LE MATEMATICHE
Vol. LXXIII (2018) – Fasc. I, pp. 161–177
doi: 10.4418/2018.73.1.12

MODULES AND THE SECOND CLASSICAL ZARISKI TOPOLOGY

SEÇIL ÇEKEN - MUSTAFA ALKAN

Let R be an associative ring with identity and $Spec^s(M)$ denote the set of all second submodules of a right R-module M. In this paper, we present a number of new results for the second classical Zariski topology on $Spec^s(M)$ for a right R-module M. We obtain a characterization of semisimple modules by using the second spectrum of a module. We prove that if R is a ring such that every right primitive factor of R is right artinian, then every non-zero submodule of a second right R-module M is second if and only if M is a fully prime module. We give some equivalent conditions for $Spec^s(M)$ to be a Hausdorff space or T_1-space when the right R-module M has certain algebraic properties. We obtain characterizations of commutative Quasi-Frobenius and artinian rings by using topological properties of the second classical Zariski topology. We give a full characterization of the irreducible components of $Spec^s(M)$ for a non-zero injective right module M over a ring R such that every prime factor of R is right or left Goldie.

Entrato in redazione: 8 marzo 2018

AMS 2010 Subject Classification: 16D10, 16D50, 16D60, 16D80, 54B99
Keywords: Second submodule, second spectrum, prime submodule, second classical Zariski topology
The second author was supported by the Scientific Research Project Administration of Akdeniz University.

1. Introduction

Throughout this paper all rings will be associative rings with identity elements and all modules will be unital right modules. Unless otherwise stated R will denote a ring. By a proper submodule N of a non-zero right R-module M, we mean a submodule N with $N \neq M$. Given a right R-module M, we shall denote the annihilator of M (in R) by $ann_R(M)$.

Prime submodules of modules over commutative rings were introduced and studied by C. P. Lu in [16]. Let R be a commutative ring. A proper submodule N of an R-module M is said to be prime if, given elements $r \in R$ and $m \in M$, if $mr \in N$ then either $m \in N$ or $Mr \subseteq N$, i.e. $r \in ann_R(M/N)$. This definition is generalized to arbitrary rings as follows. A non-zero module M over an arbitrary ring R is called a *prime module* if $ann_R(M) = ann_R(K)$ for every non-zero submodule K of M. A proper submodule N of a module M is called a *prime submodule* of M if M/N is a prime module. If R is a commutative ring, then two definitions given above are equivalent. To see this, let N be a prime submodule of M in the sense of commutative case and let N be a proper submodule of a submodule K of M. If $r \in ann_R(K/N)$, then $Kr \subseteq N$. There exists an element $m \in K \backslash N$ and we have $mr \in N$. By definition, $r \in ann_R(M/N)$ and so $ann_R(M/N) = ann_R(K/N)$. Thus N is a prime submodule of M in the sense of the general definition. Now, suppose that M/N is a prime module in the sense of the general definition. Let $r \in R$, $m \in M \backslash N$ and $mr \in N$. Then $(mR)r \subseteq N$ and so $r \in ann_R((mR+N)/N) = ann_R(M/N)$. Thus N is a prime submodule of M in the sense of the commutative case. Recall that an R-module M is said to be *fully prime* if each proper submodule of M is prime. The set of all prime submodules of a module M is called the *prime spectrum* of M and denoted by $Spec(M)$. Several authors investigated and topologized the prime spectrum of a given module (see for example [7], [8], [17], [18], [22]).

In [24], S. Yassemi introduced second submodules of modules over commutative rings as the dual notion of prime submodules. Let R be a commutative ring. A non-zero submodule N of M is called second provided that for any $r \in R$, the multiplication $f_r : N \longrightarrow N$ by r is either surjective or zero, that is, $Nr = N$ or $Nr = 0$. Second modules over arbitrary rings were defined in [2] and used as a tool for the study of attached primes over noncommutative rings. A right R-module M is called a *second module* provided $M \neq 0$ and $ann_R(M) = ann_R(M/N)$ for every proper submodule N of M. By a second submodule of a module, we mean a submodule which is also a second module. If R is a commutative ring, then two definitions given above are equivalent. To see this, let N be a second submodule of M in the sense of commutative case. Suppose that K is a proper submodule of N and $r \in ann_R(N/K)$. Then $Nr \subseteq K$. Since $K \neq N$, we $Nr = 0$ by definition. Hence $r \in ann_R(N)$ and so

$ann_R(N) = ann_R(N/K)$. Hence N is a second submodule of M in the sense of the general definition. Now, suppose that N is a second submodule of M in the sense of the general definition and let $r \in R$. If $Nr \neq N$, then Nr is a proper submodule of N as R is commutative and we have $ann_R(N) = ann_R(N/Nr)$ by definition. Thus $Nr = 0$, and so N is a second submodule of M in the sense of commutative case. If N is a second submodule of a module M, then $ann_R(N) = P$ is a prime ideal of R and in this case N is called a *P-second submodule* of M. Recently, second submodules have attracted attention of many authors and they have been extensively studied in a number of papers (see for example [1], [3], [4], [5], [6], [9], [10], [11], [12]).

