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EXISTENCE OF SOLUTION IN WEIGHTED SOBOLEV SPACES FOR A STRONGLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS HAVING NATURAL GROWTH TERMS AND L^1 DATA

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In this paper we are interested in the existence of a solution for the nonlinear degenerate elliptic equations $Lu(x) + H(x,u,\nabla u)\,\omega_2 = f$ in the setting of the weighted Sobolev space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$, where H is a nonlinear term with natural growth with respect to ∇u and $f \in L^1(\Omega)$.

1. Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ for the nonlinear degenerate elliptic problem with Dirichlet boundary conditions

$$(P)\begin{cases} Lu(x) + H(x,u,\nabla u)\,\omega_2 = f \in L^1(\Omega), \\ H(x,u,\nabla u) \in L^1(\Omega,\omega_2), \\ u \in W_0^{1,p}(\Omega,\omega_1,\omega_2), \end{cases}$$

where L is the partial differential operator $Lu = -\operatorname{div}(\omega_1\,\mathcal{A}(x,u,\nabla u))$ and the function $H(x,u,\nabla u)$ is a non linear lower order term having natural growth (of order p) with respect to $|\nabla u|$ (with respect to $|u|$ we do not assume any growth

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restrictions, but we assume the sign-condition $H(x,\eta,\xi)\,\eta\geq 0$), Ω is a bounded open set in $\mathbb{R}^n$, ω_1 and ω_2 are two weight functions, $1<p<\infty$, $f\in L^1(\Omega)$ and the functions $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ and $H:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfy the following conditions:

(H1) $x\mapsto\mathcal{A}(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$,
$(\eta,\xi)\mapsto\mathcal{A}(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$;

(H2) $|\mathcal{A}(x,\eta,\xi)|\leq K(x)+h_1(x)\,|\eta|^{p-1}+h_2(x)\,|\xi|^{p-1}$, $K\in L^{p'}(\Omega,\omega_1)$ and $h_1,h_2\in L^\infty(\Omega)$ (with $1/p+1/p'=1$);

(H3) $[\mathcal{A}(x,\eta,\xi)-\mathcal{A}(x,\eta',\xi')].(\xi-\xi')\geq 0$, whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where $\mathcal{A}(x,\eta,\xi)=(\mathcal{A}_1(x,\eta,\xi),...,\mathcal{A}_n(x,\eta,\xi))$, a dot denote here the Euclidian scalar product in $\mathbb{R}^n$;

(H4) $\mathcal{A}(x,\eta,\xi).\xi\geq\alpha\,|\xi|^p$, where α is a positive constant;

(H5) $x\mapsto H(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$,
$(\eta,\xi)\mapsto H(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$;

(H6) $|H(x,\eta,\xi)|\leq b(\eta)\big(|\xi|^p\,\dfrac{\omega_1(x)}{\omega_2(x)}+h(x)\big)$, where $h\in L^1(\Omega,\omega_2)$, $h\geq 0$ and $0\leq b(\eta)\leq\beta$ for all $\eta\in\mathbb{R}$;

(H7) $H(x,\eta,\xi)\,\eta\geq 0$;

(H8) There exist $\sigma>0$ and $\gamma>0$ ($0<\gamma<\beta$) such that

$$|H(x,\eta,\xi)|\geq\gamma|\xi|^p\,\frac{\omega_1(x)}{\omega_2(x)}$$

if $|\eta|\geq\sigma$.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $0<\omega(x)<\infty$ for a.e. $x\in\mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu_i(E)=\int_E\omega_i(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$, $i=1,2$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [5], [9],[10], [13] and [17]).

In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. This bad behaviour can be caused by the coefficients of the corresponding differential operator as well as by the solution itself. The so-called p-Laplacian is a prototype of such an operator and its character can be interpreted as a degeneration or as a singularity of the classical (linear) Laplace operator (with $p=2$). There are several very concrete problems from practice which lead to such differential equations, e.g. from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction, reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [4], [8] and [18]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [15]). These classes have found many useful applications in harmonic analysis (see [16]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [14]). There are, in fact, many interesting examples of weights (see [13] for p-admissible weights).

Note that, in the proof of our main results, many ideas have been adapted from [1], [2], [3], [5], [6] and [7]. This problem is a generalization of [3] by L. Boccardo and T. Gallouet. In [3] the existence of a solutions has been proved in $W_0^{1,p}(\Omega)$ (non degenerate case, i.e., when $\omega_1 = \omega_2 = 1$).

The following theorem will be proved in section 3.

Theorem 1.1. *Assume (H1)-(H8). If* $\omega_1, \omega_2 \in A_p$ *(with* $1 < p < \infty$*) and* $\omega_2 \leq \omega_1$*, then there exist a solution* $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ *of problem (P).*

2. Definitions and Basic Results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \leq C_{p,\omega}$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [12], [13],[14] or [17] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x;2r)) \leq C\mu(B(x;r))$ for every ball $B = B(x;r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [13]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [16]).

