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ON THE EXISTENCE OF MILD SOLUTIONS FOR NONLOCAL IMPULSIVE INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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In this paper, we prove the existence of mild solutions for a class of nonlinear impulsive integrodifferential equations with a nonlocal initial conditions. Sufficient conditions for the existence are derived with the help of the resolvent operator. In the end, an example is given to show the application of our result.

1. Introduction

Integrodifferential equations have attracted much attention because of their applications in many areas : physics, population dynamics, electrical engineering, finance, biology, ecology, sociology and other areas of science and engineering. Qualitative properties such as existence, uniqueness, controllability and stability for various integrodifferential equations have been extensively studied by many researchers, see for instance [2, 9–13].

As a practical application, we note that the following equation

$$\frac{d}{dt}\left[\rho(t)-\lambda\int_{-\infty}^{t}C(t-s)\rho(s)ds\right]=A\rho(t)+\lambda\int_{-\infty}^{t}B(t-s)\rho(s)ds-p(t)+q(t)$$

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arises in the study of the dynamics of income, employment, value of capital stock, and cumulative balance of payment ; see [6] for details. In the above system, λ is a real number, the state $\rho(t) \in \mathbb{R}^n, C(\cdot), B(\cdot)$ are $n \times n$ continuous functions matrices, A is a constant $n \times n$ matrix, $p(\cdot)$ represents the government intervention, and $q(\cdot)$ the private initiative.
Abstract integrodifferential equations also appear in the theory of heat conduction. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly on the temperature v and on its gradient $\nabla\rho$. Under these conditions, the classic heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [18, 24], the internal energy and the heat flux are described as functionals of v and v_x. The next system, see for instance [5, 7, 8, 22], has been frequently used to describe this phenomenon,

$$\begin{cases} \dfrac{d}{dt}\left[\rho(t,x) + \displaystyle\int_{-\infty}^{t} k_1(t-s)\rho(s,x)ds\right] \\ \qquad\qquad = c\displaystyle\sum_{i=1}^{n} \frac{\partial^2\rho(t,x)}{\partial x_i^2} + \int_{-\infty}^{t} k_2(t-s)\sum_{i=1}^{n}\frac{\partial^2\rho(t,x)}{\partial x_i^2}ds, \\ \rho(t,x) = 0, x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is open, bounded, and with smooth boundary ; $(t,x) \in [0,\infty[\times\Omega$; $\rho(t,x)$ represents the temperature in x at the time t ; c is a physical constant $k_i : \mathbb{R} \to \mathbb{R}, i = 1,2$, are the internal energy and the heat flux relaxation, respectively.

The aim of this work is to study the existence and uniqueness of mild and solutions for the following nonlinear impulsive integrodifferential equation with nonlocal conditions

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t \Upsilon(t-s)x(s)ds + f(t,x(t)), 0 \le t \le T, t \ne t_i, \\ x(0) + g(x) = x_0, \\ x(t_i^+) - x(t_i^-) = \mathrm{I}_i(x(t_i)), i = 1,2,\cdots,p, 0 < t_1 < t_2 < \cdots < t_p < T, \end{cases} \tag{1}$$

where A generates a C_0-semigroup on a Banach space X, $\Upsilon(t)$ is a closed linear operator on X with time independent domain $\mathcal{D}(A) \subset \mathcal{D}(\Upsilon)$. $f : [0,T] \times \mathrm{X} \to \mathrm{X}$ and $g : \mathtt{PC}([0,T],\mathrm{X}) \to \mathrm{X}$ are continuous functions where the set $\mathtt{PC}([0,T],\mathrm{X})$ is given later in Section 3.

The nonlocal condition has a better effect on the solution and is more precise for physical measurements than the classical condition $x(0) = x_0$ alone.The nonlocal condition has advantages over traditional initial value problems because it can be used to model phenomena that cannot be modeled by traditional initial value problems, such as the dynamics of populations subject to abrupt changes (harvesting, diseases, etc.). See [1, 3, 4, 17, 25, 26] and the references therein for more comments and citations.

This manuscript is particularly inspired by following works [20, 21] in which the authors investigated deeply the nonlocal problems for integrodifferential equations and the nonlocal impulsive problems for nonlinear differential equations in Banach spaces, respectively. Indeed, the arguments used in this paper generalize the work in [20] to a larger class of impulsive system. Using the compactness of the resolvent operator $(R(t))_{t\geq 0}$ (find below), we found an intermediate result needed to establish the existence of mild solutions, under various conditions on the given data, of a class of integrodifferential equation of Volterra type.

The rest of the manuscript is organized as follows. In Section 2, we give some necessary preliminaries. We study the existence of the mild solutions of equation (1) in Section 3. Finally in Section 4, an example is provided which illustrates our results.

2. Integrodifferential equations

Let X and Y be Banach spaces. $\mathcal{L}(\mathsf{X},\mathsf{Y})$ denotes the space of bounded linear operator from X to Y, simply $\mathcal{L}(\mathsf{X})$ when $\mathsf{X}=\mathsf{Y}$. For the question of existence of mild solution of the integrodifferential equations, we recall some fundamental results needed. Regarding the theory of resolvent operators, we refer the reader to [14]. Let Y be the Banach space $\mathcal{D}(A)$ equipped with the graph norm given by $|y|_Y := |Ay|+|y|$ for $y\in Y$. The notation $\mathcal{C}(\mathbb{R}^+;Y)$ stands for the space of all continuous functions from $\mathbb{R}^+$ into Y. We consider the following Cauchy problem :

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t \Upsilon(t-s)v(s)ds \quad \text{for } t\geq 0 \\ v(0) = v_0 \in \mathsf{X}. \end{cases} \tag{2}$$

Definition 2.1. [14] A resolvent operator for equation (2) is a bounded linear operator-valued function $R(t)\in\mathcal{L}(\mathsf{X})$ for $t\geq 0$,verifying the following conditions :

(i) $R(0)=\mathsf{I}$(the identity map of X) and $\|R(t)\|\leq Ne^{\eta t}$ for some constants $N>0$ and $\eta\in\mathbb{R}$.

(ii) For each $x\in\mathsf{X}$, $R(t)x$ is strongly continuous for $t\geq 0$.

(iii) $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y, R(.)x \in \mathcal{C}^1(\mathbb{R}^+;\mathsf{X})\bigcap\mathcal{C}(\mathbb{R}^+;Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t \Upsilon(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)\Upsilon(s)xds \quad for\, t \geq 0. \end{aligned}$$

From now on, we assume that

($\mathbf{R1}$) The operator A is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on X.

($\mathbf{R2}$) For all $t \geq 0$, $\Upsilon(t)$ is closed linear operator from $\mathcal{D}(A)$ to X and $\Upsilon(t) \in \mathcal{L}(Y,\mathsf{X})$. For any $y \in Y$, the map $t \mapsto \Upsilon(t)y$ is bounded, differentiable and the derivative $t \mapsto \Upsilon'(t)y$ is bounded uniformly continuous on $\mathbb{R}^+$.

The following theorem gives a satisfactory answer to the problem of existence of solutions.