The set of all second submodules of a module M is called the *second spectrum* of M and denoted by $Spec^s(M)$. As in [9], for any submodule N of a right R-module M we define $V^{s*}(N)$ to be the set of all second submodules of M contained in N. Clearly $V^{s*}(0)$ is the empty set and $V^{s*}(M)$ is $Spec^s(M)$. Note that for any family of submodules N_i $(i \in I)$ of M, $\cap_{i\in I} V^{s*}(N_i) = V^{s*}(\cap_{i\in I} N_i)$. Thus if $\zeta^{s*}(M)$ denotes the collection of all subsets $V^{s*}(N)$ of $Spec^s(M)$, then $\zeta^{s*}(M)$ contains the empty set and $Spec^s(M)$, and $\zeta^{s*}(M)$ is closed under arbitrary intersections. But in general $\zeta^{s*}(M)$ is not closed under finite unions. A module M is called a *cotop module* if $\zeta^{s*}(M)$ is closed under finite unions. In this case $\zeta^{s*}(M)$ is called the *quasi-Zariski topology* on M (see [5]). Note that in [1] a cotop module was called a top^{s}-module. More information about the class of cotop modules can be found in [1], [5] and [13].

Let M be a right R-module and $\zeta'^{s}(M) = \{V^{s*}((0 :_M I)) : I \text{ is an ideal of } R\}$. Then [1, Lemma 4.1] shows that $\zeta'^{s}(M)$ satisfies the axioms for closed sets in a topological space and so it induces a topology on $Spec^s(M)$. We call this topology the *dual Zariski topology* on $Spec^s(M)$. The dual Zariski topology of modules over commutative rings has been investigated in [5], [9] and [13].

Let M be a right R-module. As in [6], for each submodule N of M, we define $W^s(N) = Spec^s(M) - V^{s*}(N)$ and put $\Omega^s(M) = \{W^s(N) : N \leq M\}$. Let $\eta^s(M)$ be the topology on $Spec^s(M)$ by the sub-basis $\Omega^s(M)$. In fact $\eta^s(M)$ is the collection $\mathcal{U}$ of all unions of finite intersections of elements of $\Omega^s(M)$. As in [6], we call this topology the *second classical Zariski topology* of M. Note that this topology is defined in a way dual to that of defining classical Zariski topology on the prime spectrum of a module in [7]. It is clear that if M is a cotop module, then the quasi-Zariski topology on $Spec^s(M)$ coincide with the second classical Zariski topology of M. It is also clear that the second classical Zariski topology of M is finer than the dual Zariski topology of M.

In [6] the authors introduced and studied the second classical Zariski topology of a module over a commutative ring. In this paper, we study the second classical Zariski topology of a module over an arbitrary ring and we obtain a number of new results for this topology. After this introductory section, this paper is divided into two sections. In section 2, we give our preliminary results about the second classical Zariski topology which will be used in section 3.

In section 3, we present our main results for the second classical Zariski topology and the second spectrum of a module. We obtain a characterization of semisimple modules by using the second spectrum of a module (see Theorem 3.4). In Proposition 3.6, we prove that if R is a ring such that every right primitive factor of R is right artinian, then every non-zero submodule of a second right R-module M is second if and only if M is a fully prime module. We study modules whose the second classical Zariski topologies are respectively T_1, Hausdorff and cofinite. In Theorem 3.10, Proposition 3.11 and Theorem 3.12, we give some equivalent conditions for $Spec^s(M)$ to be a Hausdorff space or T_1-space when the right R-module M has certain algebraic properties. In Theorem 3.13, we obtain a characterization of commutative artinian rings and in Theorem 3.14, we give a characterization of commutative Quasi-Frobenius rings by using the second classical Zariski topology. Then we deal with the irreducible subsets of $Spec^s(M)$ for a right R-module M. In Theorem 3.16 we give some equivalent conditions for $Spec^s(M)$ to be an irreducible space when the right R-module M has certain algebraic properties. We determine all the irreducible components of $Spec^s(M)$ for a non-zero injective right module M over a ring R such that the ring R/P is right or left Goldie for every prime ideal P of R (see Theorem 3.17).

2. Preliminaries

In this section we give some preliminary results about the second classical Zariski topology which will be used in section 3.

From now on, we write $X^s(M)$ to denote the second spectrum $Spec^s(M)$ for a right R-module M and we consider $X^s(M)$ with the second classical Zariski topology unless otherwise stated

First, we will investigate when $X^s(M)$ is a T_1-space for an R-module M.

Proposition 2.1. Let M be a right R-module. Then $X^s(M)$ is a T_1-space if and only if $X^s(M) = \emptyset$ or every element of $X^s(M)$ is minimal.

Proof. This proposition can be proved as in the commutative case in [6, Theorem 2.7]. □

By using Proposition 2.1 and the fact that a finite topological space X is a T_1-space if and only if X is the discrete space, we get the following proposition.

Proposition 2.2. Let M be a right R-module such that $X^s(M)$ is finite. Then the following statements are equivalent.

(a) $X^s(M)$ is a Hausdorff space.
(b) $X^s(M)$ is a T_1-space.
(c) $X^s(M)$ is the cofinite topology.
(d) $X^s(M)$ is discrete.
(e) $X^s(M) = \emptyset$ or every element of $X^s(M)$ is minimal.

For each subset Y of $Spec^s(M)$, we will denote the closure of Y in $Spec^s(M)$ by $cl(Y)$.

Proposition 2.3. Let M be a right R-module. Then the following are true.
(a) If Y is a finite subset of $X^s(M)$, then $cl(Y) = \cup_{S\in Y} V^{s*}(S)$
(b) If Y is a closed subset of $X^s(M)$, we have $Y = \cup_{S\in Y} V^{s*}(S)$.

