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MOSER TYPE ESTIMATES IN NONLINEAR NEUMANN PROBLEMS

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Sharp exponential estimates for solutions to homogeneous Neumann problems for nonlinear elliptic equations in open subsets Ω of $\mathbb{R}^n$ are established, with data from limiting Lebesgue spaces, or, more generally, Lorentz spaces.

The aim of the present note is to announce some recent results dealing with sharp exponential estimates for solutions to nonlinear elliptic equations in limiting cases.

Exponential integrability properties of functions from borderline Sobolev spaces have been known for a long time, and go back to [22], [24]. The optimal constant in the corresponding Sobolev inequality for compactly supported functions was found in a by now classical paper by Moser [21] in the case of first order Sobolev spaces, and extended to Sobolev spaces of arbitrary order in [1]. More recently, similar properties have been shown to hold also for solutions to elliptic equations - see e.g. [5], [15]. In particular, Moser type inequalities, involving sharp constants, for solutions to Dirichlet problems have been established in [16], [6], [4], [12], [13], [14].

Here, we are concerned with solutions to homogeneous Neumann problems, which, in their basic form, read

$$\begin{cases} -\mathrm{div}\big(a(x,u,\nabla u)\big) = \mathrm{div} F & \text{in } \Omega \\ a(x,u,\nabla u)\cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a sufficiently regular bounded domain in $\mathbb{R}^n$, $n \geq 3$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function; the function $F : \Omega \to \mathbb{R}^n$ is given and satisfies suitable integrability conditions; ∇u denotes the gradient of u; ν is the unit normal vector on $\partial\Omega$, and "$\cdot$" stands for scalar product in $\mathbb{R}^n$. As an ellipticity condition, we assume that there exists $p \in (1, n]$ such that, for a.e. $x \in \Omega$,

$$a(x, t, \xi) \cdot \xi \geq |\xi|^p \qquad \text{for every } (t, \xi) \in \mathbb{R} \times \mathbb{R}^n . \tag{2}$$

We deal with weak solutions to (1), namely with functions u from the Sobolev space $W^{1,p}(\Omega)$ satisfying

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla\phi \, dx = \int_\Omega F \cdot \nabla\phi \, dx \tag{3}$$

for every $\phi \in W^{1,p}(\Omega)$.

By the Sobolev embedding theorem, the membership of $|F|$ in the Lebesgue space $L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$, entails that the right-hand side of (3) is convergent for every test function $\phi \in W^{1,p}(\Omega)$, and hence that weak solutions to (3) are well-defined, as long as $a(x, t, \xi)$ is such that the left-hand side converges as well. This is certainly true, for instance, if there exists a constant K such that, for a.e. $x \in \Omega$,

$$|a(x, t, \xi)| \leq K|\xi|^{p-1} \qquad \text{for every } (t, \xi) \in \mathbb{R} \times \mathbb{R}^n .$$

A higher summability of $|F|$ reflects on the regularity of solutions u. Specifically, u is known to belong to $L^s(\Omega)$ where $s = \frac{nq(p-1)}{n-q(p-1)}$, the Sobolev conjugate of $q(p-1)$, if $|F| \in L^q(\Omega)$ with $q < \frac{n}{p-1}$, and to $L^\infty(\Omega)$ if $q > \frac{n}{p-1}$. Roughly speaking, our main result ensures that, in the borderline cases where $q = \frac{n}{p-1}$, the function $\lambda|u|^{n'}$ is exponentially integrable for every $\lambda > 0$, provided that the domain is of class $\mathcal{C}^{1,\alpha}$, and, what is most interesting, exhibits the largest value of λ for which such an integrability property is uniform in F. In fact, our conclusions hold for data F from the larger class of Lorentz spaces $L^{\frac{n}{p-1},\frac{q}{p-1}}(\Omega)$. Recall that

$$\| \, |F| \, \|_{L^{\frac{n}{p-1},\frac{q}{p-1}}(\Omega)} = \left(\int_0^{|\Omega|} F^*(s)^{\frac{q}{p-1}} s^{\frac{q}{n}-1} ds \right)^{\frac{p-1}{q}},$$

where $|\Omega|$ is the Lebesgue measure of Ω, and F^* denotes the decreasing rearrangement of $|F|$.

Theorem 1. *Let Ω be a bounded connected domain in $\mathbb{R}^n$, $n \geq 3$, of class $\mathcal{C}^{1,\alpha}$, for some $\alpha \in (0, 1]$. Let $p \in (1, n]$ and let $|F| \in L^{\frac{n}{p-1}, \frac{q}{p-1}}(\Omega)$ for some $q \in (1, \infty]$. Let u be a weak solution to problem (1).*

(i) Case $1 < q < \infty$. A constant $C = C(\Omega, q)$ exists such that

$$\int_\Omega \exp\left(n\left(\frac{\omega_n}{2}\right)^{1/n} \frac{|u - \mathrm{m}(u)|}{\|\,|F|\,\|_{L^{\frac{n}{p-1}, \frac{q}{p-1}}}}\right)^{n'} dx \leq C\,, \tag{4}$$

where $\omega_n = \pi^{n/2}/\,\Gamma\big(1 + \frac{n}{2}\big)$, the measure of the unit ball in $\mathbb{R}^n$, and

$$\mathrm{m}(u) = \sup\big\{t \in \mathbb{R} : |\{u > t\}| \geq |\Omega|/2\big\},$$

the median of u. Moreover, the constant $n(\omega_n/2)^{\frac{1}{n}}$ is sharp. Indeed, domains $\Omega \in \mathcal{C}^{1,\alpha}$ exist such that the left-hand side of (4), with $n(\omega_n/2)^{\frac{1}{n}}$ replaced by any larger constant, cannot be uniformly bounded as $|F|$ ranges among all functions from $L^{\frac{n}{p-1}, \frac{q}{p-1}}(\Omega)$ and u is a weak solution to (1) with $a(x, t, \xi) = |\xi|^{p-2}\xi$.