Theorem 2.2. *[14] Assume that* $(\mathbf{R1})-(\mathbf{R2})$ *hold. Then there exists a unique resolvent operator for the Cauchy problem* (2)*.*

The resolvent operator gives some results for the existence of solutions for the following integrodifferential equation:

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t \Upsilon(t-s)v(s)ds + \mu(t) \quad \text{for } t \geq 0 \\ v(0) = v_0 \in \mathsf{X}, \end{cases} \tag{3}$$

where $\mu : \mathbb{R}^+ \to \mathsf{X}$ is a continuous function.

Definition 2.3. [14] A continuous function $v : \mathbb{R}^+ \to \mathsf{X}$ is said to be a strict solution of equation (3) if $v \in \mathcal{C}^1(\mathbb{R}^+;\mathsf{X})\bigcap\mathcal{C}(\mathbb{R}^+;Y)$ and v satisfies equation (3).

Theorem 2.4. *[14] Assume that* $(\mathbf{R1})-(\mathbf{R2})$ *hold. If v is a strict solution of equation* (3)*, then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)\mu(s)ds \quad for\, t \geq 0.$$

Lemma 2.5. *[15] Assume that* $(\mathbf{R1})$-$(\mathbf{R2})$ *hold.The resolvent operator* $(R(t))_{t\geq 0}$ *is compact for* $t > 0$ *if and only if the semigroup* $(S(t))_{t\geq 0}$ *is compact for* $t > 0$.

Lemma 2.6. *[19] Assume that* $(\mathbf{R1})-(\mathbf{R2})$ *hold. If the resolvent operator* $(R(t))_{t\geq 0}$ *is compact for* $t>0$ *then it is norm continuous (or continuous in the uniform operator topology) for* $t>0$.

Lemma 2.7. *[19] For any* $T>0$ *there exists a positive constant* $\gamma=\gamma(T)$ *such that*

$$\|R(t+h)-R(h)R(t)\|_{\mathcal{L}(\mathsf{X})}\leq \gamma h \quad \text{for } 0\leq h\leq t\leq T.$$

The resolvent operator plays an important role to study the existence of solutions and gives a variation of constant formula for semilinear systems. For more details on resolvent operators, we refer to [14, 16].

3. Existence of mild solutions of equation (1)

Let $\mathtt{PC}\equiv\mathtt{PC}([0,T],\mathsf{X})$ be the set of all function x from $[0,T]$ into X such that $x(t)$ is continuous at $t\neq t_i$ and left continuous at $t=t_i$ and the right limit $x(t_i^+)$ exists for $i=1,2,\cdots,p$. We recall from [20] that $\mathtt{PC}([0,T],\mathsf{X})$ is a Banach space with the following norm

$$\|x\|_{\mathtt{PC}}=\sup_{t\in[0,T]}\|x(t)\|_{\mathsf{X}}.$$

Accordingly, we make the following definition.

Definition 3.1. A function $x\in\mathtt{PC}([0,T],\mathsf{X})$ is a mild solution of equation (1) if it satisfies :

$$\begin{aligned} x(t) &= R(t)[x_0-g(x)]+\int_0^t R(t-s)f(s,x(s))ds \\ &+ \sum_{0<t_i<t} R(t-t_i)\mathsf{I}_i(x(t_i)),0\leq t\leq T. \end{aligned} \tag{4}$$

Remark 3.2. The integral term in (4) is not always well defined. It is if f sent bounded set into bounded once .

The mild solutions of (1) will be established under various conditions on the functions f, g, the functions I_i, and the resolvent operator $(R(t))_{t\geq 0}$.

3.1. Lipschitz nonlocal function

In this subsection, we prove the existence and uniqueness of mild solution of (1) by means of the fixed point theory. To develop this result, we make the following assumptions

$(\mathbf{L1})$ The functions I_i, f and g are continuous. There exist constants $L_f > 0, L_G > 0, L_i > 0, i = 1, 2, \cdots, p$, such that :

$$\begin{aligned} \|f(t,x) - f(t,y)\|_{\mathsf{X}} &\leq L_f \|x-y\|_{\mathsf{X}}, \quad t \in [0,T], x, y \in \mathsf{X} \\ \|g(u) - g(v)\|\|_{\mathsf{X}} &\leq L_G \|u-v\|_{\mathtt{PC}}, \quad u, v \in \mathtt{PC} \\ \|\mathsf{I}_i(x) - \mathsf{I}_i(y)\|\|_{\mathsf{X}} &\leq L_i \|x-y\|_{\mathsf{X}}, \quad x, y \in \mathsf{X} \end{aligned}$$

$(\mathbf{L2})$ Moreover, we assume that

$$M_T \left(L_G + L_f T + \sum_{i=1}^{p} L_i \right) < 1,$$

where $M_T = \sup_{t \in [0,T]} \|R(t)\|_{\mathcal{L}(\mathsf{X})}$.

Theorem 3.3. *Assume that* $(\mathbf{R1})$, $(\mathbf{R2})$, $(\mathbf{L1})$ *and* $(\mathbf{L2})$ *hold. Then for every* $x_0 \in \mathsf{X}$, *equation* (1) *has a unique mild solution on* $[0,T]$.

Proof. Let $x_0 \in \mathsf{X}$ be fixed. Define the operator Q on $\mathtt{PC}([0,T],\mathsf{X})$ by

$$\begin{aligned} (Qv)(t) &= R(t)[x_0 - g(v)] + \int_0^t R(t-s) f(s, v(s)) ds \\ &+ \sum_{0<t_i<t} R(t-t_i) \mathsf{I}_i(v(t_i)), 0 \leq t \leq T. \end{aligned}$$

From the strong continuity of the resolvent operator, we get that Q maps $\mathtt{PC}([0,T],\mathsf{X})$ to itself. Using the Lipschitz conditions in assumption $(\mathbf{L1})$, we have the following estimations :

$$\begin{aligned} \|(Qv)(t) - (Qw)(t)\|_{\mathsf{X}} &\leq \|R(t)[g(v) - g(w)]\|_{\mathtt{PC}} \\ &\quad + \int_0^t \|R(t-s)\|_{\mathcal{L}(\mathsf{X})} \|f(s,v(s)) - f(s,w(s))\|_{\mathsf{X}} ds \\ &\quad + \sum_{0<t_i<t} \|R(t-t_i)\|_{\mathcal{L}(\mathsf{X})} \|\mathsf{I}_i(v(t_i)) - \mathsf{I}_i(w(t_i))\|_{\mathsf{X}} \\ &\leq M_T (L_G + L_f T) \|v-w\|_{\mathtt{PC}} + \sum_{0<t_i<t} M_T L_i \|v(t_i) - w(t_i)\| \\ &\leq M_T (L_G + L_f T) \|v-w\|_{\mathtt{PC}} + M_T \|v-w\|_{\mathtt{PC}} \sum_{i=1}^{p} L_i \\ &\leq M_T (L_G + L_f T) \|v-w\|_{\mathtt{PC}} + \sum_{i=1}^{p} L_i \|v-w\|_{\mathtt{PC}}, \ v, w \in \mathtt{PC}. \end{aligned}$$

From assumption $(\mathbf{L2})$, it follows that Q is a strict contraction operator on $\mathtt{PC}([0,T],\mathsf{X})$. Consequently, by Banach's fixed point Theorem, Q has a unique fixed point which is a unique mild solution of (1). The proof is completed. □

3.2. Compact nonlocal function

This present subsection deals with the existence of mild solution where g is compact. Note that a compact operator is a continuous operator which maps a bounded set into a precompact set. We need weaker conditions on the data according to [20]. We give the following assumptions to set the problem.