Proof. This proposition can be proved as in the commutative case in [6, Proposition 3.1]. □

It is well-known that any topological space X determines a preorder $\preceq$, namely $\preceq$ is defined by setting, for $x, y \in X$,

$$x \preceq y :\Longleftrightarrow y \in cl(\{x\}).$$

Proposition 2.3-(a) yields that

$$S_1 \preceq S_2 \Longleftrightarrow S_2 \in cl(\{S_1\}) = V^{s*}(S_1) \Longleftrightarrow S_2 \subseteq S_1.$$

Thus, the preorder of the second classical Zariski topology is the reverse inclusion.

The next lemma is the noncommutative analogue of [6, Lemma 3.3].

Lemma 2.1. *Let M be a right R-module. Then for each $S \in X^s(M)$, $V^{s*}(S)$ is irreducible.*

Proof. The proof is straightforward. □

A non-zero right R-module M is second if and only if $MI = 0$ or $MI = M$ for every ideal I of R (see [10, Lemma 2.1]). By using this fact, we get the following lemma.

Lemma 2.2. *Let S be a non-zero submodule of a right R-module M. Then the following are equivalent.*
(a) *S is a second submodule of M.*
(b) *For each ideal I of R and for each submodule K of M, $SI \subseteq K$ implies that $SI = 0$ or $S \subseteq K$.*

Let M be a right R-module and $Y \subseteq X^s(M)$. As in [6], we will denote $\sum_{S\in Y} S$ by $T(Y)$.

The next theorem is the noncommutative analogue of [6, Theorem 3.5]. We give its proof as it is slightly different from the commutative case.

Theorem 2.4. Let M be a right R-module and $Y \subseteq X^s(M)$. Then we have the following.

(a) If Y is irreducible, then $T(Y)$ is a second submodule of M.

(b) If $T(Y)$ is a second submodule of M and $T(Y) \in cl(Y)$, then Y is irreducible.

Proof. (a) Suppose that Y is an irreducible subset of $X^s(M)$. Then clearly $T(Y) \neq 0$ and $Y \subseteq V^{s*}(T(Y))$. Let I be an ideal of R and K be a submodule of M such that $T(Y)I \subseteq K$. Then it is easy to show that $Y \subseteq V^{s*}((K :_M I)) \subseteq V^{s*}(K) \cup V^{s*}((0 :_M I))$. Since Y is irreducible, we have $Y \subseteq V^{s*}(K)$ or $Y \subseteq V^{s*}((0 :_M I))$. If $Y \subseteq V^{s*}(K)$, then we get that $T(Y) \subseteq K$. If $Y \subseteq V^{s*}((0 :_M I))$, then $SI = 0$ for all $S \in Y$. Thus $T(Y)I = 0$. Hence by Lemma 2.2, $T(Y)$ is a second submodule of M.

(b) Suppose that $S := T(Y)$ is a second submodule of M and $S \in cl(Y)$. By using Proposition 2.3 we can show that $cl(Y) = V^{s*}(S)$. Now let $Y \subseteq Y_1 \cup Y_2$, where Y_1 and Y_2 are closed sets. Then we have $V^{s*}(S) = cl(Y) \subseteq Y_1 \cup Y_2$. By Lemma 2.1, $V^{s*}(S)$ is irreducible. This implies that $V^{s*}(S) \subseteq Y_1$ or $V^{s*}(S) \subseteq Y_2$. Hence $Y \subseteq Y_1$ or $Y \subseteq Y_2$. So Y is irreducible. □

Let M be an R-module. The sum of all second submodules of M is called the *second radical* of M and is denoted by $sec(M)$. If there is no second submodule of M, then we define $sec(M) = 0$. A submodule N of M is called a *second radical submodule* in case $N = sec(N)$ (see [11]). Note that in [6] the second radical of M was called the second socle of M and a second radical submodule was called a socle submodule.

Corollary 2.5. Let M be a right R-module and N be a submodule of M. Then $V^{s*}(N)$ is irreducible if and only if $sec(N)$ is a second submodule of M. Consequently, $Spec^s(M)$ is irreducible if and only if $sec(M)$ is a second submodule of M.

3. Main Results

In this section, we present our main results for the second classical Zariski topology.

Let M be an R-module. For any submodule N of M, define $V(N) = \{Q \in Spec(M) : N \subseteq Q\}$. M is said to satisfy $(*)$ condition provided that if N_1, N_2 are submodules of M with $V(N_1) = V(N_2)$, then $N_1 = N_2$ (see [7]). In [7, Proposition 2.3], it is proved that a non-zero right R-module M satisfies $(*)$ condition if and only if every proper submodule of M is an intersection of prime submodules.

A right R-module M is called *cosemisimple* if every simple module is M-injective i.e., every proper submodule of M is an intersection of maximal submodules. Every semisimple module is of course cosemisimple. If R_R ($_RR$) is a cosemisimple module, then the ring R is called a right (left) *V-ring*. A commutative ring is a (right) V-ring if and only if it is Von Neumann regular (see [23, 23.5]). In [7, Theorem 2.9], it is proved that a right module M over a ring R such that every right primitive factor of R is right artinian satisfies $(*)$ condition if and only if M is cosemisimple.

We shall investigate when a module M satisfies the following condition.

$(**)$ For any submodules $N_1, N_2 \leq M$, $V^{s*}(N_1) = V^{s*}(N_2)$ implies that $N_1 = N_2$.

Let R be a simple ring and M be a non-zero right R-module. Then every non-zero submodule of M is a second submodule. Thus M satisfies $(**)$ condition.

Proposition 3.1. Let M be a non-zero right R-module. Then the following statement are equivalent.

(1) M satisfies $(**)$ condition.