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $1 \leq p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega) \subset L^1_{\rm loc}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [17]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be open and let ω_1 and ω_2 be A_p-weights ($1 < p < \infty$). We define the weighted Sobolev space $W^{1,p}(\Omega,\omega_1,\omega_2)$ as the set of functions $u \in L^p(\Omega,\omega_2)$ with weak derivatives $D_j u \in L^p(\Omega,\omega_1)$ for $j = 1,...,n$. The norm of u in $W^{1,p}(\Omega,\omega_1,\omega_2)$ is defined by

$$\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \left(\int_\Omega |u(x)|^p\, \omega_2(x)\, dx + \int_\Omega |\nabla u(x)|^p\, \omega_1(x)\, dx \right)^{1/p}. \qquad (1)$$

We also define $W^{1,p}_0(\Omega,\omega_1,\omega_2)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm (1).

Remark 2.3. (a) If $\omega \in A_p$, then $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega,\omega) = W^{1,p}(\Omega,\omega,\omega)$ (see Corollary 2.1.6 in [17]).
(b) If $\omega_2 \le \omega_1$ then $W^{1,p}_0(\Omega,\omega_1) \subset W^{1,p}_0(\Omega,\omega_1,\omega_2) \subset W^{1,p}_0(\Omega,\omega_2)$.

The spaces $W^{1,p}(\Omega,\omega_1,\omega_2)$ and $W^{1,p}_0(\Omega,\omega_1,\omega_2)$ are Banach spaces. The dual space of $W^{1,p}_0(\Omega,\omega_1,\omega_2)$ is the space $[W^{1,p}_0(\Omega,\omega_1,\omega_2)]^* = W^{-1,p'}(\Omega,\omega_1,\omega_2)$,

$$[W^{1,p}_0(\Omega,\omega_1,\omega_2)]^* = \Big\{ T = f_0 - \mathrm{div}(F),\, F = (f_1,..,f_n) : \\ \frac{f_0}{\omega_2} \in L^{p'}(\Omega,\omega_2),\, \frac{f_j}{\omega_1} \in L^{p'}(\Omega,\omega_1),\, j =, 1,..,n \Big\}.$$

If $T \in [W^{1,p}_0(\Omega,\omega_1,\omega_2)]^*$, and $\varphi \in W^{1,p}_0(\Omega,\omega_1,\omega_2)$ we denote

$$\langle T,\varphi\rangle = \int_\Omega f_0\, \varphi\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j\varphi\, dx,$$

$$\|T\|_* = \|f_0/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)},$$

$$|\langle T,\varphi\rangle| \le \|T\|_* \|\varphi\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.$$

In this article we use the following results.

Theorem 2.4. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega,\omega)$ such that*
(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$, μ-a.e. on Ω;
(ii) $|u_{m_k}(x)| \le \Phi(x)$, μ-a.e. on Ω;
(where $\mu(E) = \int_E \omega(x)\, dx$).

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [11]. □

Theorem 2.5. *(The weighted Sobolev inequality) Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ $(1 < p < \infty)$. There exist positive constants C_Ω and δ such that for all $u \in W_0^{1,p}(\Omega,\omega)$ and all θ satisfying $1 \leq \theta \leq n/(n-1)+\delta$,*

$$\|u\|_{L^{\theta p}(\Omega,\omega)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega,\omega)}. \tag{2}$$

Proof. Its suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [10]). To extend the estimates (2) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{\theta p}(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2). □

Lemma 2.6. *If $\omega \in A_p$, then $\left(\frac{|E|}{|B|}\right)^p \leq C_{p,\omega}\frac{\mu(E)}{\mu(B)}$, whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B.*

Proof. See Theorem 15.5 *Strong doubling of A_p-weights* in [13]. □

By Lemma 2.6, if $\mu(E) = 0$ then $|E| = 0$. Therefore, $\mu(E) = 0$ if and only if $|E| = 0$.

Lemma 2.7. *Let ω_1 and ω_2 be A_p-weights, $1 < p < \infty$, and a sequence $\{u_n\}$, $u_n \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ satisfies*

(i) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ and μ_1-a.e. in Ω;

(ii) $\int_\Omega \langle \mathcal{A}(x,u_n,\nabla u_n) - \mathcal{A}(x,u_n,\nabla u), \nabla(u_n - u)\rangle\, \omega_1\, dx \to 0$ with $n \to \infty$.

Then $u_n \to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Proof. The proof of this lemma follows the line of Lemma 5 in [2]. □

Definition 2.8. We say that $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is a solution of problem (P) if for any $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2) \cap L^\infty(\Omega)$ we have

$$\int_\Omega \mathcal{A}(x,u,\nabla u).\nabla\varphi\, \omega_1\, dx + \int_\Omega H(x,u,\nabla u)\, \varphi\, \omega_2\, dx = \int_\Omega f\, \varphi\, dx, \tag{3}$$

$$H(x,u,\nabla u) \in L^1(\Omega,\omega_2). \tag{4}$$

Definition 2.9. For a given constant $k>0$ we define the cut-function $T_k:\mathbb{R}\to\mathbb{R}$ as

$$T_k(s)=\begin{cases} s \text{ if } |s|\le k, \\ k\,\text{sign}(s) \text{ if } |s|>k. \end{cases}$$

Remark 2.10. (i) Note that for given $h>0$ and $k>0$ we have

$$T_h(u-T_k(u))=\begin{cases} 0 \text{ if } |u|\le k, \\ (|u|-k)\,\text{sign}(u) \text{ if } k<|u|\le k+h, \\ h\,\text{sign}(u) \text{ if } |u|>k+h. \end{cases}$$