(**C1**) f is continuous and maps a bounded set into a bounded set.

(**C2**) g, and $\mathsf{I}_i : \mathsf{X} \to \mathsf{X}, i = 1, 2, \cdots, p$, are compact operators, and $R(t)$ the resolvent operator is compact for any $t > 0$).

(**C3**) For each $x_0 \in \mathsf{X}$, there exists a constant $r > 0$ such that :

$$M_T \left(\|x_0\|_{\mathsf{X}} + \sup_{\varphi \in \mathsf{Y}_r} \|g(\varphi)\|_{\mathsf{X}} + T \sup_{s \in [0,T], \varphi \in \mathsf{Y}_r} \|f(s, \varphi(s))\|_{\mathsf{X}} \right)$$

$$+ M_T \left(\sup_{\varphi \in \mathsf{Y}_r} \sum_{i=1}^{p} \|\mathsf{I}_i(\varphi(t_i))\|_{\mathsf{X}} \right) \leq r,$$

where

$$\mathsf{Y}_r := \{\varphi \in \mathsf{PC}([0,T], \mathsf{X}) : \|\varphi(t)\| \leq r \, for \, t \in [0,T]\}. \tag{5}$$

Remark 3.4. If the operators g, f, and $\mathsf{I}_i : \mathsf{X} \to \mathsf{X}, i = 1, 2, \cdots, p$, are compact then assumption (**C1**) is satisfied.

Theorem 3.5. *Assume that the resolvent operator $(R(t))_{t \geq 0}$ is compact for $t > 0$. Let $\mathsf{m} > 1$. Define*

$$(\Theta \ell)(t) = \int_0^t R(t-s)\ell(s)ds \quad for \, t \in [0,T], \ell \in L^{\mathsf{m}}([0,T], \mathsf{X}).$$

Then $\Theta : L^{\mathsf{m}}([0,T], \mathsf{X}) \to \mathcal{C}([0,T], \mathsf{X})$ is compact.

Proof. Let $\{\ell_k\}_{k \geq 1}$ a bounded sequence on $L^{\mathsf{m}}([0,T], \mathsf{X})$ such that $\|\ell_k\|_{L^{\mathsf{m}}([0,T], \mathsf{X})} \leq 1$, for all $k \geq 1$. We need to prove that $\{\Theta \ell_k\}_{k \geq 1}$ is relatively compact in $\mathcal{C}([0,T], \mathsf{X})$. To this end, we first prove that for each $t \in [0,T]$, the set $\{(\Theta \ell_k)(t)\}_{k \geq 1}$ is relatively compact in X. In fact, the case where $t = 0$ is trivial. We let $t \in (0, T-\varepsilon]$ and $\varepsilon > 0$. Then

$$\begin{aligned}
(\Theta_\varepsilon \ell_k)(t) &:= \int_0^{t-\varepsilon} R(t-r)\ell_k(r)dr \\
&= \int_0^{t-\varepsilon} [R(t-r) - R(\varepsilon)R(t-r-\varepsilon) + R(\varepsilon)R(t-r-\varepsilon)]\,\ell_k(r)dr \\
&= \int_0^{t-\varepsilon} [R(t-r) - R(\varepsilon)R(t-r-\varepsilon)]\,\ell_k(r)dr \\
&\quad + \int_0^{t-\varepsilon} R(\varepsilon)R(t-r-\varepsilon)\ell_k(r)dr \\
&= \int_0^{t-\varepsilon} [R(t-r) - R(\varepsilon)R(t-r-\varepsilon)]\,\ell_k(r)dr \\
&\quad + R(\varepsilon)\int_0^{t-\varepsilon} R(t-r-\varepsilon)\ell_k(r)dr \\
&= K_\varepsilon + R(\varepsilon)\int_0^{t-\varepsilon} R(t-r-\varepsilon)\ell_k(r)dr, \qquad (6)
\end{aligned}$$

where

$$K_\varepsilon = \int_0^{t-\varepsilon} [R(t-r) - R(\varepsilon)R(t-r-\varepsilon)]\,\ell_k(r)dr \quad for\ \varepsilon \in (0,t].$$

Using Lemma 2.7, we have the following estimations

$$\begin{aligned}
\|K_\varepsilon\|_{\mathsf{X}} &\leq \int_0^{t-\varepsilon} \|[R(t-r) - R(\varepsilon)R(t-r-\varepsilon)]\,\ell_k(r)\|_{\mathsf{X}}\,dr \\
&\leq \int_0^{t-\varepsilon} \|R(t-r) - R(\varepsilon)R(t-r-\varepsilon)\|_{\mathcal{L}(\mathsf{X})}\|\ell_k(r)\|_{\mathsf{X}}dr \\
&\leq \int_0^{t-\varepsilon} \gamma\varepsilon\|\ell_k(r)\|_{\mathsf{X}}dr \\
&\leq \int_0^{T} \gamma\varepsilon\|\ell_k(r)\|_{\mathsf{X}}dr \\
&\leq \varepsilon\gamma T^{(\mathsf{m}-1)/\mathsf{m}}.
\end{aligned}$$

We deduce that K_ε converges to zero whenever ε goes to zero unformly in k, that is, $K_\varepsilon = \mathcal{O}(\varepsilon)$. Thus (6) takes the following form

$$(\Theta_\varepsilon \ell_k)(t) = \mathcal{O}(\varepsilon) + R(\varepsilon)\int_0^{t-\varepsilon} R(t-r-\varepsilon)\ell_k(r)dr.$$

The term $\int_0^{t-\varepsilon} R(t-r-\varepsilon)\ell_k(r)dr$ is bounded uniformly in k. By the compactness of $R(\varepsilon)$ the set $\{(\Theta_\varepsilon \ell_k)(t)\}_{k\geq 1}$ is relatively compact for $\varepsilon \in (0,t]$.