(2) Every non-zero submodule of M is second radical.

Proof. The proof is the same as the commutative case in [6, Proposition 2.2]. □

Since every simple submodule of a module M is second, we get the following corollary.

Corollary 3.2. Every semisimple module M satisfies $(**)$ condition.

Remark 3.3. In general the converse of Corollary 3.2 is not true. For example, any module M over a simple ring R satisfies $(**)$ condition, but M is not necessarily a semisimple R-module.

We shall be interested in a ring R such that the ring R/P is right artinian for every right primitive ideal P. PI rings (in particular commutative rings) and right FBN rings have this property (see [19, 13.3.8] and [14, Proposition 9.4]). Clearly semilocal rings also have this property. Note that such rings have the property that the right primitive ideals are precisely the maximal ideals.

In the following theorem we show that if R is a ring such that the ring R/P is right artinian for every right primitive ideal P, then the converse of Corollary 3.2 is true for all right R-modules.

Theorem 3.4. Let R be a ring such that the ring R/P is right artinian for every right primitive ideal P. Then a right R-module M satisfies $(**)$ condition if and only if M is semisimple.

Proof. $(\Longleftarrow)$ Clear by Corollary 3.2.

$(\Longrightarrow)$ Let $m \in M$ and K be a maximal submodule of mR. Let P denote the annihilator of mR/K in R. Then R/P is a right artinian prime ring by the hypothesis on R. This implies that P is a maximal ideal of R. By the hypothesis, $mR = \sum_{i \in I} S_i$ where $\{S_i\}_{i \in I}$ is the set of all second submodules of mR. Fix $i \in I$. We have either $(S_i + K)/K = mR/K$ or $S_i \subseteq K$. In the first case $ann_R((S_i + K)/K) = ann_R(S_i) = P$, a maximal ideal of R. The hypothesis on R implies that S_i is homogeneous semisimple with $ann_R(S_i) = P$. It follows that $mR = (\sum_{j \in I, S_j \subseteq K} S_j) + T = K + T$, where T is a homogeneous semisimple module. Since $K \cap T$ is a maximal submodule of T, there exists a simple submodule L of T such that $T = (K \cap T) \oplus L$. L cannot be a simple submodule of K and so $K \cap L = 0$. This implies that $mR = K \oplus L$. Thus every maximal submodule of mR is a direct summand. Suppose that $Soc(mR) \neq mR$. Then there exists a maximal submodule N of mR containing $Soc(mR)$ and $mR = N \oplus Q$ for some simple submodule Q of mR. This implies that $Q \subseteq N$, a contradiction. Thus $mR = Soc(mR)$. This implies that M is a semisimple module. □

Corollary 3.5. Let R be a ring such that the ring R/P is right artinian for every right primitive ideal P. Then the following statements are equivalent.

(1) R is a semisimple ring.
(2) Every right R-module satisfies $(**)$ condition.
(3) Every left R-module satisfies $(**)$ condition.
(4) The left R-module $_RR$ satisfies $(**)$ condition.
(5) The right R-module R_R satisfies $(**)$ condition.

In the following proposition, we prove that $(*)$ and $(**)$ conditions are equivalent for a second module M over a ring R such that every right primitive factor of R is right artinian.

Proposition 3.6. Let R be a ring such that the ring R/P is right artinian for every right primitive ideal P and M be a second right R-module. Then the following statements are equivalent.

(1) M satisfies $(**)$ condition.
(2) Every non-zero submodule of M is second.
(3) M is homogeneous semisimple.
(4) M is fully prime.
(5) M satisfies $(*)$ condition.

Proof. (1) $\Longrightarrow$ (2) M is a semisimple module by Theorem 3.4. Since M is a second module which contains a maximal submodule, M is homogeneous semisimple by [10, Lemma 1.3]. Every non-zero submodule of M is also homogeneous semisimple. Hence every non-zero submodule of M is second.

(2) $\Longrightarrow$ (1) By Proposition 3.1.

(1) $\Longrightarrow$ (3) This implication is proved in the proof of implication (1) $\Longrightarrow$ (2).

(3) $\Longrightarrow$ (4) Every homogeneous semisimple module is fully prime.

(4) $\Longrightarrow$ (5) M satisfies $(*)$ condition by [7, Proposition 2.3].

(5) $\Longrightarrow$ (1) M is a cosemisimple module by [7, Theorem 2.9]. Since M is a second module which contains a maximal submodule, M is semisimple by [10, Lemma 1.3]. The result follows from Corollary 3.2. □

In the following proposition, we prove that $(*)$ and $(**)$ conditions are equivalent for a prime module M over a commutative ring R.

Proposition 3.7. Let R be a commutative ring and M be a prime right R-module. Then the following statements are equivalent.

(1) M satisfies $(*)$ condition.

(2) M is fully prime.

(3) M is homogeneous semisimple.

(4) M satisfies $(**)$ condition.

(5) Every non-zero submodule of M is second.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) By [7, Corollary 2.7]

(3) $\Longrightarrow$ (4) By Corollary 3.2.

(4) $\Longrightarrow$ (5) M is a semisimple module by Theorem 3.4. Since M is a prime R-module which contains a simple submodule, M is homogeneous semisimple by [10, Lemma 1.3]. Every non-zero submodule of M is also homogeneous semisimple. Hence every non-zero submodule of M is second.

(5) $\Longrightarrow$ (4) By Proposition 3.1.