(ii) Case $q = +\infty$. For every $\gamma < n(\omega_n/2)^{\frac{1}{n}}$, a constant $C = C(\Omega, \gamma)$ exists such that

$$\int_\Omega \exp\left(\gamma \frac{|u - \mathrm{m}(u)|}{\|\,|F|\,\|_{L^{\frac{n}{p-1}, \infty}}}\right) dx \leq C. \tag{5}$$

The result is sharp. Indeed, there exist domains $\Omega \in \mathcal{C}^{1,\alpha}$, functions F, with $|F| \in L^{\frac{n}{p-1}, \infty}(\Omega)$, and weak solutions to (1), with $a(x, t, \xi) = |\xi|^{p-2}\xi$, such that the left-hand side of (5) diverges for every $\gamma \geq n(\omega_n/2)^{\frac{1}{n}}$.

Remark 1. Results in the spirit of Theorem 1 for linear Neumann problems are contained in our earlier paper [2], of which the present work is a continuation. Apart from the nonlinearity of the equations considered here, another novelty is that, in contrast to [2], right-hand sides in divergence form are taken into account.

Remark 2. Let us notice that the best constant $n(\omega_n/2)^{\frac{1}{n}}$ in (4)–(5) depends only on the dimension n. Loosely speaking, this can be explained by the fact that the boundary of smooth domains is asymptotically flat. Of course, the geometry of Ω enters in the constant C on the right-hand side. A version of Theorem 1 holds even for irregular domains Ω having singularities of conical type. However, in this case, the optimal constant in the exponential does depend on geometric properties of $\partial\Omega$ at its irregular points, and, in particular, on the minimum of the solid apertures of $\partial\Omega$ at these points.

Remark 3. Equations containing lower order terms depending on ∇u can be included in our discussion, at least for values of $p \geq 2$. The conclusions are analogous to those of Theorem 1, provided that the coefficients of the new term belong to appropriate Lorentz spaces.

Precise statements and proofs of the extensions of Theorem 1 to which we allude in Remark 2 and 3 can be found in [3]. We refer to this paper also for the proof of Theorem 1.

Let us just outline here the basic ingredients in our approach. The starting point is an estimate for $(u - \mathrm{m}(u))$ in terms of rearrangements, in the spirit of those of [23] and [9] for Dirichlet problems, and of [17], [10], [8] for Neumann problems. Unlike the case of solutions to Dirichlet problems, such an estimate depends on Ω through its isoperimetric function h_Ω. Recall that $h_\Omega : (0, |\Omega|) \to [0, +\infty)$ is defined as

$$h_\Omega(s) = \inf\{\mathcal{P}(E; \Omega) : E \subset \Omega, |E| = s\} \qquad \text{for } s \in (0, |\Omega|), \tag{6}$$

where $\mathcal{P}(E; \Omega)$ denotes the perimeter of E relative to Ω (see e.g. [7], Definition 3.35), which agrees with the $(n-1)$-dimensional Hausdorff measure of $\partial E \cap \Omega$ if E is sufficiently regular.

The importance of isoperimetric inequalities relative to a domain and of the related isoperimetric function in the study of Sobolev embeddings and of a priori estimates in Neumann problems was pointed out in the work by V.G. Maz'ya ([18], [19], [20]). The point is that the isoperimetric function h_Ω is very difficult to compute in general, and it is explicitly known only for very special domains, such as balls, half-spaces and convex cones. Nevertheless, we can show that what plays a role in connection to inequalities of type (4) and (5) is only a precise description of the asymptotic behavior of h_Ω at 0. Such a description has been recently provided in [11], motivated by the study of Moser-Trudinger inequalities for functions which do not necessarily vanish on the boundary of their domain. With this material in place, the problem of estimates (4) and (5) is reduced to one-dimensional inequalities for an integral operator whose kernel satisfies suitable properties. The relevant inequalities can be studied via techniques introduced in [21] and developed in [1], [6], [12], [13], [14].

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COMPARISON RESULTS FOR SOLUTIONS OF PARABOLIC EQUATIONS WITH A ZERO ORDER TERM

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We give a comparison result for solutions of Cauchy-Dirichlet problems for parabolic equations by means of Schwarz symmetrization. The result takes into account the influence of the zero order term which could have a singularity at the origin of the type $1/|x|^2$.

Let Ω be a bounded, open subset of $\mathbb{R}^N$ and u be a real measurable function on Ω, we define the decreasing rearrangement of u as

$$u^*(s) = \sup\{\theta \geq 0 : \mu_u(\theta) > s\}, \quad s \in (0, |\Omega|)$$

where μ_u is the distribution function of u. Furthermore, we denote by $\Omega^\#$ the ball of $\mathbb{R}^N$ centered at the origin having the same measure as Ω, by $u^\#$ and $u_\#$ the decreasing and the increasing spherical rearrangement of u. Roughly speaking, $u^\#$ and $u_\#$ are spherically symmetric functions, defined on $\Omega^\#$ which are respectively decreasing and increasing along the radius and preserve the measure of the level sets of u.