Then, for any $\varepsilon_0 > 0$, there exists a finite set $\{x_i\}_{1\le i\le m}$ in X such that

$$\{(\Theta_\varepsilon \ell_k)(t)\}_{k\ge1} \subset \bigcup_{i=1}^{m} B(x_i, \varepsilon_0/2),$$

where $B(x_i, \varepsilon_0/2)$ is an open ball in X with center x_i and radius $\varepsilon_0/2$. We have the following inequalities :

$$\begin{aligned} \|(\Theta\ell_k)(t) - (\Theta_\varepsilon \ell_k)(t)\|_{\mathsf{X}} &= \left\| \int_{t-\varepsilon}^{t} R(t-s)\ell_k(s)ds \right\|_{\mathsf{X}} \\ &\le M_T \int_{t-\varepsilon}^{t} \|\ell_k(s)\|_{\mathsf{X}} ds \\ &\le M_T \varepsilon^{(\mathsf{m}-1)/\mathsf{m}} < \varepsilon_0/2. \end{aligned}$$

Then, for any $\varepsilon_0 > 0$, we get that

$$\begin{aligned} \|(\Theta\ell_k)(t) - x_i\|_{\mathsf{X}} &\le \|(\Theta\ell_k)(t) - (\Theta_\varepsilon\ell_k)(t)\|_{\mathsf{X}} + \|(\Theta_\varepsilon\ell_k)(t) - x_i\|_{\mathsf{X}} \\ &< \varepsilon_0/2 + \varepsilon_0/2 = \varepsilon_0 \end{aligned}$$

Hence

$$\{(\Theta\ell_k)(t)\}_{k\ge1} \subset \bigcup_{i=1}^{m} B(x_i, \varepsilon_0).$$

Next, we show that $\{(\Theta\ell_k)\}_{k\ge1}$ is equicontinuous on $[0,T]$. In fact, for $0 < \tau_1 < \tau_2 \le T$ and $0 < \tau \le \tau_1$,

$$\begin{aligned} (\Theta\ell_k)(\tau_2) - (\Theta\ell_k)(\tau_1) &= \int_{\tau_1}^{\tau_2} R(\tau_2 - s)\ell_k(s)ds \\ &\quad + \int_0^{\tau_1} \left(R(\tau_2 - s) - R(\tau_1 - s)\right)\ell_k(s)ds \\ &=: \mathsf{J}_1 + \mathsf{J}_2, \end{aligned}$$

respectively. Then, we have the following estimations :

$$\begin{aligned}
\|\mathsf{J}_1\|_{\mathsf{X}} &\leq \int_{\tau_1}^{\tau_2} \|R(\tau_2 - s)\ell_k(s)\|_{\mathsf{X}} ds \\
&\leq M_T \int_{\tau_1}^{\tau_2} \|\ell_k(s)\|_{\mathsf{X}} ds \\
&\leq M_T |\tau_2 - \tau_1|^{(\mathsf{m}-1)/\mathsf{m}} \left(\int_{\tau_1}^{\tau_2} \|\ell_k(s)\|_{\mathsf{X}}^{p} ds \right)^{1/\mathsf{m}} \\
&\leq M_T |\tau_2 - \tau_1|^{(\mathsf{m}-1)/\mathsf{m}} \qquad (7) \\
\|\mathsf{J}_2\|_{\mathsf{X}} &\leq \int_0^{\tau_1} \|R(\tau_2 - s) - R(\tau_1 - s)\|_{\mathcal{L}(\mathsf{X})} \|\ell_k(s)\|_{\mathsf{X}} ds \\
&\leq \int_0^{\tau_1} \|R(\tau_2 - s) - R(\tau_1 - s)\|_{\mathcal{L}(\mathsf{X})} \|\ell_k(s)\|_{\mathsf{X}} ds \\
&\leq \left(\int_0^{\tau_1} \|R(\tau_2 - \tau_1 + s) - R(s)\|_{\mathcal{L}(\mathsf{X})}^{\mathsf{m}/(\mathsf{m}-1)} ds \right)^{(\mathsf{m}-1)/\mathsf{m}} \\
&\quad \times \left(\int_0^{\tau_1} \|\ell_k(\tau_1 - s)\|_{\mathsf{X}}^{\mathsf{m}} ds \right)^{1/\mathsf{m}} \qquad (8) \\
&\leq \left(\int_0^{\tau_1} \|R(\tau_2 - \tau_1 + s) - R(s)\|_{\mathcal{L}(\mathsf{X})}^{\mathsf{m}/(\mathsf{m}-1)} ds \right)^{(\mathsf{m}-1)/\mathsf{m}} \qquad (9)
\end{aligned}$$

which are independent of k. From Lemma 2.7, we know that the resolvent operator is continuous in the operator norm on $(0, +\infty)$. Thus, we obtain the equicontinuity of the set $\{(\Theta\ell_k)\}_{k\geq 1}$ on $[0, T]$. Then by the Ascoli-Arzelà's Theorem, we obtain the compactness of the operator Θ. □

Lemma 3.6. *Let K be a compact set of X. Then*

$$\lim_{h\to 0^+} \left(\sup_{x\in K} \|R(t+h)x - R(t)x\| \right) = 0, \quad \textit{for } t \geq 0.$$

where $(R(t))_{t\geq 0}$ is the resolvent operator.

Proof. Let $t \geq 0$ and $(h_n)_{n\geq 1} \subset \mathbb{R}^+$ be a sequence of positive numbers going to zero. We define a sequence $(\alpha_n)_{n\geq 1} \subset \mathbb{R}^+$ by

$$\alpha_n = \sup_{x\in K} \|R(t+h_n)x - R(t)x\|.$$

Since K is compact, then, there exists $x_n \in K$ such that

$$\alpha_n = \|R(t+h_n)x_n - R(t)x_n\|.$$

Thus we get, $(x_n)_{n\geq 1}$, a sequence of K. $(x_n)_{n\geq 1}$ has a converging subsequence noted $(x_{n_k})_{k\geq 0}$. Denoting y the limit of $(x_{n_k})_{k\geq 0}$, then, we have

$$\begin{aligned}
\|R(t+h_{n_k})x_{n_k} - R(t)x_{n_k}\| &\leq \|R(t+h_{n_k})x_{n_k} - R(t+h_{n_k})y\| \\
&\quad + \|R(t+h_{n_k})y - R(t)y\| + \|R(t)y - R(t)x_{n_k}\| \\
&\leq Ne^{\eta t}\left[e^{\eta h_n}+1\right]\|x_{n_k}-y\| + \|R(t+h_{n_k})y - R(t)y\|.
\end{aligned}$$

It follows that $(\alpha_{n_k})_{k\geq 1}$ converges to zero.
Claim : $(\alpha_n)_{n\geq 1}$ converges to zero.
If there exists a diverging subsequence $(\alpha_{n_k})_{k\geq 1}$, that is, there exists $\varepsilon > 0$ such that $\alpha_{n_k} > \varepsilon$.
Define the sequence $(\beta_k)_{k\geq 1}$ by $\beta_k = \alpha_{n_k}$. Repeating the previous argument, one has a subsequence $(\beta_{k_p})_{p\geq 1}$ converging to zero. Impossible, since $\beta_{k_p} = \alpha_{n_{k_p}} > \varepsilon$. The proof is completed. □

Theorem 3.7. *Assume that* (**R1**)-(**R2**)*,* (**C1**)*,* (**C2**) *and* (**C3**) *hold. Then for every* $x_0 \in \mathsf{X}$*, equation* (1) *has at least a mild solution.*

Proof. Let $x_0 \in \mathsf{X}$ be fixed. Define an operator Q on $\mathtt{PC}([0,T],\mathsf{X})$ by

$$\begin{aligned}
(Qv)(t) &= R(t)[x_0 - g(v)] + \int_0^t R(t-s)f(s,v(s))ds \\
&\quad + \sum_{0<t_i<t} R(t-t_i)\mathsf{I}_i(v(t_i)), 0 \leq t \leq T \\
&:= (Q_1v)(t) + (Q_2v)(t) + (Q_3v)(t),
\end{aligned}$$

where

$$\begin{aligned}
&(Q_1v)(t) = R(t)[x_0 - g(v)], 0 \leq t \leq T, \\
&(Q_2v)(t) = \int_0^t R(t-s)f(s,v(s))ds, 0 \leq t \leq T, \\
&(Q_3v)(t) = \sum_{0<t_i<t} R(t-t_i)\mathsf{I}_i(v(t_i)), 0 \leq t \leq T.
\end{aligned}$$

We show, by Schauder's fixed point Theorem, that Q has at least a fixed point in Y_r given by (5), which is a mild solution of equation (1).

