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PROBLEMI LINEARI E NON LINEARI DI FLUSSO INTORNO AD OSTACOLI

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We discuss the problem of the steady two-dimensional flow past fixed disturbances in an open channel of finite depth. We consider different types of obstacles, like submerged or surface-piercing bodies, and localized perturbations of a horizontal bottom; recent results on unique solvability of the linear problem and rigorous proofs of solvability of the non linear, free boundary problem are reviewed.

1. Introduction.

Let us consider the steady two-dimensional flow of a heavy, ideal fluid, past fixed obstacles in a channel of finite depth; we assume irrotational motion and neglect the effects of surface tension. Then, we can describe the flow by a holomorphic function (the complex velocity field) defined in an unbounded domain and satisfying a non linear condition (the Bernoulli condition) on a free boundary (the free surface). The problem is completed by specifying the conditions at the other boundaries of the region filled by the fluid; assuming rigid walls and obstacles, we have the no-flow condition; the same kinematic condition holds on the free surface.
The above problem has been widely studied by analytical and numerical methods; however, little is known about its solvability from a rigorous point of view, due to the difficulties related to the free boundary. The mathematical approach

to such problem, and more generally to problems involving wave-structure interactions, usually deals with linearized versions, whose solutions are called *linear water waves*; an exhaustive treatment of linear water-wave theory, including a discussion of its physical justification, can be found in [1]. One of the still open questions in the linear theory is to determine whether the problem of the stream past assigned obstacles (submerged or surface-piercing bodies, roughness of a channel's bed, etc.) is uniquely solvable for all values of the flux velocity; a positive answer is known for special geometries (see Section 3), but in general the connection between unique solvability and the geometry of the obstacles is not completely understood. As previously remarked, even less is known about the solutions of the exact, free-boundary problem; at the present time, rigorous results has been established only in a few cases, starting from the discussion of particular linear problems and using local methods of non linear analysis (see Section 4). We recall that local and global bifurcation theory has been successfully applied to the non linear problem of the *free flow* (i.e. without obstacles) in a channel with horizontal bottom of finite depth; in this case, there are non trivial solutions bifurcating from the constant parallel flow, whose properties depend critically on the parameter $\mathrm{Fr} = c/\sqrt{gH}$, the Froude number. More precisely:

For $\mathrm{Fr} < 1$ (subcritical flow) there are *periodic waves* (Stokes waves) of wave length λ near subcritical velocities satisfying the relation

$$c^2 = g\frac{\lambda}{2\pi}\tanh\frac{2\pi H}{\lambda}. \tag{1.1}$$

For $\mathrm{Fr} > 1$ (supercritical flow) there are *solitary waves* of any amplitude, vanishing at infinity.

As we will show in the present note, the parameter Fr is also critical in the discussion of the flow past fixed disturbances, both for the linear and non linear problem.

2. Formulation of the problem and linearized equations.

Let us now discuss an example of two-dimensional problem, whose geometry is depicted in Figure 1 below; the unknowns of the problem can be listed as follows:

- *two scalars* $x_+ > 0$, $x_- < 0$;
- *a real function* $h : \mathbb{R}\backslash[x_-, x_+] \to \mathbb{R}$ *(the free surface)*;

– *a complex function :* $\omega(z) = u(x, y) - iv(x, y)$ *(complex velocity field) holomorphic in* S*, the region filled with the fluid.*

We remark that S depends on x_+, x_- and h; if there are no semi-submerged obstacles, the free surface is represented by a continuous function defined on the whole real line.

