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ON SOME SEMILINEAR EQUATIONS

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The results presented here are obtained with Lucio Boccardo and regards some nonlinear elliptic problems whose prototype is the following

(0.1) $$\begin{cases} -\Delta u + \nu|u|^{p-1}u = \gamma|\nabla u|^{2\theta} + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $f(x)$ is a summable function in Ω (bounded open set in $\mathbb{R}^N$, $N > 2$), $0 < \theta < 1$ and $\gamma \in \mathbb{R}$.

We are interested in existence of weak solutions for a class of semilinear elliptic problems whose simplest model is the following

(0.2) $$\begin{cases} -\Delta u + \nu|u|^{p-1}u = \gamma|\nabla u|^{2\theta} + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N > 2$, $\gamma \in \mathbb{R}$ and $0 < \theta < 1$.

The quadratic case ($\theta = 1$) is related to the problem of minimizations of some functionals of the Calculus of Variations. As a matter of fact it is well known that the minimizations in $W_0^{1,2}(\Omega)$ of simple functionals like

$$I(v) = \frac{1}{2}\int_\Omega a(x, v)|\nabla v|^2 - \int_\Omega f(x)v(x)$$

where a is a bounded, smooth function and $f \in L^2(\Omega)$, leads to the following Euler-Lagrange equation

$$\begin{cases} -\mathrm{div}(a(x,u)\nabla u) + \dfrac{1}{2}\, a'(x,u)|\nabla u|^2 = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Anyway even if the problems considered here do not come from Calculus of Variations, they are interesting by themselves and appear in connections to Stochastic optimal control problems. In particular when $p = 1$ this type of equations is sometimes referred as stationary viscous Hamilton-Jacobi equations.
We restrict our study to the superlinear case ($2\theta > 1$) as the case $2\theta \leq 1$ was just treated in [2] and in [3]. Notice that in the previous papers ($2\theta \leq 1$) it is not necessary the presence of a lower order term in order to have existence of weak solution, i.e. the following problem

$$\begin{cases} -\Delta u = \gamma |\nabla u|^{2\theta} + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.3}$$

admits weak solutions. On the other hand, it is well known that in the quadratic case ($2\theta = 2$) there aren't weak solutions of (0.3) even for bounded data f if we don't require further conditions, like for example suitable smallness of the data (see [13]).

Hence the natural questions that arise are the following.
What happen in the superlinear case $1 < 2\theta < 2$? What is the role of the lower order term $|u|^{p-1}u$? Is it "necessary or not" to have distributional solutions of problems like

$$\begin{cases} -\Delta u + \nu |u|^{p-1}u = \gamma |\nabla u|^{2\theta} + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

when $1 < 2\theta < 2$ and $f \in L^1(\Omega)$?

Differently from the case $2\theta \leq 1$ the presence of the term $|u|^{p-1}u$ is not only crucial to prove the existence of a weak solution of (0.2) but it is in some sense necessary to guarantee the existence of a solution when the growth of the gradient is superlinear. Indeed, if we erase it, that is if we get $\nu = 0$ in the previous equation that thus becomes (0.3), then the existence of a solution requires that f is small enough and regular enough (see [1]): for example just for bounded f to have distributional solutions the L^∞ norm of f must be suitable small (hence a behaviour analogous to the quadratic case). Moreover also when the lower order term is present with $p = 1$ again necessary conditions must imposed to have solutions. In particular if f belongs to a Lebesgue space $L^m(\Omega)$ it is necessary that

$$m \geq \frac{N}{(2\theta)'}. \tag{0.4}$$

We refer to [1] and [12] for an extensive study on the necessary conditions to have weak solutions.

Notice that (when $p = 1$) the condition (0.4) is also sufficient to allow the existence of a solution as it has been recently proved in [11] if either $\nu > 0$ or $\nu = 0$ and a size condition is satisfied.
Thus if $f \in L^1(\Omega)$ the previous condition becomes

$$2\theta \leq \frac{N}{N-1}, \tag{0.5}$$

and thus there aren't solutions when $p = 1$, f is only a sommable function and 2θ is close to 2 (i.e. for $\frac{N}{N-1} < 2\theta < 2$).