Clearly Q maps $\mathtt{PC}([0,T],\mathsf{X})$ to itself. Let $u, v \in \mathtt{PC}([0,T],\mathsf{X})$, then we have the following estimations :

$$\begin{aligned}
\|(Qv)(t)-(Qu)(t)\|_{\mathsf{X}} &\leq \Big\| R(t)[g(u)-g(v)]+(Q_2u)(t)-(Q_2v)(t) \\
&\quad + \sum_{0<t_i<t} R(t-t_i)\left(\mathsf{I}_i(u(t_i))-\mathsf{I}_i(v(t_i))\right)\Big\|_{\mathsf{X}} \\
&\leq M_T\|g(u)-g(v)\|_{\mathsf{X}}+\|(Q_2u)(t)-(Q_2v)(t)\|_{\mathsf{X}} \\
&\quad + \sum_{0<t_i<t} M_T\|\left(\mathsf{I}_i(u(t_i))-\mathsf{I}_i(v(t_i))\right)\|_{\mathsf{X}} \\
&\leq M_T\|g(u)-g(v)\|_{\mathsf{X}}+\|Q_2u-Q_2v\| \\
&\quad +M_T\sum_{i=1}^{p}\|\mathsf{I}_i(u)-\mathsf{I}_i(v)\|. \qquad (10)
\end{aligned}$$

By Theorem 3.5, we have that Q_2 is compact (therefore continuous). From assumption (**C1**), all the involved functions in (10) are continuous. Then,

$$\|Qv-Qu\|_{\mathtt{PC}} \leq C\|v-u\|_{\mathtt{PC}}$$

where C is a positive constant depending $T, M_T, g, \mathsf{I}_i, i=1,2,\cdots,p$. We deduce the continuity of the map Q on $\mathtt{PC}([0,T],\mathsf{X})$. Let $v\in\mathsf{Y}_r$; we have, by assumption (**C3**), the following estimations :

$$\begin{aligned}
\|(Qv)(t)\|_{\mathsf{X}} &\leq \|R(t)\|_{\mathcal{L}(\mathsf{X})}\left(\|x_0\|_{\mathsf{X}}+\|g(v)\|_{\mathsf{X}}\right)+\int_0^t\|R(t-s)\|_{\mathcal{L}(\mathsf{X})}\|f(s,v(s))\|_{\mathsf{X}}ds \\
&\quad + \sum_{0<t_i<t}\|R(t-t_i)\|_{\mathcal{L}(\mathsf{X})}\|\mathsf{I}_i(v(t_i))\|_{\mathsf{X}}, \\
&\leq M_T\left(\|x_0\|_{\mathsf{X}}+\sup_{\varphi\in\mathsf{Y}_r}\|g(\varphi)\|_{\mathsf{X}}\right)+\int_0^t M_T\sup_{r\in[0,T],\varphi\in\mathsf{Y}_r}\|f(r,\varphi(r))\|_{\mathsf{X}}ds \\
&\quad + \sum_{0<t_i<t} M_T\sup_{\varphi\in\mathsf{Y}_r}\|\mathsf{I}_i(\varphi(t_i))\|_{\mathsf{X}}, \\
&\leq M_T\left(\|x_0\|_{\mathsf{X}}+\sup_{\varphi\in\mathsf{Y}_r}\|g(\varphi)\|_{\mathsf{X}}\right)+TM_T\sup_{r\in[0,T],\varphi\in\mathsf{Y}_r}\|f(r,\varphi(r))\|_{\mathsf{X}} \\
&\quad +M_T\sum_{0<t_i<t}\sup_{\varphi\in\mathsf{Y}_r}\|\mathsf{I}_i(\varphi(t_i))\|_{\mathsf{X}}, \\
&\leq M_T\Bigg(\|x_0\|_{\mathsf{X}}+\sup_{\varphi\in\mathsf{Y}_r}\|g(\varphi)\|_{\mathsf{X}}+T\sup_{r\in[0,T],\varphi\in\mathsf{Y}_r}\|f(r,\varphi(r))\|_{\mathsf{X}} \\
&\quad + \sum_{0<t_i<t}\sup_{\varphi\in\mathsf{Y}_r}\|\mathsf{I}_i(\varphi(t_i))\|_{\mathsf{X}}\Bigg) \\
&\leq r,
\end{aligned}$$

for $0 \leq t \leq T$. Therefore, Q is a continuous mapping from Y_r to Y_r.
Now, we establish the compactness of Q_1 and Q_3. First, note that

$$(Q_3 v)(t) = \sum_{0<t_i<t} R(t-t_i)\mathsf{I}_i(v(t_i)) = \begin{cases} 0 & if \quad t \in [0,t_1], \\ R(t-t_1)\mathsf{I}_1(v(t_1)) & if \quad t \in (t_1,t_2], \\ \vdots & \vdots \quad \vdots \\ \sum_{i=1}^{p} R(t-t_i)\mathsf{I}_i(v(t_i)) & if \quad t \in (t_p,T], \end{cases}$$

where $0 < t_1 < t_2 < \cdots < t_p < T$ a finite subintervals of $[0,T]$. We set

$$\Lambda = \{\sigma : \sigma(t) = R(t-t_1)\mathsf{I}_1(v(t_1)), t \in [t_1,t_2], v \in \mathsf{Y}_r\}$$

and prove that Λ is precompact in $\mathcal{C}([t_1,t_2],\mathsf{X})$.
From assumption (**C2**), we have that, for each $t \in [t_1,t_2]$, the set

$$\{\sigma(t) = R(t-t_1)\mathsf{I}_1(v(t_1)) : v \in \mathsf{Y}_r\}$$

is precompact in X . Next, we show that the functions in Λ are equicontinuous. Let $\sigma \in \Lambda$, for $t_1 \leq s < t \leq t_2$, we have that

$$\begin{aligned} \|\sigma(t) - \sigma(s)\| &\leq \|R(t-t_1)\mathsf{I}_1(v(t_1)) - R(s-t_1)\mathsf{I}_1(v(t_1))\| \\ &\leq \sup_{x \in K} \|R(t-t_1)x - R(s-t_1)x\| \end{aligned}$$

where $K = \{\mathsf{I}_1(v(t_1)) : v \in \mathsf{Y}_r\}$. By Lemma 3.6, the functions in Λ are equicontinuous due to the compactness of I_1 and the resolvent operator $(R(t))_{t\geq 0}$ for $t > 0$. By Ascoli-Arzela Theorem Λ is precompact in $\mathcal{C}([t_1,t_2],\mathsf{X})$. Using the same method, the precompactness in the cases for other subintervals follows. Therefore, Q_3 is a compact operator.