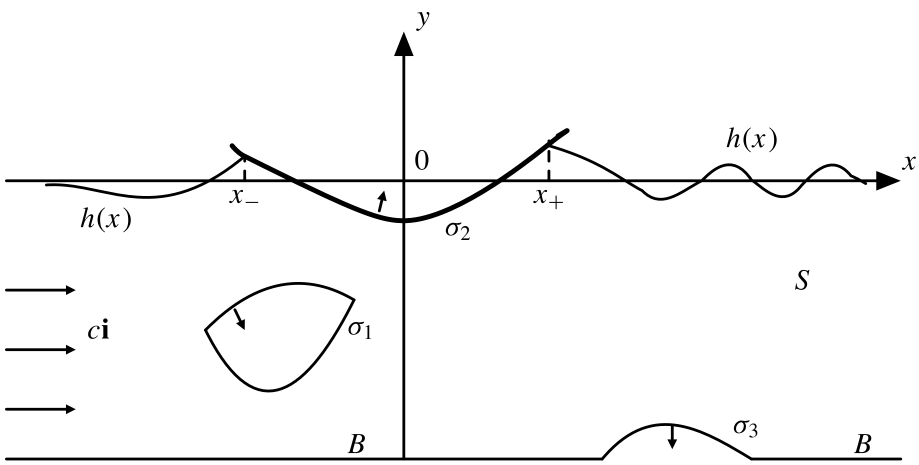

Figure 1

The boundary conditions of the problem are

(2.1) $$\frac{1}{2}|\omega(x, h(x))|^2 + gh(x) = \text{const.}, \ \ x \in \mathbb{R}/[x_-, x_+]$$

(2.2) $$v(x, h(x)) = h'(x)u(x, h(x)), \ \ x \in \mathbb{R}/[x_-, x_+]$$

(2.3) $$(u\mathbf{i} + v\mathbf{j}) \cdot \mathbf{n} = 0, \ \text{ on } \sigma = \bigcup \sigma_i;$$

(2.4) $$v = 0, \qquad \text{on } B;$$

(2.5) $$\lim_{x \to -\infty} \omega(z) = c;$$

(2.6) $$\lim_{x\to-\infty} h(x) = 0;$$

(2.7) $$\big(x_{\pm}, h(x_{\pm})\big) \in \sigma_2.$$

Equations (2.1) and (2.2) represent, respectively, the Bernoulli condition and the kinematic condition on the free surface; equations (2.3), (2.4) express the conditions on the (rigid) physical boundaries. By conditions (2.5), (2.6), the flow at upstream infinity is equal to the constant parallel flow with velocity c, while the free surface reduces to the surface of calm water. Finally, (2.7) must be satisfied at the contact points between the free surface and a surface-piercing body.

The linearized problem is obtained by inserting in (2.1)–(2.7) the formal expansion

$$\omega(x, y) = c + \epsilon\omega^{(1)}(x, y) + \epsilon^2\omega^{(2)}(x, y) + \cdots;$$
$$h(x) = \epsilon h^{(1)}(x) + \epsilon^2 h^{(2)}(x) + \cdots$$

(where ϵ is a suitable adimensional parameter [1] and by observing that for $\epsilon \to 0$ the free surface reduces to a subset of the line $y = 0$; a typical geometry of a linear problem is represented in Figure 2 below.

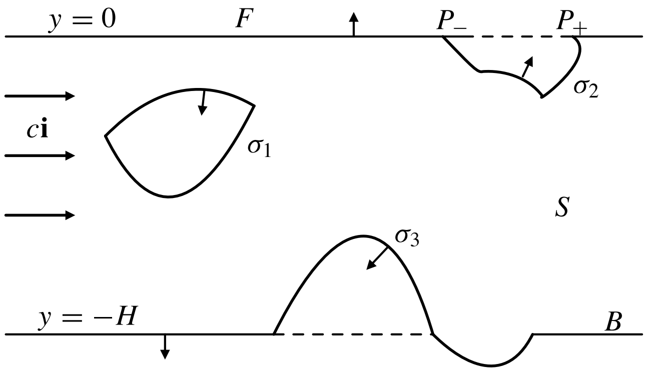

Figure 2.