The restriction (0.5) on θ can be easily justified by the following heuristic argument. It is well known that when $f \in L^1(\Omega)$ the problem

$$\begin{cases} -\Delta u + \nu u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits solutions $u \in W_0^{1,q}(\Omega)$, for every $1 \leq q < \frac{N}{N-1}$. Then if we add in the right-hand side of the previous equation a term like $\gamma|\nabla u|^{2\theta}$ it naturally doesn't improve the regularity of the solution. Thus the solutions of

$$\begin{cases} -\Delta u + \nu u = \gamma|\nabla u|^{2\theta} + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.6}$$

will be at most in $W_0^{1,q}(\Omega)$, where q is as before, and hence at most we have

$$|\nabla u|^{2\theta} \in L^{\frac{q}{2\theta}}(\Omega),$$

where

$$\frac{q}{2\theta} \geq 1 \quad \Leftrightarrow \quad 2\theta \leq \frac{N}{N-1},$$

that is the condition (0.5).

So if we want distributional solutions for every $1 < 2\theta$ and for every $f \in L^1(\Omega)$ we need a different lower order term in (0.6) that must have a regularizing effect on the solution, i.e. it must allow that it satisfies

$$|\nabla u|^{2\theta} \in L^1(\Omega)$$

also when

$$2\theta > \frac{N}{N-1}.$$

The choice of $|u|^{p-1}u$ as a lower order term solves the problem. As a matter of fact what happens is that such a term has a regularizing effect on the solution, i.e. if p is sufficiently big we obtain higher integrability on u and on its gradient with respect to the case $\nu = \gamma = 0$. This higher integrability will be occur also if $f \in L^{1+\varepsilon}(\Omega)$, $\varepsilon > 0$ and surprisingly for $p > \frac{1}{\varepsilon}$ will assure a solution in $W_0^{1,2}(\Omega)$.

We state now our results not in all their generality (see [7]). Let us consider the following problem

$$\text{(0.7)} \qquad \begin{cases} -\operatorname{div}(M(x,u)\nabla u) + g(x,u) = b(x,u,\nabla u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 3$. We assume that $M(x,s)$ is a Carathéodory matrix and $b(x,s,\xi)$ and $g(x,s)$ are Carathéodory functions (that is, measurable with respect to x for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) for almost every $x \in \Omega$) which satisfy, for some positive constants $\theta, \alpha, \beta, \gamma, \nu$ a.e. in $x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$\text{(0.8)} \qquad M(x,s)\xi \cdot \xi \geq \alpha|\xi|^2,$$

$$\text{(0.9)} \qquad |M(x,s)| \leq \beta,$$

$$\text{(0.10)} \qquad |b(x,s,\xi)| \leq \gamma|\xi|^{2\theta}, \quad 1 < 2\theta < 2.$$

On the function g, we assume

$$\text{(0.11)} \qquad g(x,s)s \geq \nu|s|^{p+1}, \qquad \text{with} \qquad p > \frac{\theta}{1-\theta}.$$

On the data in the right hand side we require that

$$\text{(0.12)} \qquad f \in L^1(\Omega).$$

Indeed our results hold true with the same proofs even if the principal part is nonlinear also with respect to the gradient.

Before enouncing our existence and regularity results we briefly introduce some notations and recall the definition of weak solution. If $m \in [1,+\infty]$ we denote with m' and m^*, the values in $[1,+\infty]$ such that $\frac{1}{m} + \frac{1}{m'} = 1$ and $\frac{1}{m^*} = \frac{1}{m} - \frac{1}{N}$, where we set "$\frac{1}{+\infty} = 0$".

Definition 0.1. We say that $u \in W_0^{1,1}(\Omega)$ is a weak solution of (0.7) if $g(x,u) \in L^1(\Omega)$, $b(x,u,\nabla u) \in L^1(\Omega)$ and for every $\varphi \in W_0^{1,\infty}(\Omega)$ we have

$$\int_\Omega M(x,u)\nabla u \nabla\varphi\, dx + \int_\Omega g(x,u)\varphi\, dx = \int_\Omega b(x,u,\nabla u)\varphi\, dx + \int_\Omega f(x)\varphi\, dx. \tag{0.13}$$

We have the following results.