To prove the compactness of Q_1, we use the same above method. For each $t \in [0,T]$, the set $\{R(t)[x_0 - g(v)] : v \in \mathsf{Y}_r\}$ is precompact in X since g is compact. Therefore, Q_1 is a compact operator by the Ascoli-Arzela Theorem. Hence Q is also a compact operator.
Schauder's fixed point Theorem implies that Q has a fixed point, which is a mild solution. The proof is completed. □

3.3. Neither Lipschtz nor compact nonlocal function

In this subsection, we establish the existence of mild solution Eq.(1) with more general nonlocal function. We assume that the values of $x(t)$ for t near zero do not affect $g(x)$. For example, it is the case when

$$g(x) = \sum_{j=1}^{q} c_j x(s_j), 0 < s_1 < s_2 < \cdots < s_q < T.$$

We established the existence of mild solutions under the following assumptions.

(**N1**) f is continuous, and there exists a constant $L_f > 0$ such that

$$\|f(t,x) - f(t,y)\|_{\mathsf{X}} \le L_f \|x-y\|_{\mathsf{X}}, \quad t \in [0,T], x,y \in \mathsf{X}.$$

(**N2**) $\mathsf{I}_i : \mathsf{X} \to \mathsf{X}, i = 1,2,\cdots,p$, are compact and the resolvent operator $(R(t))_{t\ge 0}$ is also compact for $t > 0$.

(**N3**) For each $x_0 \in \mathsf{X}$, there exists a constant $r > 0$ such that

$$M_T\left(\|x_0\|_{\mathsf{X}} + \sup_{\varphi\in \mathsf{Y}_r} \|g(\varphi)\|_{\mathsf{X}} + T \sup_{s\in[0,T],\varphi\in\mathsf{Y}_r} \|f(s,\varphi(s))\|_{\mathsf{X}}\right)$$

$$+M_T\left(\sup_{\varphi\in\mathsf{Y}_r}\sum_{i=1}^{p}\|\mathsf{I}_i(\varphi(t_i))\|_{\mathsf{X}}\right) \le r.$$

(**N4**) $g : \mathtt{PC}([0,T],\mathsf{X}) \to \mathsf{X}$ is continuous, maps Y_r into a bounded set, and there is a $\delta = \delta(r) \in (0,t_1)$ such that $g(\varphi) = g(\psi)$ for any $\varphi,\psi \in \mathsf{Y}_r$ with $\varphi(s) = \psi(s), s \in [\delta,T]$.

Theorem 3.8. *Assume that* (**R1**)-(**R2**), (**N1**)-(**N3**) *and* (**N4**) *hold. Then for every $x_0 \in \mathsf{X}$, equation* (1) *has at least a mild solution.*

Proof. For $\delta = \delta(r) \in (0,t_1)$, set

$$\mathsf{Y}(\delta) := \mathtt{PC}([\delta,T],\mathsf{X}) = \text{restrictions of functions in } \mathtt{PC}([0,T],\mathsf{X}) \text{ on } [\delta,T],$$
$$\mathsf{Y}_r(\delta) := \{\varphi \in \mathsf{Y}(\delta) : \|\varphi(t)\| \le r \text{ for } t \in [\delta,T]\}.$$

For $v \in \mathsf{Y}_r(\delta)$ fixed, we define a mapping $\mathcal{F}_v$ on Y_r by

$$(\mathcal{F}_v\varphi)(t) = R(t)(x_0 - g(\tilde{v})) + \int_0^t R(t-s)f(s,\varphi(s))ds + \sum_{0<t_i<t} R(t-t_i)\mathsf{I}_i(v(t_i)),$$

$t \in [0,T]$, where $\tilde{v}(t) = \begin{cases} v(t) & \text{if} \quad t \in [\delta,T], \\ v(\delta) & \text{if} \quad t \in [0,\delta]. \end{cases}$
From our assumptions, $\mathcal{F}_v$ is a continuous mapping from Y_r to Y_r.

Moreover, by iterative process involving repeated substitution of the expression of $\mathcal{F}_v$ into itself we obtain after m iterations the following inequality

$$\|(\mathcal{F}_v^m\varphi)(t) - (\mathcal{F}_v^m\psi)(t)\| \le \frac{(ML_f t)^m}{m!} \sup_{s\in[0,t]} \|\varphi(s) - \psi(s)\|,$$

$t \in [0,T], \varphi,\psi \in \mathsf{Y}_r, m = 1,2,\cdots.$

Therefore, for m large enough $\mathcal{F}_v^m$ is a contraction operator on Y_r. Thus, by Banach's fixed Theorem $\mathcal{F}_v$ has a unique fixed point $\varphi_v \in \mathsf{Y}_r$, i.e.,

$$\varphi_v(t) = R(t)(x_0 - g(\tilde{v})) + \int_0^t R(t-s)f(s,\varphi_v(s))ds + \sum_{0<t_i<t} R(t-t_i)\mathsf{I}_i(v(t_i)),$$

$t \in [0,T]$. Using the above procedure, we define a mapping $\mathcal{G}$ from $\mathsf{Y}_r(\delta)$ to itself by :

$$\begin{aligned}(\mathcal{G}v)(t) &= \varphi_v(t)|_{[\delta,T]}\\ &= R(t)(x_0 - g(\tilde{v})) + \int_0^t R(t-s)f(s,\varphi_v(s))ds\\ &\quad + \sum_{0<t_i<t} R(t-t_i)\mathsf{I}_i(v(t_i)), \quad t \in [\delta,T].\end{aligned}$$

In fact, let $v \in \mathsf{Y}_r(\delta)$, we have that

$$(\mathcal{G}v)(t) = \varphi_v(t)|_{[\delta,T]} = (\mathcal{F}_v\varphi)(t)|_{[\delta,T]}\,.$$

In order to prove the Theorem, we need the following Lemmas.

Lemma 3.9. *Under the assumptions* (**N1**)-(**N4**)*, the linear map* $v \mapsto \tilde{v}$ *on* $\mathtt{PC}([0,T],\mathsf{X})$ *to itself is bounded and* $\{\tilde{v}, v \in \mathsf{Y}_r\} \subset \mathsf{Y}_r$*.*

Proof. Let $u,v \in \mathtt{PC}([0,T],\mathsf{X})$ and $\alpha \in \mathbb{R}$, we get the following relations

$$\begin{aligned}\widetilde{u+v}(t) &= \begin{cases}(u+v)(t) & \text{if} \quad t\in[\delta,T]\\ (u+v)(\delta) & \text{if} \quad t\in[0,\delta]\end{cases}\\ &= \begin{cases}\tilde{u}(t)+\tilde{v}(t) & \text{if} \quad t\in[\delta,T]\\ \tilde{u}(\delta)+\tilde{v}(\delta) & \text{if} \quad t\in[0,\delta]\end{cases}\\ \widetilde{u+v}(t) &= (\tilde{u}+\tilde{v})(t), \quad t\in[0,T] \quad \Rightarrow \quad \widetilde{u+v} = \tilde{u}+\tilde{v}\\ \widetilde{\alpha v}(t) &= \alpha v(t) = \alpha\tilde{v}(t), \quad t\in[0,T] \quad \Rightarrow \quad \widetilde{\alpha v} = \alpha\tilde{v}\\ \|\tilde{v}(t)\|_{\mathsf{X}} &\le \|v\|_{\mathtt{PC}}, \quad t\in[0,T] \quad \Rightarrow \quad \|\tilde{v}\|_{\mathtt{PC}} \le \|v\|_{\mathtt{PC}}\end{aligned}$$