It is well known that by using rearrangements sharp bounds for solutions of elliptic and parabolic equations can be found. Indeed, for large class of equations the solutions may be compared to the solution of a problem of the same type with spherical symmetry (the so-called "symmetrized" problem). The first results in this direction were obtained by G.Talenti [9] for elliptic equations and C. Bandle [5] for parabolic equations and since then, have been extended in different directions by various authors (see, for instance, [1], [3], [10], [12]). We consider the problem

$$\begin{cases} u_t - \left(a_{ij}\,(x,t)\,u_{x_i}\right)_{x_j} + cu = f & \text{in } \ \Omega \times (0,T) \\ u = 0 & \text{on } \ \partial\Omega \times (0,T) \\ u\,(x,0) = u_0\,(x) & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, $T > 0$.
Assume that the coefficients a_{ij} are measurable, bounded functions satisfying the condition

$$a_{ij}\,(x,t)\,\xi_i\xi_j \geq |\xi|^2 \quad \text{for a.e.} \quad (x,t) \in \Omega \times (0,T)\,, \quad \forall \xi \in \mathbb{R}^N, \tag{2}$$

$$c \in L^r\,(\Omega) \quad \text{with } \ r > N/2 \ \text{ if } \ N \geq 2, \ r \geq 1 \ \text{ if } \ N = 1, \tag{3}$$

$$f \in L^2\,(\Omega \times (0,T))\,, \qquad u_0 \in L^2\,(\Omega)\,. \tag{4}$$

Our aim is to find a comparison result between the solution u of the problem (1) and the solution v of a spherically symmetric problem which keeps in mind the zero order term. The candidate problem is the following

$$\begin{cases} v_t - \Delta v + \left(\left(c^+\right)_\# - \left(c^-\right)^\#\right) v = f^\# & \text{in } \ \Omega^\# \times (0,T) \\ v = 0 & \text{on } \ \partial\Omega^\# \times (0,T) \\ v\,(x,0) = u_0^\#\,(x) & x \in \Omega^\#, \end{cases} \tag{5}$$

where c^+, c^- are the positive and the negative part of c and $f^\#$ is the decreasing spherical rearrangement of f with respect to the space variable, for t fixed. We deal with weak solutions of the problem (1) i.e. functions $u \in L^2\left(0,T;H_0^1\,(\Omega)\right) \cap C\left([0,T]\,;L^2\,(\Omega)\right)$ such that $u_t \in L^2\,(\Omega \times (0,T))$ and

$$\begin{cases} \displaystyle\int_\Omega u_t\varphi\,dx + \int_\Omega a_{ij}u_{x_i}\varphi_{x_j}\,dx + \int_\Omega cu\varphi\,dx = \int_\Omega f\varphi\,dx \\ u\,(0) = u_0, \end{cases} \tag{6}$$

for all $\varphi \in H_0^1\,(\Omega)$ and for a.e. $t \in [0,T]$. The existence and the required regularity of such a solution is guaranteed under suitable assumptions on the data. We prove the following:

Theorem 1. *Let Ω be a bounded, open subset of $\mathbb{R}^N$, assume that the data of problem* (1) *satisfy* (2)–(4). *Let u and v be the weak solutions of problems* (1) *and* (5) *respectively, then for all $t \in [0, T]$,*

$$\int_0^s u^* (\sigma, t)\, d\sigma \leq \int_0^s v^* (\sigma, t)\, d\sigma, \quad \forall s \in [0, |\Omega|] \tag{7}$$

where u^* and v^* are the decreasing rearrangements of u and v with respect to the space variable, for t fixed.

The proof of Theorem 1 is given in [4]. In the case $c \in L^\infty(\Omega)$, the estimate (7) improves the estimate given in the papers 5, 8 in which the influence of the zero order term cu is neglected, since it is essentially omitted in the "symmetrized" problem using the sign assumption $c(x) \geq 0$. The first difficulty that appears in the proof of Theorem 1 and in general in the parabolic case is the presence of the time derivative term. This term can be treated by two different methods. Following the approach contained in a paper of C. Bandle (see [5]), one has to prove a delicate derivation formula with respect to the time variable for functions defined by integrals. In [5], such a formula is proved under strong regularity assumptions on the solutions. These hypotheses have been removed later in a paper of Mossino-Rakotoson (see [8]), where the formula is proved for fuctions $u \in H^1\left(0, T; L^2(\Omega)\right)$ by using the notion of relative rearrangement. Recently, generalizations of this result have been obtained in [2], where a formula concerning the second derivatives is also given. Another approach uses the implicit time discretization scheme. In this way the study is reduced to the case of an elliptic operator with a zero order term, for which comparison results are known (see [1], [10]). Obviously the case c bounded from below can be reduced to the case $c \geq 0$. In fact, if $c(x) \geq \lambda$ for a.e. $x \in \Omega$, we can replace the function u with the function $e^{\lambda t} u$. This situation was already studied in [12]. We give a different proof that avoids to proceed by means of the approximation used in [12]. More delicate is the proof when c is not bounded from below, since we prefer to work straight on problem (1) and do not want to use approximating problems having the troncations of c as coefficients of the zero order terms. The motivation of this study, besides its intrinsic interest, is also connected to some recent results obtained by various authors (see [6], [7]), related to the existence of solutions of parabolic equations when Ω is bounded, open subset of $\mathbb{R}^N$ $(N > 2)$ containing the origin and $c(x) = -\lambda/|x|^2$. The equation