And if $\alpha\in\mathbb{R}$, $\alpha\neq 0$, we have $T_k(\alpha u)=\alpha\,T_{k/|\alpha|}(u)$.
(ii) If $u\in W^{1,1}_{\text{loc}}(\Omega,\omega)$ then we have $\nabla T_k(u)=\chi_{\{|u|\le k\}}\nabla u$, where χ_E denotes the characteristic function of a measurable set $E\subset\mathbb{R}^n$.
(iii) If $u_1,u_2\in W^{1,p}_0(\Omega,\omega)$ then $\varphi=T_k(u_1+u_2)\in W^{1,p}_0(\Omega,\omega)\cap L^\infty(\Omega)$, and we have $\nabla\varphi=\nabla T_k(u_1+u_2)=\nabla(u_1+u_2)\,\chi_{\{|u_1+u_2|\le k\}}$.

3. Proof of Theorem 1.1

To prove Theorem 1.1. we will apply the same technique as introduced in [3] (which is the non degenerate case when $\omega_1=\omega_2\equiv 1$).
Step 1. If $f\in L^{p'}(\Omega,\omega_2)$ then the problem (P) has a solution (see [6]). Considering a sequence of functions $\{f_\varepsilon\}$, $f_\varepsilon\in C_0^\infty(\Omega)$ ($f_\varepsilon\in L^{p'}(\Omega,\omega_2)$), $\varepsilon>0$, which $f_\varepsilon\to f$ in $L^1(\Omega)$ and $\|f_\varepsilon\|_{L^1(\Omega)}\le\|f\|_{L^1(\Omega)}$, there exists a unique solution $u_\varepsilon\in W^{1,p}_0(\Omega,\omega_1,\omega_2)$ to the Dirichlet problem

$$(P_\varepsilon)\begin{cases} L(u_\varepsilon)+H(x,u_\varepsilon,\nabla u_\varepsilon)\,\omega_2=f_\varepsilon \text{ in } \Omega, \\ u_\varepsilon\in W^{1,p}_0(\Omega,\omega_1,\omega_2), \\ H(x,u_\varepsilon,\nabla u_\varepsilon)\in L^1(\Omega,\omega_2). \end{cases}$$

Hence,

$$\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla\varphi\,\omega_1\,dx+\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,\varphi\,\omega_2\,dx=\int_\Omega f_\varepsilon\,\varphi\,dx, \qquad (5)$$

for all $\varphi\in W^{1,p}_0(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$. In particular, for $\varphi=T_k(u_\varepsilon)$ we have

$$\begin{aligned}\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla T_k(u_\varepsilon)\,\omega_1\,dx &+ \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,T_k(u_\varepsilon)\,\omega_2\,dx \\ &= \int_\Omega f_\varepsilon\,T_k(u_\varepsilon)\,dx. \qquad (6)\end{aligned}$$

Since $\nabla T_k(u_\varepsilon) = \nabla u_\varepsilon \, \chi_{\{|u_\varepsilon| \le k\}}$ and by (H4) we have

$$\begin{aligned} \int_\Omega \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla T_k(u_\varepsilon)\, \omega_1\, dx &= \int_\Omega \mathcal{A}(x, u_\varepsilon, \nabla T_k(u_\varepsilon)).\nabla T_k(u_\varepsilon)\, \omega_1\, dx \\ &\ge \alpha \int_\Omega |\nabla T_k(u_\varepsilon)|^p\, \omega_1\, dx, \end{aligned}$$

and using (H7) we have that

$$\begin{aligned} H(x, u_\varepsilon, \nabla u_\varepsilon)\, T_k(u_\varepsilon) &= \begin{cases} H(x, u_\varepsilon, \nabla u_\varepsilon)\, u_\varepsilon & \text{if } |u_\varepsilon| \le k, \\ k \operatorname{sign}(u_\varepsilon)\, H(x, u_\varepsilon, \nabla u_\varepsilon) & \text{if } |u_\varepsilon| > k, \end{cases} \\ &\ge 0. \end{aligned}$$

Hence, in (6) we obtain

$$\begin{aligned} \alpha \int_\Omega |\nabla T_k(u_\varepsilon)|^p\, \omega_1\, dx &\le \int_\Omega f_\varepsilon\, T_k(u_\varepsilon)\, dx \le \int_\Omega |f_\varepsilon|\, |T_k(u_\varepsilon)|\, dx \\ &\le k \int_\Omega |f_\varepsilon|\, dx = k \|f_\varepsilon\|_{L^1(\Omega)}. \end{aligned} \tag{7}$$

Step 2. We will prove that, for any $t > 0$,

$$\int_{\{|u_\varepsilon|>t\}} |H(x, u_\varepsilon, \nabla u_\varepsilon)|\, \omega_2\, dx \le \int_{\{|u_\varepsilon|>t\}} |f_\varepsilon|\, dx.$$