(4) $\Longrightarrow$ (1) M is a semisimple module by Theorem 3.4. M is of course a cosemisimple module. Now the result follows from [7, Corollary 2.4]. □

Remark 3.8. In [6], among other nice results, Corollary 2.6 states that a module M over a commutative ring R satisfies $(**)$ condition if and only if M is a cosemisimple module. Unfortunately, this result is not true in general. Let R be the direct product $R = \Bbbk^{\mathbb{N}}$, where $\Bbbk$ is a field and $\mathbb{N}$ is the set of natural numbers. Clearly, R is a Von Neumann regular ring which is not semisimple. Since R is commutative, R_R is a cosemisimple module. But R_R does not satisfy $(**)$ condition by Theorem 3.4. However [6, Corollary 2.6] is true when M is an artinian module. For, if M is artinian, then it is semisimple if and only if it is cosemisimple, and the result follows from Theorem 3.4.

The set of all minimal submodules of a module M will be denoted by $Min(M)$. The following theorem generalizes [6, Theorem 2.11].

Theorem 3.9. Let M be a right R-module. Suppose that one of the following conditions is satisfied,

(a) R is a right perfect ring or

(b) R is a ring such that the ring R/P is right artinian for every right primitive ideal P and M is a noetherian right R-module,

Then $X^s(M)$ is a T_1-space if and only if either $X^s(M) = \emptyset$ or $X^s(M) = Min(M)$.

Proof. First suppose that $X^s(M)$ is a T_1-space. Then $X^s(M) = \emptyset$ or every element of $X^s(M)$ is minimal by Proposition 2.1. If $X^s(M) = \emptyset$ then we are done. Let S be a second submodule of M.

(a) If R is a right perfect ring, then S is a semisimple R-module by [10, Corollary 1.4] and hence S has a minimal submodule. Since every element of $X^s(M)$ is minimal, S is a minimal submodule of M. This implies that $X^s(M) = Min(M)$ since every minimal submodule of M is second.

(b) If condition (b) is satisfied, then $sec(M) = soc(M)$ by [11, Corollary 2.6]. Thus S is a semisimple R-module. The proof is completed as in part (a).

Conversely suppose that $X^s(M) = \emptyset$ or $X^s(M) = Min(M)$. Then $X^s(M)$ is a T_1-space by Proposition 2.1. □

By a *maximal second submodule* of a module M, we mean a second submodule L of M such that L is not properly contained in another second submodule of M. In [10, Corollary 4.3] it is shown that every second submodule of a non-zero module M is contained in a maximal second submodule of M. We will use this fact in our next proofs without further comment.

The following theorem generalizes [6, Theorem 2.19].

Theorem 3.10. Let R be a ring such that the ring R/P is right artinian for every right primitive ideal P and M be a noetherian right R-module. Then the following statements are equivalent.

(1) $X^s(M)$ is a Hausdorff space.

(2) $X^s(M)$ is a T_1-space.

(3) $X^s(M)$ is the cofinite topology.

(4) $X^s(M)$ is discrete.

(5) Either $X^s(M) = \emptyset$ or $X^s(M) = Min(M)$.

Proof. (1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (3) By Proposition 2.1, $X^s(M) = \emptyset$ or every element of $X^s(M)$ is minimal. If $X^s(M) = \emptyset$, then the result is clear. Assume that $X^s(M) \neq \emptyset$. Then every second submodule of M is maximal second. By [11, Theorem 3.1], M

contains only a finite number of maximal second submodules. Thus $X^s(M)$ is finite. By Proposition 2.2, $X^s(M)$ is the cofinite topology.

(3) $\Longrightarrow$ (4) Since every cofinite topology satisfies T_1-axiom, $X^s(M) = \emptyset$ or every element of $X^s(M)$ is minimal by Proposition 2.1. We may assume that $X^s(M) \neq \emptyset$. As we proved in the proof of (2) $\Longrightarrow$ (3), $X^s(M)$ is finite. Therefore $X^s(M)$ is discrete by Proposition 2.2.

(4) $\Longrightarrow$ (5) By Theorem 3.9.

(5) $\Longrightarrow$ (1) $X^s(M)$ is a T_1-space by Theorem 3.9. If $X^s(M) = \emptyset$, then we are done. If $X^s(M) = Min(M)$, then every second submodule of M is maximal second. By [11, Theorem 3.1], M contains only a finite number of maximal second submodules. Thus $X^s(M)$ is finite. Now the result follows from Proposition 2.2. □

Proposition 3.11. Let R be a right perfect ring and M be a right R-module. Then the following statements are equivalent.

(1) $X^s(M)$ is a Hausdorff space.
(2) $X^s(M)$ is a T_1-space.
(3) $X^s(M)$ is the cofinite topology.
(4) $X^s(M)$ is discrete.
(5) Either $X^s(M) = \emptyset$ or $X^s(M) = Min(M)$.

Proof. (1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (3) By Theorem 3.9, $X^s(M) = \emptyset$ or $X^s(M) = Min(M)$. If $X^s(M) = \emptyset$, then we are done. Assume that $X^s(M) \neq \emptyset$. Since any sum of P-second submodules for a prime ideal P of R is also P-second, M contains no more than one copy of each simple submodule. Since R is a right perfect ring, it is a semilocal ring and hence there are only finitely many non-isomorphic simple right R-modules. Thus $X^s(M)$ is finite. By Proposition 2.2, $X^s(M)$ is the cofinite topology.

(3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (5) Similar to the proof of Theorem 3.10.