$$u_t - \Delta u - \frac{\lambda}{-} |x|^2 u = f \tag{8}$$

is a borderline case in the classic theory of parabolic equations, indeed the potential $\lambda/|x|^2$ belongs to $L_w^{N/2}$, therefore it is not possible to use traditional uniqueness and regularity results. This kind of problems were firstly studied by Baras and Goldstein in [6], with the assumptions $f, u_0 \geq 0,\ f, u_0 \neq 0$. They prove that the behaviour of solutions depends on the value of the parameter λ. More precisely, there exists a critical value $\lambda_N := (N-2)^2/4$, corresponding to the best constant in the classical Hardy inequality, such that for $\lambda \leq \lambda_N$, the Cauchy-Dirichlet problem associated to equation (8) has a solution, while in the case $\lambda > \lambda_N$ the same problem has no local solution for any $f, u_0 \neq 0$. Afterwards, this problem was studied in [11] removing the sign assumptions on the data and pursuing a deeper analysis of the critical case $\lambda = \lambda_N$, and in [7], where the corresponding nonlinear case is treated. The subcritical case $\lambda < \lambda_N$ is easier to study than the case $\lambda = \lambda_N$. Indeed it is possible to use the classical methods of the Calculus of Variations, since by the classical Hardy inequality it follows that the operator $-\Delta u - \lambda/(|x|^2)u$ is coercive; then for any $f \in L^2(\Omega \times (0,T))$, $u_0 \in L^2(\Omega)$ there exists a unique solution $u \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$ (see [6], [7], [11]). The situation is very different in the critical case $\lambda = \lambda_N$, in which, there is a solution in L^2 but *not in* H_0^1 (see [7], [11]).

The above mentioned results, pursue us to obtain in [13] comparison results for problem (1) under the weaker regularity assumption $c \in L_w^{N/2}$ $(N > 2)$. We send back to [13] for the detailed results concerning both the subcritical and the critical case. In the subcritical case under the assumption

$$c^+ = 0 \quad \left(c^-\right)^{\#}(x) \leq \frac{\lambda}{|x|^2} \quad \forall x \in \Omega^{\#} \setminus \{0\}, \quad 0 < \lambda < \lambda_N \tag{9}$$

the solution u of (1) is compared with the solution v of the following problem

$$\begin{cases} v_t - \Delta v - \dfrac{\lambda}{|x|^2} v = f^{\#} & \text{in } \Omega^{\#} \times (0,T) \\ v = 0 & \text{on } \partial\Omega^{\#} \times (0,T) \\ v(x,0) = u_0^{\#}(x) & x \in \Omega^{\#}. \end{cases} \tag{10}$$

More precisely, we prove:

Theorem 2. *Let Ω be a bounded, open subset of $\mathbb{R}^N$, such that $0 \in \Omega$, assume that the data of problem* (1) *satisfy* (2), (4), (9). *Let u and v be the weak solutions of problems* (1) *and* (10) *respectively, then for all $t \in [0,T]$,* (7) *holds.*

Much more delicate is the study of the critical case: we have to make suitable assumptions on the data in order to introduce a functional space in which, there exists a unique solution of the problem (see [11]).

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FORMULA DI TAYLOR NEI GRUPPI DI CARNOT E APPLICAZIONI

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In '82 Folland and Stein defined, for a given function $f \in C^n(\mathbb{G})$, the Taylor polynomial, where $\mathbb{G}$ is a connected and simply connected real Lie group whose algebra is endowed with a family of dilations δ_λ. Although the extensive studies in this field, particularly in the case of stratified groups nowadays called *Carnot groups*, an explicit representation of the n^{th}-Taylor polynomial in this last setting seems to be missing. In this note we announce an explicit expression for the Taylor polynomial when $\mathbb{G}$ is the Heisenberg group $\mathbb{H}^1$, and we present one application.

1. Introduzione.

In questo annuncio ci occupiamo di una particolare classe di gruppi di Lie reali, i *gruppi di Carnot*, oggetto di intensi studi nell'ultimo trentennio per via dei legami con molti settori della matematica pura ed applicata come, per esempio, equazioni differenziali alle derivate parziali ipoellittiche (vedi, per esempio [12], [6], [21]), equazioni differenziali alle derivate parziali completamente

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I risultati qui presentati sono stati comunicati dal secondo autore: quelli relativi alla formula di Taylor sono stati ottenuti con G. Arena ed A. Causa, le applicazioni con G. Arena. Gli enunciati in forma completa e le relative dimostrazioni appariranno altrove.

non lineari (vedi per esempio [15] e relativa bibliografia), funzioni debolmente differenziabili ed argomenti correlati, con particolare riferimento alla teoria geometrica della misura (vedi [9], [4], [19], [14] per una estesa bibliografia), alla geometria sub–Riemanniana (vedi [2], [18]) e le relative applicazioni alla teoria geometrica del controllo (vedi per esempio [1] e relativa bibliografia), ai modelli matematici della visione (vedi [3] e relativa bibliografia).

I gruppi di Carnot, che devono il loro nome ad un lavoro di Charatheodory su una formulazione matematica del secondo principio della termodinamica, si rivelano essere naturalmente gli spazi metrici tangenti nei punti regolari di una varietà sub–Riemanniana: in pratica i gruppi di Carnot costituiscono per le varietà sub–Riemanniane quello che gli spazi vettoriali Euclidei sono per le varietà Riemanniane. Una varietà sub–Riemanniana (M, D, g) è una varietà Riemanniana (M, g) sulla quale è stata assegnata una distribuzione D di sottospazi vettoriali m–dimensionali del fibrato tangente. Una curva assolutamente continua $\gamma : \mathbb{R} \supset I \to M$ congiugente due dati punti p e q in M e tale che il vettore $\dot{\gamma}(t)$ appartiene, per q.o. $t \in I$, al sottospazio m–dimensionale $D_{\gamma(t)}$, si dirà *orizzontale*. Una condizione che garantisce che, per una fissata distribuzione D, tali curve esistano, è la cosiddetta condizione di Hörmander. Sotto tale condizione, possiamo definire una distanza (oggi detta C–C, ovvero di *Carnot–Caratheodory*), $d(p, q)$ come l'infimum delle lunghezze delle curve orizzontali congiungenti p e q. La topologia indotta da d è quella originale, tuttavia d e d_g, la metrica Riemanniana, non sono in generale equivalenti. Tale condizione, che appare in un fondamentale lavoro di Hörmander (vedi [12]), richiede che, assegnata una famiglia finita $X_1, X_2, \ldots, X_m$ di campi vettoriali lisci su una varietà n–dimensionale M, esista un intero r tale che, fra i campi vettoriali X_i, $[X_{i_1}, X_{i_2}]$, $[X_{i_1}, [X_{i_2}, X_{i_3}]]$, $\ldots$, $[X_{i_1}, [X_{i_2}, [X_{i_3}, \cdots [X_{i_{r-1}}, X_{i_r}] \cdots]]]$, almeno n siano linearmente indipendenti in ogni punto di M. L'aspetto metrico tipico della geometria sub–Riemanniana, appare chiaramente in una serie di lavori tra cui, per citarne alcuni, quello fondamentale di A. Nagel, E.M. Stein e S. Wainger (vedi [20]), ed altri di C. Fefferman e D.H. Phong (vedi [5]), e B. Franchi e E. Lanconelli (vedi [10], ed anche [13], [8]); passando poi dagli aspetti puramente metrici a quelli geometrici, ad esempio quello che realizza tali gruppi come strutture tangenti alle varietà sub–Riemanniane, una serie di idee successive in prevalenza dovute alla scuola facente capo a M.Gromov, ha condotto la teoria generale di tali varietà allo stadio attuale (per approfondimenti vedi [2], [18], vedi anche [17], [23], [24], [16], e relative bibliografie).