We follow a technique of [3]. Let $\{\psi_i\}$ be a sequence of real smooth increasing bounded functions with $\psi_i' \in L^\infty(\mathbb{R})$ and $\psi_i(0) = 0$. We have $\nabla(\psi_i(u_\varepsilon)) = \psi_i'(u_\varepsilon)\, \nabla u_\varepsilon$. Then $\psi_i(u_\varepsilon) \in W_0^{1,p}(\Omega, \omega_1, \omega_2) \cap L^\infty(\Omega)$, and using $\psi_i(u_\varepsilon)$ as test function in (5) we obtain

$$\begin{aligned} \int_\Omega \mathcal{A}(x, u_\varepsilon \nabla u_\varepsilon).\nabla(\psi_i(u_\varepsilon))\, \omega_1\, dx &+ \int_\Omega H(x, u_\varepsilon, \nabla u_\varepsilon)\, \psi_i(u_\varepsilon)\, \omega_2\, dx \\ &= \int_\Omega f_\varepsilon\, \psi_i(u_\varepsilon)\, dx. \end{aligned}$$

By (H4) and $\nabla(\psi_i(u_\varepsilon)) = \psi_i'(u_\varepsilon)\, \nabla u_\varepsilon$ we have

$$\int_\Omega \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla(\psi_i(u_\varepsilon))\, \omega_1\, dx = \int_\Omega \Big(\mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla u_\varepsilon\Big)\, \psi_i'(u_\varepsilon) \omega_1\, dx \ge 0,$$

and we obtain

$$\int_\Omega H(x, u_\varepsilon, \nabla u_\varepsilon)\, \psi_i(u_\varepsilon)\, \omega_2\, dx \le \int_\Omega f_\varepsilon\, \psi_i(u_\varepsilon)\, dx. \tag{8}$$

Now, we can choose a sequence $\{\psi_i\}$ that converges to the function ψ,

$$\psi(s) = \begin{cases} 1 & \text{if } s \ge t, \\ 0 & \text{if } -t < s < t, \\ -1 & \text{if } s \le -t, \end{cases}$$

(where $t>0$). Hence, we obtain

$$\int_{\{|u_\varepsilon|>t\}} |H(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx \leq \int_{\{|u_\varepsilon|>t\}} |f_\varepsilon|\,dx. \tag{9}$$

Step 3. The sequence $\{u_\varepsilon\}$ is bounded in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$. In fact, by (H8) we obtain for $t\geq\sigma$

$$\begin{aligned}\int_{\{|u_\varepsilon|>t\}} |H(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx &\geq \gamma\int_{\{|u_\varepsilon|>t\}} |\nabla u_\varepsilon|^p\,\frac{\omega_1}{\omega_2}\,\omega_2\,dx\\ &= \gamma\int_{\{|u_\varepsilon|>t\}} |\nabla u_\varepsilon|^p\,\omega_1\,dx,\end{aligned}$$

and by (9) we obtain (for $t\geq\sigma$)

$$\gamma\int_{\{|u_\varepsilon|>t\}} |\nabla u_\varepsilon|^p\,\omega_1\,dx \leq \int_{\{|u_\varepsilon|>t\}} |f_\varepsilon|\,dx.$$

Hence, if $t\geq\sigma$, we have

$$\int_{\{|u_\varepsilon|>t\}} |\nabla u_\varepsilon|^p\,\omega_1\,dx \leq \frac{1}{\gamma}\int_{\{|u_\varepsilon|>t\}} |f_\varepsilon|\,dx. \tag{10}$$

Using (7) we have, for all $k>0$,

$$\alpha\int_{\{|u_\varepsilon|\leq k\}} |\nabla u_\varepsilon|^p\,\omega_1\,dx \leq k\int_\Omega |f_\varepsilon|\,dx, \tag{11}$$

and from (10) and (11) we obtain

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^p\,\omega_1\,dx &= \int_{\{|u_\varepsilon|\leq\sigma\}} |\nabla u_\varepsilon|^p\,\omega_1\,dx + \int_{\{|u_\varepsilon|>\sigma\}} |\nabla u_\varepsilon|^p\,\omega_1\,dx\\ &= \int_\Omega |\nabla T_\sigma(u_\varepsilon)|^p\,\omega_1\,dx + \int_{\{|u_\varepsilon|>\sigma\}} |\nabla u_\varepsilon|^p\,\omega_1\,dx\\ &\leq \frac{\sigma}{\alpha}\int_\Omega |f_\varepsilon|\,dx + \frac{1}{\gamma}\int_{\{|u_\varepsilon|>\sigma\}} |f_\varepsilon|\,dx\\ &\leq \left(\frac{\sigma}{\alpha}+\frac{1}{\gamma}\right)\|f_\varepsilon\|_{L^1(\Omega)}\\ &\leq \left(\frac{\sigma}{\alpha}+\frac{1}{\gamma}\right)\|f\|_{L^1(\Omega)}.\end{aligned}$$

By Theorem 2.5 (with $\theta=1$) and $\omega_2\leq\omega_1$, we have that the sequence $\{u_\varepsilon\}$ is bounded in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Therefore, there exits a subsequence (still denoted by $\{u_\varepsilon\}$) and a function $u\in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ such that

$$u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega,\omega_1,\omega_2), \tag{12}$$

$$u_\varepsilon \to u \text{ in } L^p(\Omega,\omega_i)\ (i=1,2), \tag{13}$$

$$u_\varepsilon \to u\ \mu_i - a.e. \tag{14}$$

Step 4. We will prove that $u_\varepsilon^+ \to u^+$ and $u_\varepsilon^- \to u^-$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

First, we will prove that $u_\varepsilon^+ \to u^+$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$. For this, we split the demonstration in two parts.