Theorem 0.2. *Assume that* (0.8)–(0.12) *hold true. Then there exists a weak solution* $u \in W_0^{1,q} \cap L^p(\Omega)$ *of* (0.7) *for every* $1 \le q < q_1$ *where*

$$q_1 = \max\{1^*, \frac{2p}{1+p}\}.$$

Notice that it results

$$2\theta < 1^* \Leftrightarrow 2\theta < \frac{N}{N-1}$$

and

$$2\theta < \frac{2p}{1+p} \Leftrightarrow p > \frac{\theta}{1-\theta}$$

that is exactly our assumption on p.
Moreover by Sobolev imbedding Theorem the solution constructed in Theorem 0.2 belongs also to $L^s(\Omega)$, for every $s < \max\{1^{**}, (\frac{2p}{p+1})^*\}$.
Finally we can consider also more general data (see [7]).

If f has an higher integrability then we have more regular solutions. More in details we have the following result.

Theorem 0.3. *Assume that* (0.8)–(0.11) *hold true and that* f *belongs to* $L^m(\Omega)$*, where* $1 < m < \frac{N}{2}$*. Then there exists a weak solution* $u \in W_0^{1,q}(\Omega) \cap L^{pm}(\Omega) \cap L^{m^{**}}(\Omega)$ *of* (0.7) *where*

$$q = \min\left\{2, \max\{m^*, \frac{2pm}{1+p}\}\right\}.$$

Hence if

$$p(m-1) - 1 \ge 0 \quad hboxor\ if \quad \frac{2N}{N+2} \le m < \frac{N}{2},$$

then $u \in W_0^{1,2}(\Omega)$.

Our restriction $m < \frac{N}{2}$ in Theorem 0.3 is due to the fact that the case $m > \frac{N}{2}$ was just considered in [6] (bounded solutions), while the case $m = \frac{N}{2}$ can be derived from the results in [10] and in [9].

Notice that it results

$$pm \geq m^{**} \quad \Longleftrightarrow \quad p \geq \frac{N}{N-2m}.$$

Moreover we have

$$q = \min\left\{2, \frac{2pm}{1+p}\right\} \quad \Longleftrightarrow \quad p \geq \frac{N}{N-2m}.$$

Remark 0.4. Really in the proofs of Theorems 0.2 and 0.3 we don't use the assumption $\theta > \frac{1}{2}$, i.e. the results of these theorems hold true for every $0 < \theta < 1$. Hence the lower order term $g(x,u)$ that, as just noticed, is not necessary to have existence results when $0 < \theta \leq \frac{1}{2}$, has a regularizing effect on the solutions also when the growth of $b(x,u,\nabla u)$ is not superlinear.

1. Sketch of the proof of the existence Theorem.

In order to prove the existence of a weak solution of (0.7) let us define for $n \in \mathbb{N}$, the approximations

$$b_n(x,s,\xi) = \frac{b(x,s,\xi)}{1+\frac{1}{n}|b(x,s,\xi)|}, \qquad f_n(x) = \frac{f(x)}{1+\frac{1}{n}|f(x)|}.$$

Notice that it results

$$|b_n(x,s,\xi)| \leq |b(x,s,\xi)|, \quad |f_n(x)| \leq |f(x)|,$$

and

$$|b_n(x,s,\xi)| \leq n, \quad |f_n(x)| \leq n.$$

Consider the approximate boundary value problems

$$\begin{cases} u_n \in W_0^{1,2}(\Omega): \\ \displaystyle\int_\Omega M(x,u_n)\nabla u_n \nabla\varphi + \int_\Omega g(x,u_n)\varphi \\ \qquad = \displaystyle\int_\Omega b_n(x,u_n,\nabla u_n)\varphi + \int_\Omega f_n\,\varphi, \quad \forall\varphi \in W_0^{1,2}\cap L^\infty(\Omega). \end{cases} \tag{1.1}$$

The existence of weak, bounded (see [15]) solution $u_n \in W_0^{1,2}(\Omega)$ of (1.1) follows by the classical results of [14] (see also [8]).

The proof of Theorem 0.2 proceed by steps. We give here only an a priori estimate, (that is the easier part of the proof), as it shows clearly the regularizing effect of the lower order term.