By considering the first order terms in ϵ of (2.1), (2.2), we can eliminate the unknown h (by differentiation) and obtain

$$u_x^{(1)}(x, 0) + \frac{g}{c^2} v^{(1)}(x, 0) = 0, \qquad \text{on} \quad F \quad (h = 0).$$

Then, by defining the potential ϕ of the perturbed (zero circulation) flow, $\nabla\phi = u^{(1)}\mathbf{i} + v^{(1)}\mathbf{j}$, we readily obtain the equations of the *linear problem (Neumann-Kelvin problem)*:

$$\Delta\phi = 0, \quad \text{in } S; \tag{2.8}$$

$$\phi_{xx} + \nu\phi_y = 0, \quad \text{on } F; \tag{2.9}$$

$$\frac{\partial\phi}{\partial n} = k, \quad \text{on } \sigma = \bigcup \sigma_i; \tag{2.10}$$

$$\phi_y = 0, \quad \text{on } B; \tag{2.11}$$

$$\lim_{x\to-\infty} |\nabla\phi| = 0; \tag{2.12}$$

where $\nu = g/c^2$ and $k = -cn_x$ on σ. In order to obtain well-posed formulations of the problem with semi submerged obstacles, one needs *supplementary conditions* at the bow and stern points, which somehow replace (2.7); for example, one can choose

$$\phi_x(P_+) = \alpha_+, \quad \phi_x(P_-) = \alpha_-, \tag{2.13}$$

which can be interpreted as fixing the elevation of the free surface at $P_\pm$ [1]. By requiring that there is no additional flux at infinity from the perturbed flow (see the asymptotic properties of the solutions in section 3) the data must be chosen to satisfy the *compatibility conditions*

$$\alpha_+ - \alpha_- + \nu\int_\sigma k = 0. \tag{2.14}$$

3. Solvability of the linear problem.

We now discuss the solvability of the previous problem for all values of the quantity $c/\sqrt{gH}$ (Froude number); to this aim, we first recall the solutions of the *free problem* (no obstacles):

– if $c > \sqrt{gH}$, i.e. $\nu H < 1$ (supercritical flow) we have only the trivial solution $\phi(x, y) = \phi_0$ (ϕ_0 arbitrary constant);
– if $c < \sqrt{gH}$, i.e. $\nu H > 1$ (subcritical flow) there are two independent solutions

$$\begin{aligned} S(x, y) &= \sin(\nu_0 x)\cosh[\nu_0(y + H)], \\ C(x, y) &= \cos(\nu_0 x)\cosh[\nu_0(y + H)], \end{aligned} \tag{3.1}$$

where $\nu_0 > 0$ satisfies $\nu_0/\nu = \tanh(\nu_0 H)$, i.e. $\nu_0 = 2\pi/\lambda$ is the wave number of the dispersion relation (1.1).

The same difference between supercritical and subcritical flows also concerns the (a priori) *asymptotic properties* of the solutions of the Neumann-Kelvin problem (2.8)–(2.13); in fact, for $|x|$ large enough (away from the obstacles) we have:

– if $\nu H < 1$: $\sup e^{\mu_1|x|}|\nabla\phi(x, y)| < +\infty$, where μ_1 is the first positive solution of $\mu/\nu = \tan(\mu H)$.
– if $\nu H > 1$: $\phi(x, y) \approx \phi_0 + A\, S(x, y) + B\, C(x, y)$ for $x \to +\infty$, where A, B are suitable constants.

From the above discussion, it follows in particular that in the subcritical regime one will not get, in general, solutions of finite energy; hence the proof of unique solvability of the linear problem is more delicate in this regime. In any case, the standard approach to solvability relies on integral equation techniques [1], which apply when the boundaries of the obstacles are sufficiently regular, including piecewise smooth contours with corner points (but no cusps). Recently, an alternative variational approach has been proposed for the problem of *ship waves* generated by submerged or partially submerged bodies in uniform motion (including the limit case of a surface beam, see [4] and references therein).
As a result, one can prove *unique solvability for all supercritical velocities* both in the case of semi submerged and submerged obstacles. The situation is different in the subcritical case: for a given obstacle, there could be a sequence (for surface-piercing bodies) or a finite number (for immersed bodies) of ”singular” velocities such that unique solvability does not hold. Unique solvability for *all* subcritical velocities has been proved only for special obstacles, see [3] (submerged circular cylinder), [4] (surface-piercing symmetric body), [7] (surface

beam). Actually, there are examples of non uniqueness of the solution (trapped modes) for exceptional values of ν in the Neumann-Kelvin problem for a *surface piercing tandem* [2] and in the problem of the *flow over a submerged hollow* (of rectangular shape) in a channel's bed [8].