La problematica qui presentata è collegata all'espressione esplicita della formula di Taylor nei gruppi di Carnot (per una introduzione a questi, si rimanda a [7], [22], [11], ed anche a [14], [19]). Infatti, la definizione originale di polinomio di Taylor in tali gruppi, dovuta a Folland e Stein (vedi la Sezione 2, oltre che le pagg. 26–27 di [7], non consente, a causa della particolare natura dei campi dell'algebra, di poter calcolare agevolmente l'azione delle k–derivazioni su di un fissato monomio: così la definizione non sembra particolarmente conveniente qualora si voglia esplicitamente scrivere il polinomio di Taylor di grado fissato. È tuttavia possibile aggirare il problema per via teorica; precisamente, partendo dall'usuale espansione in serie di Taylor in un gruppo di Lie, è possibile congetturare la forma esplicita dell'n-esimo polinomio di Taylor e, tramite considerazioni di carattere algebrico, oltre che adattando argomenti standard della geometria sub-Riemanniana, caratterizzarlo come nel caso euclideo.

Per semplicità annunciamo i risultati nel caso del più noto gruppo di Carnot, il gruppo di Heisenberg $\mathbb{H}^1$, che presentiamo rapidamente nella sezione seguente; il lettore potrà tuttavia facilmente realizzare le estensioni al caso generale.

2. Notazioni e Preliminari.

$\mathbb{H}^1$ è l'unico gruppo di Lie reale nilpotente connesso e semplicemente connesso di dimensione 3 la cui algebra $\mathfrak{h}^1$ ammette una decomposizione del tipo $\mathfrak{h}^1 = \mathfrak{h} \oplus \mathfrak{t}$, con $\mathfrak{h}$ generata dai campi vettoriali invarianti a sinistra $X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$, detti *orizzontali*, e $\mathfrak{t}$ generato dal commutatore $T = -\frac{[X,Y]}{4}$, dove, con (x, y, t), denotiamo le coordinate di un elemento di $\mathfrak{h}^1$ e, al solito, identifichiamo i campi vettoriali con le derivazioni associate dell'algebra. Dalla formula di Baker–Campbell–Hausdorff segue che, dette $p = (x, y, t)$ e $q = (x', y', t')$ le coordinate esponenziali di due elementi di $\mathbb{H}^1$, l'operazione di gruppo opera secondo la legge $p\,q = \big(x + x', y + y', t + t' + 2(yx' - xy')\big)$: in particolare $p\,0 = 0\,p = p$ e $p^{-1} = -p$, dove, se $p = (x, y, t)$, allora $-p = (-x, -y, -t)$. Inoltre, la struttura nilpotente di $\mathfrak{h}^1$ induce su $\mathbb{H}^1$ una famiglia di dilatazioni $\{\delta_\lambda\}_{\lambda\geq 0}$, definite dalla posizione $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. Infine, una funzione $f : \mathbb{H}^1 \to \mathbb{R}$ si dirà omogenea di grado $\alpha \in \mathbb{R}$, se $f(\delta_\lambda(p)) = \lambda^\alpha f(p)$ per ogni $\lambda > 0$ e per ogni $p \in \mathbb{H}^1$; in particolare, per ogni $l, m, r = 0, 1, \ldots$, un monomio del tipo $x^l y^m t^r$ risulta essere una funzione omogenea di grado $l + m + 2r$.

Introduciamo ora una metrica sub–Riemanniana in $\mathbb{H}^1$ come segue. Fissati due punti p e q, sia $\gamma : [0, T] \to \mathbb{H}^1$ una curva assolutamente continua, tangente, per q.o. $t \in [0, T]$, alla distribuzione dei sottospazi generati dai campi X ed Y nei punti $\gamma(t)$. Più precisamente, cerchiamo due funzioni reali $\lambda, \mu : [0, T] \to \mathbb{R}$, misurabili, tali che $\lambda^2(t) + \mu^2(t) \leq 1$ e per le quali risulti $\dot\gamma(t) = \lambda(t)X(\gamma(t)) + \mu(t)Y(\gamma(t))$, per q.o. $t \in [0, T]$: tali curve, dette *orizzontali*, esistono grazie alla stratificazione nilpotente di $\mathfrak{h}^1$. Chiamiamo $d(p, q)$

l'estremo inferiore delle suddette $T > 0$. È possibile verificare che d è una metrica inducente in $\mathbb{H}^1$ la topologia euclidea, ma non equivalente alla distanza euclidea; in analogia a questa, tuttavia, d è invariante per traslazioni ed omogenea di grado 1, si ha cioè $d(qp_1, qp_2) = d(p_1, p_2)$ e $d(\delta_\lambda(p_1), \delta_\lambda(p_2)) = \lambda\, d(p_1, p_2)$, per ogni $q, p_1, p_2 \in \mathbb{H}^1$ e per ogni $\lambda > 0$.