Proposition 1.1. *There exists a positive constant c_0, independent on n, such that*

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda} \le c_0, \quad \forall 1 < \lambda < p\frac{(1-\theta)}{\theta}, \tag{1.2}$$

and

$$\int_\Omega |u_n|^p \le c_0. \tag{1.3}$$

A first and immediate consequence of (1.2) and (1.3) is the following estimate

$$\int_\Omega |\nabla u_n|^q \le c_1, \quad \forall 1 \le q < \frac{2p}{1+p}, \tag{1.4}$$

where c_1 is a constant independent on n.
As a matter of fact, for every $q < 2$, using Young's inequality we deduce

$$\int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{q\frac{\lambda}{2}}}(1+|u_n|)^{q\frac{\lambda}{2}} \le \int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda} + \int_\Omega (1+|u_n|)^{\frac{q\lambda}{2}\left(\frac{2}{2-q}\right)},$$

and thus, thanks to (1.2) and (1.3), we have (1.4) choosing

$$\frac{q\lambda}{2}\left(\frac{2}{2-q}\right) = p \quad \Leftrightarrow \quad q = \frac{2p}{\lambda+p} \quad \Leftrightarrow \quad \forall\, q < \frac{2p}{1+p}.$$

A second consequence of Proposition 1.1 (or better of (1.2)) is that there exists a positive constant c_2, independent on n, such that

$$\int_\Omega |\nabla u_n|^q \le c_2, \quad \forall 1 \le q < \frac{N}{N-1}.$$

Proof of Proposition 1.1. Use as a test function in (1.1)

$$\varphi = [1-(1+|u_n|)^{1-\lambda}]\mathrm{sgn}(u_n),$$

where $\lambda > 1$ will be chosen later. Notice that $|\varphi| \leq 1$. We obtain

$$\alpha(\lambda-1)\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda} + \nu\int_\Omega |u_n|^p[1-(1+|u_n|)^{1-\lambda}] \leq \tag{1.5}$$
$$\gamma\int_\Omega \frac{|\nabla u_n|^{2\theta}}{(1+|u_n|)^{\lambda\theta}}(1+|u_n|)^{\lambda\theta} + \int_\Omega |f| \leq$$
$$\frac{\alpha}{2}(\lambda-1)\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda} + \frac{\gamma^{\frac{1}{1-\theta}}}{[\frac{\alpha}{2}(\lambda-1)]^{\frac{\theta}{1-\theta}}}\int_\Omega (1+|u_n|)^{\frac{\lambda\theta}{1-\theta}} + \int_\Omega |f|.$$

Let T such that $1-(1+T)^{1-\lambda} = \frac{1}{2}$. We have

$$\frac{1}{2}\int_{|u_n|>T} |u_n|^p \leq \int_{|u_n|>T} |u_n|^p[1-(1+|u_n|)^{1-\lambda}] \leq \int_\Omega |u_n|^p[1-(1+|u_n|)^{1-\lambda}]$$

which implies

$$\frac{1}{2}\int_\Omega |u_n|^p \leq \frac{1}{2}\int_{|u_n|\leq T} |u_n|^p + \frac{1}{2}\int_{|u_n|>T} |u_n|^p \leq$$
$$\frac{1}{2}T^p|\Omega| + \int_\Omega |u_n|^p[1-(1+|u_n|)^{1-\lambda}].$$

Thus from the previous estimates we deduce

$$\frac{\alpha}{2}(\lambda-1)\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda} + \frac{\nu}{2}\int_\Omega |u_n|^p \leq$$
$$\frac{\gamma^{\frac{1}{1-\theta}}}{[\frac{\alpha}{2}(\lambda-1)]^{\frac{\theta}{1-\theta}}}\int_\Omega (1+|u_n|)^{\frac{\lambda\theta}{1-\theta}} + \int_\Omega |f| + \frac{\nu}{2}T^p|\Omega|.$$

Choose λ such that $\frac{\lambda\theta}{1-\theta} < p$ that is $1 < \lambda < \frac{p(1-\theta)}{\theta}$. Notice that such a choice of λ is possible as by assumption $p > \frac{\theta}{1-\theta}$. Such a choice used in the previous estimate gives the following inequality

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda} \leq c_0, \tag{1.6}$$

and

$$\int_\Omega |u_n|^p \leq c_0, \tag{1.7}$$

and thus the proof of the proposition is completed. $\square$

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