It is worthwhile to illustrate the application of the variational method in the case of a supercritical flow, $\nu H < 1$. For the sake of brevity, we only consider a problem with completely submerged obstacles as represented in Figure 3 below.

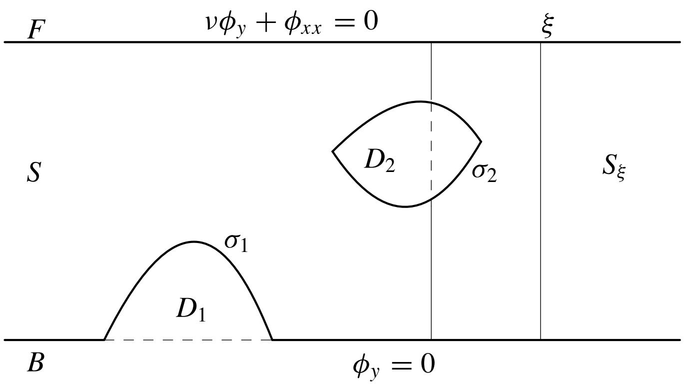

Figure 3

On the obstacles' boundaries σ_i, $i = 1, 2$, we require the conditions $\frac{\partial\phi}{\partial n_i} = k_i$.
A variational form of the problem can be stated in the functional space

$$H(S) := \left\{\phi \; : \; \iint_S |\nabla\phi|^2 + \int_F \phi_x^2 < \infty\right\}.$$

Then, by standard methods we get from (2.8)–(2.11):

$$\iint_S \nabla\phi\nabla\chi - \frac{1}{\nu}\int_F \phi_x\chi_x = \sum_{i=1}^{2}\int_{\sigma_i} k_i\chi, \quad \forall\chi \in H(S). \tag{3.2}$$

It is readily verified that the left hand side of (3.2) is a continuous bilinear form on $H(S)$; however, coercivity does not hold, due to the minus sign between the two terms. We get over to this problem by restricting the form (3.2) to a *closed subspace* of $H(S)$, whose definition is suggested by suitable *a priori conditions* satisfied by finite energy solutions. In order to find these conditions, let us consider the semi infinite strip S_ξ represented in fig. 3 above; if ϕ is a solution with finite energy, by the relation $\int_{\partial S_\xi} \frac{\partial\phi}{\partial n} = 0$ and by the boundary conditions on F and on B, we readily get

$$\phi_x(\xi, 0) = \nu \int_{-H}^{0} \phi_x(\xi, y)\, dy. \tag{3.3}$$

Hence, by Hölder inequality and integrating from ξ to infinity, we get the estimate

$$\frac{1}{\nu} \int_{\xi}^{+\infty} \phi_x(x, 0)^2\, dx \le \nu H \|\nabla\phi\|^2_{L^2(S_\xi)}. \tag{3.4}$$

A similar bound can be obtained when ξ lies above the obstacles by extending ϕ to $S \cup D_1 \cup D_2$ (D_1, D_2 open sets) with $\Delta\phi = 0$ in $D_1 \cup D_2$ and

$$-\frac{\partial\phi^{\text{int}}}{\partial n}\bigg|_{\sigma_i} = \frac{\partial\phi^{\text{ext}}}{\partial n}\bigg|_{\sigma_i} = k_i, \quad i = 1, 2.$$

In fact, by the divergence theorem it follows that the so extended solution still satisfies (3.3); we conclude that (3.4) holds for (almost) every real ξ.