Part 1. Let $k > \sigma$, and considering the test function $\varphi = T_k((u_\varepsilon^+ - u^+)^+)$ in (5) we obtain

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla T_k((u_\varepsilon^+ - u^+)^+)\,\omega_1\,dx\\
&+\int_\Omega H(x,u_\varepsilon,\nabla u_\varepsilon)\,T_k((u_\varepsilon^+ - u^+)^+)\,\omega_2\,dx\\
&=\int_\Omega f_\varepsilon\,T_k((u_\varepsilon^+ - u^+)^+)\,\omega\,dx.
\end{aligned}\tag{15}$$

If $T_k((u_\varepsilon^+(x) - u^+(x))^+) > 0$ then $u_\varepsilon^+(x) > 0$ and we have $u_\varepsilon(x) > 0$. Hence, by (H7) we have $H(x,u_\varepsilon,\nabla u_\varepsilon) \geq 0$. Therefore, in (15) we obtain

$$\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla T_k((u_\varepsilon^+ - u^+)^+)\,\omega_1\,dx \leq \int_\Omega f_\varepsilon T_k((u_\varepsilon^+ - u^+)^+)\,dx. \tag{16}$$

Since $u_\varepsilon(x) = u_\varepsilon^+(x)$ on the set $\{x\in\Omega : u_\varepsilon^+(x) - u^+(x) > 0\}$ and $\nabla T_k((u_\varepsilon^+ - u^+)^+) = 0$ on the set $\{x\in\Omega : u_\varepsilon^+(x) - u^+(x) \leq 0\}$, we can also write (16) as

$$\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+).\nabla T_k((u_\varepsilon^+ - u^+)^+)\,\omega_1\,dx \leq \int_\Omega f_\varepsilon\,T_k((u_\varepsilon^+ - u^+)^+)\,dx, \tag{17}$$

which implies

$$\lim_{\varepsilon\to 0}\int_\Omega \big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+)\big).\nabla T_k((u_\varepsilon^+ - u^+)^+)\,\omega_1\,dx = 0. \tag{18}$$

Using that $u_\varepsilon^+(x) = u_\varepsilon(x)$ where $(u_\varepsilon^+(x) - u^+(x))^+ > 0$, and also by condition

(H2), we obtain

$$\begin{aligned}
&\int_{\{u_\varepsilon^+-u^+>k\}} \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla((u_\varepsilon^+-u^+)^+)\,\omega_1\,dx\\
&\le \int_{\{u_\varepsilon>k\}} \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla(u_\varepsilon-u^+)\,\omega_1\,dx\\
&\le \int_{\{u_\varepsilon>k\}} \Big(|\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)|+|\mathcal{A}(x,u_\varepsilon,\nabla u^+)|\Big)\,|\nabla(u_\varepsilon-u^+)|\,\omega_1\,dx\\
&\le \int_{\{u_\varepsilon>k\}} \Big((K(x)+h_1|u_\varepsilon|^{p-1}+h_2|\nabla u_\varepsilon|^{p-1})\\
&\quad +(K(x)+h_1|u_\varepsilon|^{p-1}+h_2|\nabla u^+|^{p-1})\Big)(|\nabla u_\varepsilon|+|\nabla u^+|)\,\omega_1\,dx\\
&= \int_{\{u_\varepsilon>k\}} \Big(2K(x)+2h_1|u_\varepsilon|^{p-1}+h_2|\nabla u_\varepsilon|^{p-1}+h_2|\nabla u^+|^{p-1}\Big)\\
&\quad \times(|\nabla u_\varepsilon|+|\nabla u^+|)\,\omega_1\,dx\\
&= \int_{\{u_\varepsilon>k\}} \Big(2K(x)|\nabla u_\varepsilon|+2h_1|u_\varepsilon|^{p-1}|\nabla u_\varepsilon|+h_2|\nabla u_\varepsilon|^{p}+h_2|\nabla u^+|^{p-1}|\nabla u_\varepsilon|\Big)\,\omega_1\,dx\\
&+ \int_{\{u_\varepsilon>k\}} \Big(2K(x)|\nabla u^+|+2h_1|u_\varepsilon|^{p-1}|\nabla u^+|+h_2|\nabla u_\varepsilon|^{p-1}|\nabla u^+|+h_2|\nabla u^+|^{p}\Big)\,\omega_1\,dx\\
&= I,
\end{aligned}$$