(5) $\Longrightarrow$ (1) $X^s(M)$ is a T_1-space by Theorem 3.9. According to the proof of implication (2) $\Longrightarrow$ (3), $X^s(M)$ is finite. Now the result follows from Proposition 2.2. □

The next theorem is the generalization of [6, Theorem 2.18]. Although its statement is the same as [6, Theorem 2.18], its proof is somewhat different.

Theorem 3.12. Let M be an artinian right R-module. Then the following statements are equivalent.

(1) $X^s(M)$ is a Hausdorff space.
(2) $X^s(M)$ is a T_1-space.
(3) $X^s(M)$ is the cofinite topology.

(4) $X^s(M)$ is discrete.
(5) $X^s(M) = Min(M)$.

Proof. We may assume that M is non-zero. Note that since M is a non-zero artinian module, $X^s(M) \neq \emptyset$.

(1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (3) Every element of $X^s(M)$ is minimal by Proposition 2.1. This implies that every second submodule of M is a maximal second submodule of M. Since M is artinian, it contains only a finite number of maximal second submodules by [10, Theorem 4.4]. Thus $X^s(M)$ is finite. $X^s(M)$ is the cofinite topology by Proposition 2.2.

(3) $\Longrightarrow$ (4) Since every cofinite topology satisfies T_1-axiom, every element of $X^s(M)$ is minimal by Proposition 2.1. As we proved in the proof of (2) $\Longrightarrow$ (3), $X^s(M)$ is finite. Now the result follows from Proposition 2.2.

(4) $\Longrightarrow$ (5) Since $X^s(M)$ is a T_1-space, every element of $X^s(M)$ is minimal by Proposition 2.1. Every second submodule of M contains a minimal submodule as M is artinian. Therefore every second submodule of M is minimal. This implies that $X^s(M) = Min(M)$.

(5) $\Longrightarrow$ (1) Every second submodule of M is maximal second as $X^s(M) = Min(M)$. Since M is artinian, it contains only a finite number of maximal second submodules by [10, Theorem 4.4]. Thus $X^s(M)$ is finite. The result follows from Proposition 2.2. □

In the next theorem we obtain a characterization of commutative artinian rings by using the second classical Zariski topology.

Theorem 3.13. (a) If R is a right perfect ring, then $Spec^s(E(N))$ is a singleton set for every simple right R-module N where $E(N)$ is the injective hull of N.

(b) Let R be a commutative noetherian ring. Then, $Spec^s(E(N))$ is a T_1-space for every simple R-module N if and only if R is an artinian ring.

Proof. (a) Clearly $Spec^s(E(N)) \neq \emptyset$. Let S be a second submodule of $E(N)$. Then S is a homogeneous semisimple submodule of $E(N)$ by [10, Corollary 1.4]. Since $soc(E(N)) = N$, we have $S = N$. Thus $Spec^s(E(N)) = \{N\}$.

(b) Suppose that $Spec^s(E(N))$ is a T_1-space for every simple R-module N. Let P be a prime ideal of R and Q be a maximal ideal of R such that $P \subseteq Q$. Consider the R-module $M = E(R/Q)$. M is an artinian R-module by [20, p. 121, exercise 4.18]. We have $R/Q \subseteq (0 :_M Q) \subseteq (0 :_M P)$ whence $(0 :_M P) \neq 0$. [20, Proposition 2.27] implies that $(0 :_M P)$ is a P-second submodule of M. By Theorem 3.12, $(0 :_M P)$ is a simple R-module whence $R/Q = (0 :_M P)$. This implies that $P = Q$, a maximal ideal of R. Since R is noetherian and $dimR = 0$, R is an artinian ring. Conversely, if R is an artinian ring, the result follows from part (a). □

A ring R is called *Quasi-Frobenius* if it is right noetherian and right self-injective. In the next theorem, we give a characterization of commutative Quasi-Frobenius rings by using the second classical Zariski topology.

Theorem 3.14. Let R be a commutative artinian ring. Then $X^s(R)$ is a T_1-space if and only if R is a Quasi-Frobenius ring.

Proof. Suppose that $X^s(R)$ is a T_1-space. Then $X^s(R) = Min(R)$ by Theorem 3.12. Since any sum of P-second ideals for a prime ideal P of R is also P-second, $soc(R)$ contains no more than one copy of each simple ideal. By [15, (15.27) Theorem], R is a Quasi-Frobenius ring. Conversely assume that R is a Quasi-Frobenius ring. Let S be second ideal of R. Then $ann_R(S)$ is a maximal ideal of R whence S is homogeneous semisimple. But, again by [15, (15.27) Theorem], $soc(R)$ contains no more than one copy of each simple ideal. Thus S must be simple whence $X^s(R) = Min(R)$. The result follows from Theorem 3.12. □

Example 3.15. It is well-known that the group algebra $F[G]$ where F is a field and G is a finite abelian group, and R/aR where R is a commutative principal ideal domain and a is a non-zero non-unit element of R are commutative Quasi-Frobenius rings. Consider these rings as modules over themselves. By Theorem 3.14, $Spec^s(F[G])$ and $Spec^s(R/aR)$ are T_1-spaces.

Now we deal with the irreducible subsets of $Spec^s(M)$ for a right R-module M. In the following theorem we characterize the second radical of a module M over a right perfect ring and the second radical of a noetherian module M over a ring R such that every right primitive factor of R is right artinian for which $Spec^s(M)$ is irreducible.

Theorem 3.16. Let R be a ring and M be a non-zero right R-module. Suppose that one of the following conditions is satisfied.

(a) R is a right perfect ring or

(b) R is a ring such that the ring R/P is right artinian for every right primitive ideal P and M is a noetherian right R-module.