Now, in the new space

$$H(S \cup D_i) = \left\{\phi \;:\; \iint_{S\cup D_i} |\nabla\phi|^2 + \int_F \phi_x^2 < \infty\right\},$$

we can formulate the *weak problem for the extended solution*:

$$\iint_{S\cup D_i} \nabla\phi\nabla\chi - \frac{1}{\nu}\int_F \phi_x\chi_x = \sum_{i=1}^{2}\int_{\sigma_i} k_i(\chi^{\text{ext}} - \chi^{\text{int}}), \quad \forall\chi \in H(S \cup D_i), \tag{3.5}$$

where χ^{ext}, χ^{int} are the traces of χ on σ_i, respectively from outside and inside the domain D_i. By taking the limit for $\xi \to -\infty$ in (3.4) and recalling that $\nu H < 1$, we can now prove:

Theorem. *The bilinear form* (3.5) *is coercive in the subspace*

$$V := \left\{\phi \;:\; \phi_x(\xi, 0) = \nu \int_{-H}^{0} \phi_x(\xi, y) dy, \;\text{ a.e. } \xi \in \mathbb{R}\right\}.$$

Thus, we have unique solvability for the variational problem in V; moreover, one can show that the weak solution is harmonic in S and satisfies the boundary conditions of the original problem.
We remark that an analogous argument applies to the problem of a *subcritical flow past a semi submerged body*; in this case, the subspace of $H(S)$ where coercivity holds is defined by a different condition, involving the functions (3.1). The resulting variational solution, however, is (in general) *not harmonic* in S and a regularization procedure is needed in order to obtain a solution of (2.8)–(2.13); as a consequence, the proof of unique solvability for all subcritical velocities requires additional assumptions on the shape of the obstacle [4].

4. Solvability of the non linear problem.

In this final section, we briefly review some results about solvability of the non linear, free boundary problem (2.1)–(2.7). Existence of a solution to this problem has been established in the study of the ship waves generated by the uniform motion of a single "thin" body. In the case of a surface-piercing obstacle, we have solvability *both in the supercritical and subcritical regime* [6], [7], while for a completely submerged body in a *supercritical flow* we have a unique solution with prescribed circulation around the body [5]. All these results are obtained by assuming that the equation of the hull's profile is $y = \epsilon f(x)$, ($y = \epsilon f_{\pm}(x)$ for a submarine) with $\epsilon > 0$, and by considering the linear problem which turns out in the limit $\epsilon \to 0$; then, the solution of the nonlinear problem is obtained for small ϵ by local methods (implicit function theorems and bifurcation theory) which can be applied after an appropriate reformulation of the problem in the *hodograph plane*. Clearly, a crucial step in this approach is the study of the linearized limit problem; in the case of a semi submerged obstacle, this is the Neumann-Kelvin problem for a surface beam and it is uniquely solvable for *all values* of the flow velocity; on the contrary, it is not clear whether unique solvability holds for all subcritical velocities in the problem of a submerged beam. Hence, the solvability of the non linear problem for a submerged (thin) body in a *subcritical stream* is still an open problem.

Moreover, in order to provide a correct functional formulation of the nonlinear boundary conditions, we need some *regularity* of the weak solutions of the limit problem. In particular, a careful investigation is required in the Neumann-Kelvin problem for a surface beam, where two different boundary conditions meet at the end points of the beam; actually, one can show that there are uniquely determined $\alpha_{\pm}$ in conditions (2.13)–(2.14), such that the velocity field $\nabla\phi$ is *continuous at* $P_{\pm}$.

Finally, we point out that we are able to derive some qualitative features of the solutions to the non linear problem; in particular, in the problem with a surface-piercing obstacle, we find that the free surface h and the submerged hull ϵf form a $\mathcal{C}^1$ streamline, which is exponentially vanishing for $x \to -\infty$. Furthermore, in the supercritical case, the function h is negative (that is, the free surface lies below the level of calm water), monotone increasing for $x > x_+$ and decreasing for $x < x_-$; in the subcritical case, we have oscillations of the free surface for $x \to +\infty$, with wave length as in (1.1). We remark that these properties of the free surface agree with existing numerical results.

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