Introduciamo infine le funzioni di classe $C^n(\mathbb{H}^1)$, e la definizione di polinomio di Taylor. Se $f : \mathbb{H}^1 \to \mathbb{R}$ e $p_0 \in \mathbb{H}^1$, in accordo con l'invarianza dei campi X ed Y, diciamo che f è derivabile lungo X (risp. Y) in p_0, e scriveremo $Xf(p_0)$ (risp. $Yf(p_0)$), se l'applicazione $\lambda \to f(p_0\, \delta_\lambda(1, 0, 0))$ (risp. $\lambda \to f(p_0\, \delta_\lambda(0, 1, 0))$) è derivabile in $\lambda = 0$. Diciamo dunque che $f \in C^1(\mathbb{H}^1)$ se Xf ed Yf esistono in ogni punto e sono continue in $\mathbb{H}^1$. Posto poi $X_1 = X$, $X_2 = Y$ e $X_3 = T$, diciamo *k-derivazione orizzontale* (risp. *k-derivazione*) un operatore differenziale del tipo $Z_{J_k} = X_{j_1} \cdots X_{j_k}$, con $J_k = (j_1, \ldots, j_k)$ multi indice e $j_i = 1, 2$ (risp. $j_i = 1, 2, 3$) per ogni $i = 1, \ldots, k$; diciamo dunque che $f \in C^n(\mathbb{H}^1)$ se esiste ed è continua $Z_{J_k} f$, per ogni k–derivazione orizzontale, con $0 \le k \le n$. Infine, dati $f \in C^n(\mathbb{H}^1)$ e $p \in \mathbb{H}^1$, un polinomio P in $\mathbb{H}^1$ di grado n dicesi polinomio di Taylor di f in p, se risulta $Z_{J_k}(P - f)(p) = 0$, per ogni k-derivazione orizzontale, con $0 \le k \le n$.

3. Polinomio di Taylor nel gruppo di Heisenberg ed applicazioni.

Allo scopo di ricavare esplicitamente la formula di Taylor in $\mathbb{H}^1$, supponiamo dapprima che $\mathbb{G}$ sia un gruppo di Lie analitico e che f sia definita ed analitica in un intorno dell'identità $e \in \mathbb{G}$: poichè operiamo con campi invarianti a sinistra, possiamo cercare, senza perdita di generalità, il polinomio di Taylor in e. Se $X, Y, T \in \mathfrak{g}$, l'algebra di $\mathbb{G}$, sussiste la seguente espansione in serie di Taylor (vedi per esempio [25], ed anche l'Appendice di [20]):

$$(1) \qquad f(x, y, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\left(xX + yY + tT\right)^k f\right](e)$$
$$= \sum_{k=0}^{\infty}\left[\sum_{\substack{n_1, n_2, n_3 \ge 0 \\ n_1+n_2+n_3=k}} \frac{\left(\frac{\sigma(X^{n_1}, Y^{n_2}, T^{n_3})}{(n_1+n_2+n_3)!} f\right)(e)}{n_1! n_2! n_3!}\, x^{n_1} y^{n_2} t^{n_3}\right],$$

dove il simbolo $\sigma(X^{n_1}, Y^{n_2}, T^{n_3})$ si definisce come segue.
Per ogni $n_1, n_2, n_3 = 0, 1, \ldots$, poniamo $X_1 = \cdots = X_{n_1} = X$, $X_{n_1+1} = \cdots = X_{n_1+n_2} = Y$, $X_{n_1+n_2+1} = \cdots = X_{n_1+n_2+n_3} = T$, e

$$\sigma(X^{n_1}, Y^{n_2}, T^{n_3}) = \sum_{\pi \in S_{n_1+n_2+n_3}} X_{\pi(1)} \cdots X_{\pi(n_1+n_2+n_3)},$$

dove $S_{n_1+n_2+n_3}$ è il gruppo di permutazioni su $\{1, \dots, n_1 + n_2 + n_3\}$. Avendo in mente il caso $\mathbb{G} = \mathbb{H}^1$, osserviamo che, nella (1), il grado del monomio $x^{n_1} y^{n_2} t^{n_3}$ che appare nella k-esima somma, è $n_1 + n_2 + 2n_3$, che in generale è diverso da k; è dunque naturale modificare la (1) come segue

$$f(x, y, t) = \sum_{k=0}^{\infty} \Bigg[\sum_{\substack{n_1, n_2, n_3 \geq 0 \\ n_1+n_2+2n_3=k}} \frac{\left(\frac{\sigma(X^{n_1}, Y^{n_2}, T^{n_3})}{(n_1+n_2+n_3)!} f\right)(0)}{n_1! n_2! n_3!} x^{n_1} y^{n_2} t^{n_3} \Bigg].$$

Le precedenti considerazioni suggeriscono il candidato n-esimo polinomio di Taylor di una data $f \in C^n(\mathbb{H}^1)$:

$$(P_{n,f}) \qquad P_{n,f}(x, y, t) = \sum_{k=0}^{n} \Bigg[\sum_{\substack{n_1, n_2, n_3 \geq 0 \\ n_1+n_2+2n_3=k}} \frac{\left(\frac{\sigma(X^{n_1}, Y^{n_2}, T^{n_3})}{(n_1+n_2+n_3)!} f\right)(0)}{n_1! n_2! n_3!} x^{n_1} y^{n_2} t^{n_3} \Bigg].$$