(c) R is a ring such that the ring R/P is right artinian for every right primitive ideal P and M is a cosemisimple right R-module.

Then the following statements are equivalent.

(1) $Spec^s(M)$ is irreducible

(2) $sec(M)$ is a non-zero homogeneous semisimple module.

(3) $Spec^s(M) \neq \emptyset$ and for each submodule N of M, either $V^{s*}(N) = \emptyset$ or $V^{s*}(N)$ irreducible.

Proof. (a) Let R be a right perfect ring.

$(1) \Longrightarrow (2)$ Assume that $Spec^s(M)$ is irreducible. Then $sec(M)$ is a second submodule by Corollary 2.5. Since R is right perfect, $sec(M)$ is a non-zero homogeneous semisimple module by [10, Corollary 1.4].

$(2) \Longrightarrow (3)$ Clearly $Spec^s(M) \neq \emptyset$. Let N be a submodule of M and $V^{s*}(N) \neq \emptyset$. Then (2) implies that $sec(N)$ is a non-zero homogeneous semisimple module. Hence $sec(N)$ is a second submodule of M. By Corollary 2.5, $V^{s*}(N)$ is irreducible.

$(3) \Longrightarrow (1)$ Clear (since $V^{s*}(M) = Spec^s(M)$).

Suppose that condition (b) holds.

$(1) \Longrightarrow (2)$ Assume that $Spec^s(M)$ is irreducible. Then $sec(M)$ is a second submodule by Corollary 2.5. Since M is noetherian, $sec(M)$ contains a maximal submodule. By [10, Lemma 1.3], $sec(M)$ is a non-zero homogeneous semisimple module.

The proofs of implications $(2) \Longrightarrow (3)$ and $(3) \Longrightarrow (1)$ are the same as the proofs in part (a).

Suppose that condition (c) holds.

$(1) \Longrightarrow (2)$ Assume that $Spec^s(M)$ is irreducible. Then $sec(M)$ is a second submodule by Corollary 2.5. Since M is cosemisimple, $Rad(M) = 0$. [11, Proposition 2.5] implies that $sec(M)$ is semisimple. Thus $sec(M)$ is a second module which contains a maximal submodule. By [10, Lemma 1.3], $sec(M)$ is a non-zero homogeneous semisimple module.

The proofs of implications $(2) \Longrightarrow (3)$ and $(3) \Longrightarrow (1)$ are the same as the proofs in part (a). □

In the following theorem we characterize all the irreducible components of $X^s(M)$ for a non-zero injective right module M over a ring R such that the ring R/P is right or left Goldie for every prime ideal P of R. Note that if R is a ring satisfying a polynomial identity, in particular a commutative ring, then the ring R/P is right and left Goldie for every prime ideal P of R (see, for example, [19, Corollary 13.6.6]).

Theorem 3.17. Let R be a ring such that the ring R/P is right or left Goldie for every prime ideal P of R and let M be a non-zero injective right R-module. Then every irreducible component of $X^s(M)$ is of the form $V^{s*}((0 :_M p))$ for some minimal prime ideal p of $ann_R(M)$. If p is a minimal prime ideal of $ann_R(M)$ such that $(0 :_M p) \neq 0$, then $V^{s*}((0 :_M p))$ is an irreducible component of $X^s(M)$.

Proof. Let Y be an irreducible component of $X^s(M)$. Then $T(Y)$ is a second submodule of M by Theorem 2.4-(a). $ann_R(T(Y)) := p$ is a prime ideal of R which contains $ann_R(M)$. Let q be a minimal prime ideal of $ann_R(M)$ contained in p. Then $(0 :_M q)$ is a non-zero injective right (R/q)-module by [20, Proposition 2.27]. By [10, Corollary 2.7], $(0:_M q)$ is a second (R/q)-module. [10, Corollary 2.4] implies that $(0:_M q)$ is a second R-submodule of M. By Lemma 2.1, $V^{s*}((0:_M q))$ is an irreducible closed subset of $X^s(M)$. Moreover, we have $Y \subseteq V^{s*}(T(Y)) \subseteq V^{s*}((0:_M q))$. By the maximality of Y, we have $Y = V^{s*}((0:_M q))$.

For the last assertion, suppose that p is a minimal prime ideal of $ann_R(M)$ such that $(0:_M p) \neq 0$. Then we can show that $(0:_M p)$ is a second submodule of M as in the proof the first assertion. $V^{s*}((0:_M p))$ is an irreducible closed subset of $X^s(M)$ by Lemma 2.1. There exists an irreducible component Y' of $X^s(M)$ containing $V^{s*}((0:_M p))$. $Y' = V^{s*}((0:_M q))$ for some minimal prime ideal of $ann_R(M)$ by the first assertion. Since $V^{s*}((0:_M P)) \subseteq V^{s*}((0:_M q))$, we have $(0:_M p) \subseteq (0:_M q)$ and hence $q \subseteq p$. By the minimality of p, we get that $q = p$ and hence $V^{s*}((0:_M p)) = Y'$. Thus $V^{s*}((0:_M p))$ is an irreducible component of $X^s(M)$. □

Given a prime ideal P of R, a proper submodule K of an R-module M is called P-primary provided

(i) $(K:N) \subseteq P$ for every submodule N of M such that $N \not\subseteq K$ and

(ii) $P^n \subseteq (K:M)$ for some positive integer n.

