_{\omega\in\Omega\setminus\Omega_0}$ has the finite intersections property. Then, by the compactness Lemma, $\bigcap_{\omega\in\Omega\setminus\Omega_0} S(\omega) \neq \emptyset$ and then contains a point $\bar{x}$ which is in turns an almost sure equilibrium for f over K. The last point of the conclusion is immediate from Remark 4.1.

□

Beyond the convexity framework, finding the stochastic equilibrium points is more difficult. However, we are able to reduce this problem in the next result to the one where Ω is a finite subset. To do that, we require one of the following hypotheses: For every finite realizations $\omega_1,\dots,\omega_k$ in a subset of Ω whose measure is not equal to zero:

($\mathbb{H}_8$): The problems $EP(f_{\omega_i}), i \in \{1,\dots,k\}$ have a common equilibrium solution;

($\mathbb{H}_9$): The Minty problems $MEP(f_{\omega_i}), i \in \{1,\dots,k\}$ have a common solution.

Example 4.5. Consider the following:

1. Let $g : K\times K \to \mathbb{R}$ and $f : \Omega\times K\times K \to \mathbb{R}$ defined by
$$f(\omega,x,y) = g(x,y) + \mathbb{P}(\omega).$$
We can check easily that f satisfies ($\mathbb{H}_8$).

2. Assume that $(f_\omega)_{\omega\in\Omega_f}$ is decreasing or increasing for every finite subset of events $\Omega_f = \{\omega_1,\dots,\omega_k\}$ i.e.,
$$f_{\sigma(\omega_1)} \leqslant f_{\sigma(\omega_2)} \leqslant \dots \leqslant f_{\sigma(\omega_k)}$$
for some one to one map $\sigma : \{1,\dots,k\} \to \{1,\dots,k\}$. Then, f satisfies ($\mathbb{H}_8$).

Theorem 4.6. *Let K be a convex and compact of the normed space X. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $f : \Omega\times K\times K \longrightarrow \mathbb{R}$ a (random) real-valued function such that for* ***almost*** *all $\omega\in\Omega$, $f(\omega,.,.)$ is continuous and moreover satisfies either ($\mathbb{H}_8$) or ($\mathbb{H}_9$) and one of the following assumptions*

i) ($\mathbb{H}_i$), $i \in \{3\}$;

ii) ($\mathbb{H}_i$), $i \in \{1,4\}$;

iii) ($\mathbb{H}_i$), $i \in \{0,5\}$:

Then the almost sure stochastic problem (25) admits a solution. If in addition f is integrable in ω then the expected-value stochastic problem (24) has also a solution.

Proof. Let $\mu \geqslant 0$. Set Ω_0 the subset of Ω for which the assumptions of the Theorem are not satisfied. Since f is continuous in y, the assumption $(\mathbb{H}_2)$ is satisfied with φ defined by

$$\varphi(x,y) = \mu\|x-y\|^2, x,y \in K.$$

Moreover, the continuity of f implies the global μ-upper sign continuity of f in x. Then thanks to the assertion 1) of Theorem 3.8, in all cases i), ii), iii), for all $\omega \in \Omega\backslash\Omega_0$, $\mu - MEP(f_\omega, K)$ admits a solution $x(\omega)$ which is in turns a solution to $EP(f_\omega, K)$ i.e., $S(\omega) \neq \emptyset$ for all $\omega \in \Omega\backslash\Omega_0$. Now, for the case of the assumption $(\mathbb{H}_8)$, for every finite set $\Omega_f \subset \Omega\backslash\Omega_0$ of realizations $\{\omega_1, \ldots, \omega_n\}$, there exists $\omega_0 \in \Omega_f$ such that $x(\omega_0)$ is also a solution to $EP(f_\omega, K)$ for all $\omega \in \Omega_f$. Consequently, the family $(S(\omega))_{\omega\in\Omega\backslash\Omega_0}$ has the finite intersections property. For the case of the assumption $(\mathbb{H}_9)$, we obtain that the family $(M(\omega))_{\omega\in\Omega\backslash\Omega_0}$ has the finite intersections property, which implies (by the continuity assumption on f) that $(M(\omega))_{\omega\in\Omega\backslash\Omega_0}$ has the finite intersections property.
Then, by the compactness Lemma, $\bigcap_{\omega\in\Omega\backslash\Omega_0} S(\omega) \neq \emptyset$ and then contains a point $\bar{x}$ which is in turns an almost sure equilibrium for f over K. The last point of the conclusion is immediate from Remark 4.1.