•

Lemma 3.10. *Under the assumptions* (**N1**)-(**N4**)*, let* x_0 *be fixed. The map* $v \mapsto \varphi_v$ *on* Y_r *to itself is continuous.*

Proof. Let $u, v \in \mathsf{Y}_r$, by Lemma 3.9 and the assumptions (**N1**)-(**N4**) we have that

$$\begin{aligned}
\|\varphi_u(t)-\varphi_v(t)\|_{\mathsf{X}} &= \Bigg\| R(t)(-g(\tilde{u})+g(\tilde{v}))+\int_0^t R(t-s)\,[f(s,\varphi_u(s))-f(s,\varphi_v(s))]\,ds \\
&\quad + \sum_{0<t_i<t} R(t-t_i)\,[\mathsf{I}_i(u(t_i))-\mathsf{I}_i(v(t_i))] \Bigg\|_{\mathsf{X}} \\
&\leq M_T\|(-g(\tilde{u})+g(\tilde{v}))\|_{\mathsf{X}}+M_T L_f\int_0^t \|\varphi_u(s)-\varphi_v(s)\|_{\mathsf{X}} ds \\
&\quad + M_T\sum_{0<t_i<t}\|\mathsf{I}_i(u(t_i))-\mathsf{I}_i(v(t_i))\|_{\mathsf{X}} \\
&\leq C\|u-v\|_{\mathtt{PC}}+M_T L_f\int_0^t \|\varphi_u(s)-\varphi_v(s)\|_{\mathsf{X}} ds
\end{aligned}$$

where the positive constant C involve $g, M_T, \mathsf{I}_i, i=1,\cdots,p$. Gronwall's Lemma implies that

$$\|\varphi_u(t)-\varphi_v(t)\|_{\mathsf{X}} \leq e^{M_T L_f t} C\|u-v\|_{\mathtt{PC}}$$

The continuity result follows. •

Using the Lemmas 3.9 and 3.10, we get that $\mathcal{G}$ is continuous on $\mathsf{Y}_r(\delta)$. Similarly to the above, using the Lemmas 3.9 and 3.10, we established the equicontinuity and then apply the Ascoli-Arzela Theorem to get that $\mathcal{G}$ is a compact operator. Therefore, by Schauder's fixed point Theorem, we conclude that $\mathcal{G}$ has a fixed point $v_* \in \mathsf{Y}_r(\delta)$. Put $x=\varphi_{v_*}$. Then

$$\begin{aligned}
x(t) = R(t)(x_0-g(\tilde{v}_*)) &+ \int_0^t R(t-s)f(s,x(s))ds \\
&+ \sum_{0<t_i<t} R(t-t_i)\mathsf{I}_i(v_*(t_i)), \quad t\in[0,T]. \qquad (11)
\end{aligned}$$

But $g(\tilde{v}_*)=g(x)$ and $\tilde{v}_*(t_i)=x(t_i)$, since $v_*(t)=(\mathcal{G}v_*)(t)=\varphi_{v_*}(t)=x(t), t\in[\delta,T]$, by the definition of $\mathcal{G}$. This concludes, together with (11), that x is a mild solution of equation (1). The proof is completed. □

4. Example

In this section, we apply the abstract results which we have obtained in the preceding sections to study the existence of solutions for a partial differential equation submitted to nonlocal initial conditions. This type of equation arises in the study of heat conduction in materials with memory see [21, 23]. We study the impulsive effects on the following problem heat conduction in materials with memory :

$$\begin{cases} \frac{\partial}{\partial t}u(t,y) = \sum_{i=1}^{N} \frac{\partial^2}{\partial y_i^2}u(t,y) + \sum_{i=1}^{N} \int_0^t b(t-s)\frac{\partial^2}{\partial y_i^2}u(s,y) + c_0 \sin(u(t,y)), \\ \quad t \in [0,T], t \neq t_i, y \in \Omega \\ \frac{\partial}{\partial n}u(t,y) = 0, \quad t \in [0,T], y \in \partial\Omega \\ u(0,y) + g(u(\cdot,\cdot))(y) = u_0(y), \quad y \in \Omega \\ u(t_i^+,y) - u(t_i^-,y) = \mathrm{I}_i(u(t_i,y)), \quad y \in \Omega,\ i = 1,2,\cdots,p, \end{cases} \tag{12}$$

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, $0 < s_1 < s_2 < \cdots < s_q < T, 0 < t_1 < t_2 < \cdots < t_p < T, c_j \in \mathbb{R}$ $(j = 0,1,2,\cdots,q), h \in \mathcal{C}([0,T] \times \overline{\Omega}, \mathbb{R}), \alpha_i > 0, \rho_i \in \mathcal{C}(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$ for each $i = 1,2,\cdots,p$. The nonlocal and impulsive function will be precised in the next.

To illustrate our above abstract results, we set the following conditions

(E1) $\mathsf{X} = \mathcal{C}\left(\overline{\Omega}\right), A = \sum_{i=1}^{N} \frac{\partial^2}{\partial y_i^2}$ and $\Upsilon(t) = b(t)A$ with

$$\mathcal{D}(A) := \left\{ \varphi \in \bigcap_{k \geq 1} W^{2,k}(\Omega); \varphi, \sum_{i=1}^{N} \frac{\partial^2 \varphi}{\partial y_i^2} \in \mathsf{X}, \frac{\partial \varphi}{\partial n} = 0 \right\},$$

$x(t)(y) = u(t,y)$ and $b : \mathbb{R}^+ \to \mathbb{R}$ is a bounded and $\mathcal{C}^1$ function such that b' is bounded and uniformly continuous.

(E2) $f(t,\psi)(\xi) = c_0 sin(\psi(\xi)), \quad t \in [0,T], \xi \in \overline{\Omega}, \psi \in \mathsf{X}.$

Then, we obtain from [22, Corollary 3.1.24] A generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X which is compact for $t > 0$. Since b is a bounded and $\mathcal{C}^1$ function such that b' is bounded and uniformly continuous. (E1) saisfies the assumptions of Theorem 2.2, then there exists a unique resolvent operator for equation (1). By Lemma 2.5 the resolvent operator $(R(t))_{t\geq 0}$ is compact for $t > 0$. Under condition (E1), equation (1) has a resolvent operator $(R(t))_{t\geq 0}$ which is compact for $t > 0$.