Note that if K is P-primary, then $P^n \subseteq (K:M) \subseteq P$ for some positive integer n. A submodule L of an R-module M is called primary if L is P-primary for some prime ideal P of R. A submodule H of M has a primary decomposition if H is the intersection of a finite collection of primary submodules of M. Note that if H has a primary decomposition then H is a proper submodule of M (see [21]).

Let N be a submodule of an R-module M such that N has a primary decomposition. Then N is said to have a normal decomposition if there exist a positive integer n, distinct prime ideals $P_i (1 \leq i \leq n)$ of R and P_i-primary submodules $K_i (1 \leq i \leq n)$ of M such that $N = K_1 \cap \ldots \cap K_n$ and $N \neq K_1 \cap \ldots \cap K_{i-1} \cap K_{i+1} \cap \ldots \cap K_n$ for all $1 \leq i \leq n$. In [21, Corollary 2], it is shown that if N is a submodule of an R-module M such that N has a primary decomposition then N has a normal decomposition.

Corollary 3.18. Let R be a ring such that the ring R/P is right or left Goldie for every prime ideal P of R and let M be a non-zero injective right R-module. If the zero submodule of M has a primary decomposition, then $X^s(M)$ has finitely many irreducible components.

Proof. Let $\cap_{i=1}^n Q_i$ be a normal decomposition of the zero submodule of M, where Q_i is a p_i-primary submodule of M for each i. Then every minimal prime ideal of $ann_R(M)$ belongs to the set $\{p_1, ..., p_n\}$ by [21, Lemma 4]. The result follows from Theorem 3.17. □

REFERENCES

[1] J. Y. Abuhlail, *Zariski topologies for coprime and second submodules*, Algebra Colloquium, 22 (1), (2015), DOI: http://dx.doi.org/10.1142/S1005386715000061.

[2] S. Annin, *Attached primes over noncommutative rings*, J. Pure Appl. Algebra, 212 (3) (2008), 510-521.

[3] H. Ansari-Toroghy, F. Farshadifar, *On the dual notion of prime submodules*, Algebra Colloq., 19 (1) (2012), 1109-1116.

[4] H. Ansari-Toroghy, F. Farshadifar, *On the dual notion of prime submodules II*, Mediterr. J. Math., 9 (2) (2012), 327-336.

[5] H. Ansari-Toroghy and F. Farshadifar, *The Zariski topology on the second spectrum of a module*, Algebra Colloquium, 671 (2014), DOI: 10.1142/S1005386714000625.

[6] H. Ansari-Toroghy, S. Keyvani and F. Farshadifar, *On the second spectrum of a module (II)*, Bull. Malays. Math. Sci. Soc. 39 (3) (2016), 1089–1103.

[7] M. Behboodi and M. R. Haddadi, *Classical Zariski topology of modules and spectral spaces I,* Inter. Electronic J. Alg. 4 (2008), 104-130.

[8] M. Behboodi and M. R. Haddadi, *Classical Zariski topology of modules and spectral spaces II,* Inter. Electronic J. Alg. 4 (2008), 131-148.

[9] S. Çeken and M. Alkan, *Dual of Zariski topology for modules*, Book Series: AIP Conference Proceedings, 1389 (1) (2011), 357-360.

[10] S. Çeken, M. Alkan and P. F. Smith, *Second modules over noncommutative rings*, Communications in Algebra, 41 (1) (2013), 83-98.

[11] S. Çeken, M. Alkan and P. F. Smith, *The dual notion of the prime radical of a module,* Journal of Algebra, 392 (2013), 265–275.

[12] S. Çeken, M. Alkan, *On graded second and coprimary modules and graded secondary representations*, Bull. Malays. Math. Sci. Soc. 38 (4) (2015), 1317–1330.

[13] S. Çeken, M.Alkan, *Second Spectrum Of Modules And Spectral Spaces*, Bulletin of the Malaysian Mathematical Sciences Society, (2017), DOI: 10.1007/s40840-017-0473-0.

[14] K. R. Goodearl, R. B. Warfield, *An introduction to noncommutative noetherian rings,* London Math. Soc. Student Texts 16, Cambridge Univ. Press, Cambridge, 2004.

[15] T. Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag, New York, 1999.

[16] C.P. Lu, *Prime submodules of modules*. Comm. Math. Univ. Sancti Pauli 33, (1984), 61-69.

[17] C.P. Lu, *The Zariski topology on the prime spectrum of a module*, Houston J. Math. 25 (3) (1999), 417-432.

[18] R. L. McCasland, M. E. Moore and P. F. Smith, *On the spectrum of a module over a commutative ring*, Comm. Algebra 25 (1997), 79-103.

[19] J.C. McConnell, J.C. Robson, *Noncommutative noetherian rings*, Wiley, Chichester, 1987.

[20] D.W. Sharpe and P. Vamos, *Injective modules*, Cambridge University Press, London, 1972.

[21] P. F. Smith, *Uniqueness of Primary Decompositions*, Turk J Math. 27 (2003), 425-434.

[22] Ü. Tekir, *The Zariski topology on the prime spectrum of a module over noncommutative rings*, Algebra Colloquium 16 (4) (2009), 691-698.

[23] R. Wisbauer, *Foundations of Modules and Ring Theory*, Gordon and Breach Reading 1991.

[24] S. Yassemi, *The dual notion of prime submodules*, Arch. Math (Brno) 37 (2001), 273–278.

SEÇIL ÇEKEN
Department of Mathematics,
Faculty of Science,
Trakya University, Edirne-TURKEY

MUSTAFA ALKAN
Department of Mathematics, Faculty of Science,
Akdeniz University,
Antalya-TURKEY