Si osservi subito che, se $\mathrm{Sym}(X^{n_1}, Y^{n_2}, T^{n_3})$ denota la somma di tutte le $(n_1 + n_2 + n_3)$-derivazioni contenenti ciascuna n_1 volte la X, n_2 volte la Y ed n_3 volte la T, è chiaro che

$$\mathrm{Sym}(X^{n_1}, Y^{n_2}, T^{n_3}) = \frac{\sigma(X^{n_1}, Y^{n_2}, T^{n_3})}{n_1! n_2! n_3!}.$$

Volendo ora riscrivere più esplicitamente $(P_{n,f})$ in termini di derivazioni, osserviamo dapprima che, per ogni $k = 0, 1, \dots$, nella potenza formale $(X+Y+T)^k$, è opportuno raccogliere insieme tutti gli addendi contenenti n_1 volte X, n_2 volte Y ed n_3 volte T, per i quali $n_1 + n_2 + 2n_3 = k$, ed interpretarli come un unica $(n_1+n_2+n_3)$–derivazione, precisamente $\frac{n_1! n_2! n_3!}{(n_1+n_2+n_3)!} \cdot \mathrm{Sym}(X^{n_1}, Y^{n_2}, T^{n_3})$; detta così $(n_1+n_2+n_3)$-*derivazione simmetrizzata di ordine* $n_1+n_2+n_3$, *contenente* n_1 *volte la* X, n_2 *volte la* Y *ed* n_3 *volte la* T, quella definita da

$$\frac{\partial^k}{\partial X^{n_1} \partial Y^{n_2} \partial T^{n_3}} = \frac{n_1! n_2! n_3!}{(n_1 + n_2 + n_3)!} \cdot \mathrm{Sym}(X^{n_1}, Y^{n_2}, T^{n_3}),$$

dove, in tale scrittura, k denota il grado di derivazione effettivo, possiamo riscrivere $(P_{n,f})$ come segue

$$P_{n,f}(x, y, t) = \sum_{k=0}^{n} \Bigg[\sum_{\substack{n_1, n_2, n_3 \geq 0 \\ n_1+n_2+2n_3=k}} \frac{\left(\frac{\partial^k}{\partial X^{n_1} \partial Y^{n_2} \partial T^{n_3}} f\right)(0)}{n_1! n_2! n_3!} x^{n_1} y^{n_2} t^{n_3} \Bigg].$$

Infine, osservando che, nel caso particolare di $\mathbb{H}^1$,

$$\mathrm{Sym}(X^{n_1}, Y^{n_2}, T^{n_3}) = \binom{n_1+n_2+n_3}{n_3} \cdot \mathrm{Sym}(X^{n_1}, Y^{n_2})\, T^{n_3}$$

poichè T appartiene al centro di $\mathfrak{h}^1$, possiamo anche scrivere

$$P_{n,f}(x,y,t) = \sum_{k=0}^{n} \Bigg[\sum_{\substack{n_1,n_2,n_3 \geq 0 \\ n_1+n_2+2n_3=k}} \frac{\left(\frac{\partial^{n_1+n_2}}{\partial X^{n_1}\partial Y^{n_2}} T^{n_3} f\right)(0)}{n_1!n_2!n_3!} x^{n_1} y^{n_2} t^{n_3} \Bigg].$$

Enunciamo dunque il risultato principale relativo alla formula di Taylor.

Teorema. *Sia $f \in C^n(\mathbb{H}^1)$. Sussistono i seguenti fatti equivalenti:*

i) $P_{n,f}$ è l'n-esimo polinomio di Taylor di f in zero;
ii) $P_{n,f}(p) - f(p) = o\big[d(p,0)\big]^n$ per $p \to 0$.

Sottolineiamo che, per verificare che $P_{n,f}$ è l'n-esimo polinomio di Taylor, facciamo ricorso ad argomentazioni di carattere puramente algebrico sui generatori lineari graduati dell'algebra libera $\mathbb{K}\langle X, Y\rangle$.

Concludiamo questo annuncio presentando una generalizzazione al caso $\mathbb{H}^1$ del classico teorema di H. Whitney sulla prolungabilità di funzioni regolari definite in un chiuso di $\mathbb{R}^n$ (vedi [26]), qui enunciato per funzioni $g \in C^2(F)$, $F \subseteq \mathbb{H}^1$ chiuso.

Qualche notazione. Sia $g : \mathbb{H}^1 \to \mathbb{R}$; se esistono Xg ed Yg, dicesi *gradiente orizzontale* di g la funzione vettoriale $\nabla g = (Xg, Yg)$; se poi esistono X_iX_jg, per ogni $i, j = 1, 2$, dicesi *Hessiana orizzontale* la funzione a valori in $\mathbb{R}^{2\times 2}$ $Hg = (X_iX_jg)_{i,j=1,2}$.