hence

$$
\begin{aligned}
I \leq\, & 2\left(\int_{\{u_\varepsilon>k\}} K^{p'}\omega_1\,dx\right)^{1/p'}\left(\int_{\{u_\varepsilon>k\}}|\nabla u_\varepsilon|^p\omega_1\,dx\right)^{1/p}\\
&+2\|h_1\|_{L^\infty(\Omega)}\left(\int_{\{u_\varepsilon>k\}}|u_\varepsilon|^p\omega_1\,dx\right)^{1/p'}\left(\int_{\{u_\varepsilon>k\}}|\nabla u_\varepsilon|^p\omega_1\,dx\right)^{1/p}\\
&+\|h_2\|_{L^\infty(\Omega)}\int_{\{u_\varepsilon>k\}}|\nabla u_\varepsilon|^p\omega_1\,dx\\
&+\|h_2\|_{L^\infty(\Omega)}\left(\int_{\{u_\varepsilon>k\}}|\nabla u^+|^p\omega_1\,dx\right)^{1/p'}\left(\int_{\{u_\varepsilon>k\}}|\nabla u_\varepsilon|^p\omega_1\,dx\right)^{1/p}\\
&+2\left(\int_{\{u_\varepsilon>k\}} K^{p'}\omega_1\,dx\right)^{1/p'}\left(\int_{\{u_\varepsilon>k\}}|\nabla u^+|^p\omega_1\,dx\right)^{1/p}\\
&+2\|h_1\|_{L^\infty(\Omega)}\left(\int_{\{u_\varepsilon>k\}}|u_\varepsilon|^p\omega_1\,dx\right)^{1/p'}\left(\int_{\{u_\varepsilon>k\}}|\nabla u^+|^p\omega_1\,dx\right)^{1/p}\\
&+\|h_2\|_{L^\infty(\Omega)}\left(\int_{\{u_\varepsilon>k\}}|\nabla u_\varepsilon|^p\omega_1\,dx\right)^{1/p'}\left(\int_{\{u_\varepsilon>k\}}|\nabla u^+|^p\omega_1\,dx\right)^{1/p}\\
&+\|h_2\|_{L^\infty(\Omega)}\int_{\{u_\varepsilon>k\}}|\nabla u^+|^p\omega_1\,dx = II, \qquad (19)
\end{aligned}
$$

and by Young's inequality and (10), we obtain

$$
\begin{aligned}
II \leq\, & C_1\left(\int_{\{u_\varepsilon>k\}}|\nabla u_\varepsilon|^p\omega_1\,dx+\int_{\{u_\varepsilon>k\}}|u_\varepsilon|^p\omega_1\,dx+\int_{\{u_\varepsilon>k\}}|\nabla u^+|^p\omega_1\,dx\right.\\
&\left.+\int_{\{u_\varepsilon>k\}}K^{p'}\omega_1\,dx\right)\\
\leq\, & C_1\left(\frac{1}{\gamma}\int_{\{u_\varepsilon>k\}}|f_\varepsilon|\,dx+\int_{\{u_\varepsilon>k\}}|u_\varepsilon|^p\omega_1\,dx+\int_{\{u_\varepsilon>k\}}|\nabla u^+|^p\omega_1\,dx\right.\\
&\left.+\int_{\{u_\varepsilon>k\}}K^{p'}\omega_1\,dx\right)=R_\varepsilon(k), \qquad (20)
\end{aligned}
$$

where C_1 is a positive constant which depends only of $\|h_1\|_{L^\infty(\Omega)}$, $\|h_2\|_{L^\infty(\Omega)}$ and

p. We have $\lim_{k\to\infty} R_\varepsilon(k) = 0$ (uniformly with respect to ε) and from (18) we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla((u_\varepsilon^+ - u^+)^+)\,\omega_1\,dx \\
&= \lim_{\varepsilon\to 0}\Bigg[\int_{\{0\le (u_\varepsilon^+-u^+)^+\le k\}} \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla((u_\varepsilon^+ - u^+)^+)\,\omega_1\,dx \\
&+ \int_{\{(u_\varepsilon^+-u^+)^+ > k\}} \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla((u_\varepsilon^+ - u^+)^+)\,\omega_1\,dx\Bigg] \\
&= \lim_{\varepsilon\to 0}\Bigg[\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla T_k((u_\varepsilon^+ - u^+)^+)\,\omega_1\,dx \\
&+ \int_{\{(u_\varepsilon^+-u^+)^+ > k\}} \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla((u_\varepsilon^+ - u^+)^+)\,\omega_1\,dx\Bigg] \\
&= 0.
\end{aligned} \tag{21}$$

Part 2. Now, let us consider the function $g_\varepsilon^- = (u_\varepsilon^+ - T_k(u^+))^-$. We have $0\le g_\varepsilon^- \le k$, hence $g_\varepsilon^- \in L^\infty(\Omega)$.

We define the function $v_\varepsilon = \varphi_\lambda(g_\varepsilon^-)$, where $\varphi_\lambda(s) = s\,e^{\lambda s^2}$, and $\lambda = \beta^2/4\alpha^2 \in \mathbb{R}$ (where α is the constant in (H4) and β is the constant in (H6)).

If $v_\varepsilon(x)\neq 0$ then $0\le u_\varepsilon^+(x)\le k$. Hence, $v_\varepsilon \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$ and using as test function v_ε in (5), we obtain

$$\begin{aligned}
\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla g_\varepsilon^-\,\varphi_\lambda'(g_\varepsilon^-)\,\omega_1\,dx \quad &+ \quad \int_\Omega H(x,u_\varepsilon,\nabla u_\varepsilon)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_2\,dx \\
&= \quad \int_\Omega f_\varepsilon\,\varphi_\lambda(g_\varepsilon^-)\,dx.
\end{aligned} \tag{22}$$

Now we can follow the proof of [1],[3] and [6], because the left-hand side of (22) is exactly the left-hand side of (3.10) of [6].