□

Let us apply Theorem 4.3 or Theorem 4.6 to the following stochastic quasiconvex programming problem: given a (random) quasiconvex function $g : \Omega \times K \to \mathbb{R}$, we seek at finding $\bar{x} \in K$ such that

$$g(\omega,\bar{x}) \leqslant g(\omega,x),\ \forall x \in K,\ \text{for } a.e.\ \omega. \tag{26}$$

Definition 4.7. A point $\bar{x}$ satisfying (26) is called an *almost-sure minimizer* of g.

When g is a Carathéodory function, K is a separable, closed, convex and Suslin subset of X and X is Suslin normed space X, then from Theorem 3.8 we can obtain random minimizers or even a family of random minimizers dense in the set of all random minimizers of g, and hence under an integrability assumption, the integral minimizers are also possible. However, in the following, we are rather interested in stochastic minimizers that are also minimizers for the expected-value formulation.

Corollary 4.8. *Let K be a convex and compact of the normed space X. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $g:\Omega\times K\longrightarrow\mathbb{R}$ be a real-valued Carathéodory function. Assume that one of the following conditions is satisfied: There exists a subset Ω_0 of Ω with a null measure such that:*

i) for all $\omega\in\Omega\backslash\Omega_0$, $g:\Omega\times K\to\mathbb{R}$ is convex and concave in ω;

ii) for all $\omega\in\Omega\backslash\Omega_0$, $g(\omega,.)$ is quasiconvex and for every finite subset Ω_f of $\Omega\backslash\Omega_0$, the family $\{g(\omega,.)|\ \omega\in\Omega_f\}$ has a common stochastic solution.

Then the almost-sure stochastic minimization problem (26) admits a solution.

Proof. Set $f_g:\Omega\backslash\Omega_0\times K\times K\to\mathbb{R}$ defined for all $x,y\in K$ and $\omega\in\Omega\backslash\Omega_0$ by

$$f_g(\omega,x,y)=g(\omega,y)-g(\omega,x).$$

Clearly, $f_g(\omega,.,.)$ is a monotone bifunction for every $\omega\in\Omega\backslash\Omega_0$. For the case *i*), f_g satisfies the assumptions of Theorem 4.3 while it satisfies those of Theorem 4.6 in the case ii). Then, in both of these two cases, f_g admits an almost-sure equilibrium which is automatically an almost-sure minimizer for g. This completes the proof.

□

Remark 4.9.

1. In Corollary 4.8, if g is in addition integrable in ω then the corresponding expected-value stochastic to (26) has also a solution.
2. For a discussion of examples of stochastic quasiconvex together with its great interest in the context of real applications, we refer to the very recent paper [15] and references therein.

5. Conclusion and perspectives

We have considered random equilibrium problems and posed new formulations of stochastic equilibrium points for random bifunctions. The presented results on stochastic formulations have been obtained under two key assumptions: convexity and the property of finite realizations (($\mathbb{H}_8$) and ($\mathbb{H}_9$)). Thus, we believe that a good perspective of this research is to envisage relaxations of these hypotheses. In particular, the next efforts should be focused on the approximation of the stochastic equilibrium problem by a deterministic one and then examine the impact on the existing known results in stochastic optimization and stochastic variational inequalities in the nonmonotone and nonconvex framework. Examples and applications will be equally of a special relevance in a forthcoming research.

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