For the function g, we propose the following forms

(E3) $g(\varphi)(\xi) = \sum_{j=1}^{q} c_j \varphi(s_j)(\xi), \quad \xi \in \overline{\Omega}, \varphi \in \mathrm{PC}([0,T],\mathsf{X}).$

(E4) $g(\varphi)(\xi) = \int_0^T h(t,\xi)\gamma(|\varphi(t)|)dt, \quad \xi \in \overline{\Omega}, \varphi \in \mathrm{PC}([0,T],\mathsf{X}), \gamma : \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

(E5) $g(\varphi)(\xi) = \sum_{j=2}^{q\wedge p} c_j\varphi^2(t_j)(\xi), \quad \xi \in \overline{\Omega}, \varphi \in \mathtt{PC}([0,T],\mathsf{X}).$

and for the functions I_i, we take the following functions

(E6) $\mathsf{I}_i(\psi)(\xi) = \alpha_i/(|\psi(\xi)|+t_i), \quad \xi \in \overline{\Omega}, \psi \in \mathsf{X}, 1 \le i \le p.$

(E7) $\mathsf{I}_i(\psi)(\xi) = \int_\Omega \rho_i(\xi,y)\cos^2(\psi(y))dy, \quad \xi \in \overline{\Omega}, \psi \in \mathsf{X}, 1 \le i \le p.$

When c_j $(j = 0,1,\cdots,q)$ and $\alpha_i(i = 1,\cdots,p)$ are small enough, the conditions (E1)-(E3) and (E6) imply the assumptions of Theorem 3.3. Then, we have

Proposition 4.1. *The nonlocal impulsive problem* (1) *has a unique mild solution on* $[0,T]$.

Proof. One can compute and see that $L_f = c_0, L_G = \sum_{j=1}^{q} c_j, L_i = \frac{\alpha_i}{t_i}$. Therefore when c_j $(j = 0,1,\cdots,q)$ and $\alpha_i(i = 1,\cdots,p)$ are small enough, we get $M_T\left(c_0T + \sum_{j=1}^{q} c_j + \sum_{i=1}^{p}\frac{\alpha_i}{t_i}\right) < 1$ where $M_T = \sup_{t\in[0,T]} \|R(t)\|_{\mathcal{L}(\mathsf{X})}$. Theorem 3.3 gives the desired conclusion. □

The conditions (E1), (E2), (E4) and (E7) give the assumptions in Theorem 3.7 for large $r > 0$. Then, we have

Proposition 4.2. *The nonlocal impulsive problem* (1) *has at least a mild solution on* $[0,T]$.

Proof. Let $\varepsilon > 0$ et condisder $g(\mathsf{Y}_\varepsilon) = \{g(\varphi) : \varphi \in \mathsf{Y}_\varepsilon\}$, it is clear that for each $\xi \in \overline{\Omega}$

$$|g(\varphi)(\xi)| \le T \sup_{(t,y)\in[0,T]\times\overline{\Omega}} |h(t,y)| \sup_{z\in\mathbb{R}} |\gamma(z)| \quad \text{for any } \varphi \in \mathsf{Y}_\varepsilon.$$

For any $\varphi \in \mathsf{Y}_\varepsilon$ we have that

$$|g(\varphi)(\xi) - g(\varphi)(\xi')| \le \sup_{z\in\mathbb{R}} |\gamma(z)| \int_0^T |h(t,\xi) - h(t,\xi')|dt \longrightarrow 0 \quad \text{when} \quad \xi \longrightarrow \xi'.$$

Thus g is a compact operator.

Further, denoting $B(0,\varepsilon) \subset \mathsf{X}$ the ball of radius ε centred at 0 we have that for each $\xi \in \overline{\Omega}$

$$|\mathsf{I}_i(\psi)(\xi)| \le |\Omega| \sup_{(z,y)\in\overline{\Omega}\times\overline{\Omega}} |\rho_i(t,y)| \quad \text{for any } \psi \in B(0,r).$$

For any $\psi \in B(0,r)$ we have that

$$|\mathsf{I}_i(\psi)(\xi)-\mathsf{I}_i(\psi)(\xi')| \leq \int_\Omega |\rho_i(\xi,y)-\rho_i(\xi',y)|dy \longrightarrow 0 \quad \text{when} \quad \xi \longrightarrow \xi'.$$

The compactness of I_i follows for each $i=1,2,\cdots,p$. By verification, we have for $r>0$ that

$$\begin{aligned}
&M_T\left(\|x_0\|_{\mathsf{X}}+\sup_{\varphi\in\mathsf{Y}_r}\|g(\varphi)\|_{\mathsf{X}}+T\sup_{s\in[0,T],\varphi\in\mathsf{Y}_r}\|f(s,\varphi(s))\|_{\mathsf{X}}+\sup_{\varphi\in\mathsf{Y}_r}\sum_{i=1}^{p}\|\mathsf{I}_i(\varphi(t_i))\|_{\mathsf{X}}\right)\\
&\leq M_T\left(\|x_0\|_{\mathsf{X}}+T\sup_{(t,y)\in[0,T]\times\overline{\Omega}}|h(t,y)|\sup_{z\in\mathbb{R}}|\gamma(z)|+Tc_0+|\Omega|\sum_{i=1}^{p}\sup_{z,y\in\overline{\Omega}}|\rho_i(z,y)|\right)\\
&\quad=\eta.
\end{aligned}$$

Therefore if $r>\eta$ assumption (**C3**) is satisfied. All the other assumptions of Theorem 3.7 being held the desired conclusion follows. □

The conditions (E1), (E2), (E5) and (E7) make the assumptions in Theorem 3.8 satisfied for large $r>0$. Therefore, we give

Proposition 4.3. *The nonlocal impulsive problem* (1) *has at least a mild solution on* $[0,T]$.

Proof. Among assumption (**N3**) all the assumptions of Theorem 3.8 are satisfied. Nevertheless, we have that

$$\begin{aligned}
&M_T\left(\|x_0\|_{\mathsf{X}}+\sup_{\varphi\in\mathsf{Y}_r}\|g(\varphi)\|_{\mathsf{X}}+T\sup_{s\in[0,T],\varphi\in\mathsf{Y}_r}\|f(s,\varphi(s))\|_{\mathsf{X}}\right.\\
&\left.+\sup_{\varphi\in\mathsf{Y}_r}\sum_{i=1}^{p}\|\mathsf{I}_i(\varphi(t_i))\|_{\mathsf{X}}\right)-r\\
&\leq M_T\left(\|x_0\|_{\mathsf{X}}+\sum_{j=2}^{q\wedge p}c_j\sup_{\varphi\in\mathsf{Y}_r}\|\varphi\|_{\mathsf{PC}}^2+Tc_0+|\Omega|\sum_{i=1}^{p}\sup_{z,y\in\overline{\Omega}}|\rho_i(z,y)|\right)-r\\
&\leq M_T\left(\|x_0\|_{\mathsf{X}}+\sum_{j=2}^{q\wedge p}c_jr^2+Tc_0+|\Omega|\sum_{i=1}^{p}\sup_{z,y\in\overline{\Omega}}|\rho_i(z,y)|\right)-r\\
&\leq M_T\left(\|x_0\|_{\mathsf{X}}+Tc_0+|\Omega|\sum_{i=1}^{p}\sup_{z,y\in\overline{\Omega}}|\rho_i(z,y)|\right)+M_T\sum_{j=2}^{q\wedge p}c_jr^2-r\\
&=: ar^2-r+b
\end{aligned}$$

Choosing c_j $(j=2,\cdots,q\wedge p)$ small enough, we get

$$r\in(0,+\infty)\bigcap\left[\frac{1-\sqrt{1-4ab}}{2a};\frac{1\sqrt{1+4ab}}{2a}\right]$$

and assumption (**N3**) is held. Consequently the existence of mild solution follows. □

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