Since $\varphi_\lambda(g_\varepsilon^-)\neq 0$ where $0\le u_\varepsilon^+(x)\le k$, we have $\varphi_\lambda(g_\varepsilon^-)\in L^\infty(\Omega)$, and then

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon\,\varphi_\lambda(g_\varepsilon^-)\,dx = \int_\Omega f\,\varphi_\lambda((u^+ - T_k(u^+))^-)\,dx = 0.$$

Now, analogously to (3.18) of [6] we obtain (for k fixed)

$$\overline{\lim_{\varepsilon\to 0}}\int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla T_k(u^+))\Big).\nabla((u_\varepsilon^+ - T_k(u^+))^-)\,\omega_1\,dx \le 0. \tag{23}$$

We can write

$$\begin{aligned}
&\int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla((u_\varepsilon^+-u^+)^-)\,\omega_1\,dx\\
&=\int_{\{T_k(u^+)<u_\varepsilon^+\le u^+\}}\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla(u_\varepsilon^+-u^+)\,\omega_1\,dx\\
&+\int_{\{u_\varepsilon^+\le T_k(u^+)\}}\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla(u_\varepsilon^+-u^+)\,\omega_1\,dx=III,
\end{aligned}$$

and using that

$$\begin{aligned}
&\{x\in\Omega: T_k(u^+)<u_\varepsilon^+\le u^+\}\\
&=\{x\in\Omega: k<u_\varepsilon^+\le u^+\}\cup\{x\in\Omega: T_k(u^+)<u_\varepsilon^+\le u^+,\ \text{with } u_\varepsilon^+\le k\}
\end{aligned}$$

and also that $\{x\in\Omega: T_k(u^+)<u_\varepsilon^+\le u^+,\ \text{with } u_\varepsilon^+\le k\}=\emptyset$, we obtain

$$\begin{aligned}
III &= \int_{\{k<u_\varepsilon^+=u_\varepsilon\le u^+\}}\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla(u_\varepsilon^+-u^+)\,\omega_1\,dx\\
&+ \int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla T_k(u^+))\Big).\nabla(u_\varepsilon^+-T_k(u^+))\,\omega_1\,dx\\
&+ \int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,\nabla T_k(u^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla((u_\varepsilon^+-T_k(u^+))^-)\,\omega_1\,dx\\
&+ \int_{\{u_\varepsilon^+\le T_k(u^+)\}}\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla(T_k(u^+)-u^+)\,\omega_1\,dx\\
&= III_1+III_2+III_3+III_4. \qquad (24)
\end{aligned}$$

We have

(i) For III_1,

$$III_1=\int_{\{k<u_\varepsilon^+=u_\varepsilon\le u^+\}}\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla(u_\varepsilon^+-u^+)\,\omega_1\,dx\to 0$$

for $k\to\infty$, uniformly with respect to ε, analogously to (20),

(ii) For III_2,

$$III_2=\int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla T_k(u^+))\Big).\nabla(u_\varepsilon^+-T_k(u^+))^-\,\omega_1\,dx$$

we have the limit (23),

(iii) For III_3,

$$III_3=\int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,\nabla T_k(u^+))-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla((u_\varepsilon^+-T_k(u^+))^-)\omega_1 dx\to 0,$$

for k fixed and $\varepsilon\to 0$,

(iv) For III_4 we have

$$
\begin{aligned}
&|III_4| \\
&= \left| \int_{\{u_\varepsilon^+ \le T_k(u^+)\}} \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+) \Big).\nabla(T_k(u^+) - u^+)\,\omega_1\,dx \right| \\
&\le \left(\int_\Omega |\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+)|^{p'}\,\omega_1\,dx \right)^{1/p'} \\
&\times \left(\int_\Omega |\nabla(T_k(u^+) - u^+)|^p\,\omega_1\,dx \right)^{1/p} \to 0 \text{ for } k\to\infty.
\end{aligned} \tag{25}
$$

Therefore, by (24), (i), (ii), (iii) and (iv) we obtain

$$
\lim_{\varepsilon\to 0} \int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+) \Big).\nabla((u_\varepsilon^+ - u^+)^-)\,\omega_1\,dx = 0. \tag{26}
$$

Hence, from (21) and (26) we obtain

$$
\lim_{\varepsilon\to 0} \int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u^+) \Big).\nabla(u_\varepsilon^+ - u^+)\,\omega_1\,dx = 0.
$$

Therefore, by Lemma 2.7, we have

$$
u_\varepsilon^+ \to u^+ \text{ in } W_0^{1,p}(\Omega,\omega_1,\omega_2). \tag{27}
$$

Step 5. We will prove that $u_\varepsilon^- \to u^-$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Using the test function $\varphi = T_k((u_\varepsilon^- - u^-)^+)$ and $\tilde{v_\varepsilon} = \varphi_\lambda((u_\varepsilon^- - T_k(u^-))^-)$ we obtain, analogously to Step 4, that

$$
u_\varepsilon^- \to u^- \text{ in } W_0^{1,p}(\Omega,\omega_1,\omega_2). \tag{28}
$$

Step 6. By (27) and (28) there exists a subsequence (which will be denoted by $\{u_\varepsilon\}$) such that

$$
\nabla u_\varepsilon \to \nabla u \text{ in } L^p(\Omega,\omega_1), \tag{29}
$$

$$
\nabla u_\varepsilon \to \nabla u\ \mu_1 - a.e. \text{ in } \Omega, \tag{30}
$$

(and, by Lemma 2.6, $\nabla u_\varepsilon \to \nabla u$ a.e. in Ω). Using (H5), $H(x,\eta,\xi)$ is continuous in (η,ξ), we have

$$
H(x,u_\varepsilon(x),\nabla u_\varepsilon(x)) \to H(x,u(x),\nabla u(x)\ a.e., \tag{31}
$$

and $H(x,u_\varepsilon(x),\nabla u_\varepsilon(x)) \to H(x,u(x),\nabla u(x)\ \mu_2 - a.e.$.

Step 7. We will prove that $H(x,u_\varepsilon,\nabla u_\varepsilon) \to H(x,u,\nabla u)$ in $L^1(\Omega,\omega_2)$. For $m > 0$ we define

$$
X_m^\varepsilon = \{x\in\Omega : |u_\varepsilon(x)| \le m\},
$$

$$
Y_m^\varepsilon = \{x\in\Omega : |u_\varepsilon(x)| > m\}.
$$

For any measurable subset $E \subset \Omega$ we have

$$\int_E |H(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx = \int_{E\cap X_m^\varepsilon} |H(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx + \int_{E\cap Y_m^\varepsilon} |H(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx.$$

Using (H6) and (9), we obtain

$$\begin{aligned}
&\int_E |H(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx \\
&\leq \beta \int_E \left(|\nabla u_\varepsilon|^p \frac{\omega_1}{\omega_2} + h(x) \right) \omega_2\,dx + \int_{\{|u_\varepsilon|>m\}} |f_\varepsilon|\,dx \\
&= \beta \left(\int_E |\nabla u_\varepsilon|^p\,\omega_1\,dx + \int_E h(x)\,\omega_2\,dx \right) + \int_{\{|u_\varepsilon|>m\}} |f_\varepsilon|\,dx.
\end{aligned}$$

Hence, for $m \to \infty$, we have

$$\int_E |H(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx \leq \beta \left(\int_E |\nabla u_\varepsilon|^p\,\omega_1\,dx + \int_E h(x)\,\omega_2\,dx \right),$$

and since $|\nabla u_\varepsilon| \to |\nabla u|$ in $L^p(\Omega,\omega_1)$, we obtain

$$\lim_{\varepsilon\to 0} \int_E |H(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx \leq \beta \left(\int_E |\nabla u|^p\,\omega_1\,dx + \int_E h(x)\,\omega_2\,dx \right) < \infty.$$

Now, by Vitali's Theorem we have $\lim_{\mu_2(E)\to 0} \int_E |H(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx = 0$, uniformly in ε. Hence,

$$H(x,u_\varepsilon,\nabla u_\varepsilon) \to H(x,u,\nabla u) \text{ in } L^1(\Omega,\omega_2). \tag{32}$$

Step 8. In (5) we have

$$\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla\varphi\,\omega_1\,dx + \int_\Omega H(x,u_\varepsilon,\nabla u_\varepsilon)\,\varphi\,\omega_2\,dx = \int_\Omega f_\varepsilon\,\varphi\,dx,$$

for any $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$. Since $u_\varepsilon \to u$ in $L^p(\Omega,\omega_2)$, $\nabla u_\varepsilon \to \nabla u$ in $L^p(\Omega,\omega_1)$, $f_\varepsilon \to f$ in $L^1(\Omega)$ and (32), for $\varepsilon \to 0$ we obtain

$$\int_\Omega \mathcal{A}(x,u,\nabla u).\nabla\varphi\,\omega_1\,dx + \int_\Omega H(x,u,\nabla u)\,\varphi\,\omega_2\,dx = \int_\Omega f\,\varphi\,dx,$$

for any $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$.

Therefore, $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$ is a solution to problem (P).

Example 3.1. Let $\Omega = \{(x,y)\in\mathbb{R}^2 \ : x^2+y^2<1\}$, and consider the weights $\omega_1(x,y)=(x^2+y^2)^{-1/2}$ and $\omega_2(x,y)=(x^2+y^2)^{1/2}$ ($\omega_1,\omega_2\in A_2$ and $p=2$), the functions $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$ and $H:\Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ defined by

$$\mathcal{A}((x,y),\eta,\xi)=h_2(x,y)\,\xi,$$
$$H((x,y),\eta,\xi)=\left(|\xi|^2\,\frac{(\sin^2(xy)+1)}{(x^2+y^2)}+h(x,y)\right)\arctan(\eta),$$

where $h_2(x,y)=\mathrm{e}^{x^2+y^2}$, $h(x,y)=(x^2+y^2)^{1/2}\cos^2(xy)$, $b(\eta)=|\arctan(\eta)|$, $\alpha=1$, $\beta=\dfrac{\pi}{2}$, $\sigma=1$ and $\gamma=\dfrac{\pi}{8}$. Let us consider the partial differential operator

$$Lu(x,y)=-\mathrm{div}\Big[\omega(x,y)\mathcal{A}((x,y),u,\nabla u)\Big]$$

and $f(x,y)=(x^2+y^2)^{-1/4}\cos(1/(x^2+y^2))\in L^1(\Omega)$. Therefore, by Theorem 1.1, the problem

$$(P)\begin{cases} Lu(x,y)+H(x,u,\nabla u)\,\omega_2=f \text{ in } \Omega,\\ H(x,u,\nabla u)\in L^1(\Omega,\omega_2),\\ u\in W_0^{1,p}(\Omega,\omega_1,\omega_2),\end{cases}$$

has a solution.

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