Teorema. (di estensione di Whitney) *Siano $F \subseteq \mathbb{H}^1$ un chiuso, e siano $f : F \to \mathbb{R}$, $\overline{\nabla} : F \to \mathbb{R}^2$, ed $\overline{H} = (\overline{H}_{ij})_{i,j=1,2} : F \to \mathbb{R}^{2\times 2}$ funzioni continue. Poniamo*

$$\overline{T} = -\frac{(\overline{H}_{12} - \overline{H}_{21})}{4}, \quad \overline{H}^{\mathrm{S}} = \frac{\overline{H} + \overline{H}^{\mathrm{T}}}{2},$$

$$R_0(p,q) = \frac{1}{[d(p,q)]^2} \Bigg\{ f(p) - \Bigg[f(q) + \sum_{i=1}^{2} \overline{\nabla}_i(q)(q^{-1}p)_i + $$
$$+ \frac{1}{2} \sum_{i,j=1}^{2} \overline{H}^{\mathrm{S}}_{ij}(q)(q^{-1}p)_i(q^{-1}p)_j + \overline{T}(q)(q^{-1}p)_3 \Bigg]\Bigg\},$$

e, per ogni $i = 1, 2$,

$$R_i(p, q) = \frac{\overline{\nabla}_i(p) - \overline{\nabla}_i(q) - \sum_{j=1}^{2} \overline{H}_{ji}(q)(q^{-1}p)_j}{d(p, q)}.$$

Supponiamo che per ogni compatto $C \subseteq F$ *e per ogni* $i = 0, 1, 2$, *posto* $\rho_i(\delta) = \sup\left\{|R_i(p, q)| : p, q \in C,\ 0 < d(p, q) < \delta\right\}$, *risulti* $\lim_{\delta \to 0^+} \rho_i(\delta) = 0$.

Allora, esiste $\overline{f} : \mathbb{H}^1 \to \mathbb{R}$, $\overline{f} \in C^2(\mathbb{H}^1)$, *tale che* $\overline{f}_{|_F} = f$, $(\nabla \overline{f})_{|_F} = \overline{\nabla}$, $(H\overline{f})_{|_F} = \overline{H}$.

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THIN INCLUSIONS IN AN ELASTIC BODY

ELENA BERETTA - ELISA FRANCINI

We consider a plane isotropic homogeneous elastic body containing a "thin" elastic inclusions in the form of a neighborhood of thickness 2ϵ of some line segment. We derive an asymptotic expansion of the boundary displacement field as $\epsilon \to 0$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain representing the region occupied by an elastic material. Suppose that this material contains an inclusion ω_ϵ, with different elastic properties, that can be represented as a small neighborhood of a line segment σ_0:

$$\omega_\epsilon = \{x \in \Omega \ : \ d(x, \sigma_0) < \epsilon\} .$$

Let $\mathbb{C}_0$ and $\mathbb{C}_1$ be the elastic tensor fields in $\Omega \setminus \overline{\omega}_\epsilon$ and ω_ϵ respectively.

Given a traction field $g : \partial\Omega \to \mathbb{R}^2$ on the boundary of Ω, the displacement field u_ϵ, generated by this traction in the body containing the inclusion ω_ϵ, solves the following system of linearized elasticity

$$(1) \qquad \begin{cases} \operatorname{div}\left(\mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon\right) = 0 & \text{in } \Omega \\ (\mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon) \cdot \nu = g & \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_\epsilon = 0, \quad \int_\Omega \left(\nabla u_\epsilon - (\nabla u_\epsilon)^T\right) = 0, & \end{cases}$$

where $\mathbb{C}_\epsilon = \mathbb{C}_0 \chi_{\Omega \setminus \omega_\epsilon} + \mathbb{C}_1 \chi_{\omega_\epsilon}$, $\widehat{\nabla} u_\epsilon = \frac{1}{2}\left(\nabla u_\epsilon + (\nabla u_\epsilon)^T\right)$ is the symmetric deformation tensor and ν denotes the outward unit normal to $\partial\Omega$.

Let us also introduce the background displacement u_0, namely the displacement field generated by the traction g in the body without the inclusion, that is the solution to

$$
\begin{cases}
\operatorname{div}\left(\mathbb{C}_0\widehat{\nabla}u_0\right)=0 & \text{in } \Omega\\
(\mathbb{C}_0\widehat{\nabla}u_0)\cdot\nu=g & \text{on } \partial\Omega,\\
\displaystyle\int_{\partial\Omega}u_0=0,\quad \int_{\Omega}\left(\nabla u_0-(\nabla u_0)^T\right)=0, &
\end{cases}
\tag{2}
$$

The goal of this investigation is to find an asymptotic expansion for $(u_\epsilon - u_0)_{|\partial\Omega}$ as $\epsilon \to 0$. An analogous expansion has been derived in [6] for the case of thin conductivity inclusions. These expansions represent a powerful tool to solve the inverse problem of identifying the inclusions from the knowledge of boundary measurements (see, [1] and [2] for the case of thin conductivity inclusions and [3] for further references). In [4] the authors derive an asymptotic expansion for the boundary displacement field $(u_\epsilon - u_0)_{|\partial\Omega}$ in the case of diametrically small inclusions, namely inclusions of the form $z+\epsilon B$, where z is a point in Ω and B is a bounded domain containing the origin. The approach they use does not seem to work for thin inclusions.

These are our main assumptions:

(H1) The segment σ_0 is far from $\partial\Omega$ and has positive length, i.e., there is a positive constant d_0 such that

$$d(\sigma_0,\partial\Omega)\geq d_0,\quad \text{and}\quad \text{length}(\sigma_0)\geq d_0.$$

(H2) Ω and ω_ϵ are both homogeneous and isotropic, i.e. the elastic tensor fields $\mathbb{C}_0$ and $\mathbb{C}_1$ are of the following form

$$(\mathbb{C}_m)_{ijlk}=\lambda_m\delta_{ij}\delta_{kl}+\mu_m(\delta_{ki}\delta_{lj}+\delta_{kj}\delta_{li}),\ \text{for } i,j,k,l=1,2,\ \ m=0,1,$$

where (λ_0,μ_0) and (λ_1,μ_1) are the Lamè coefficients corresponding to $\Omega\setminus\overline{\omega}_\epsilon$ and ω_ϵ, respectively, and $(\lambda_0-\lambda_1)^2+(\mu_0-\mu_1)^2\neq 0$.

(H3) There are two positive constants α_0 and β_0 such that

$$\min(\mu_0,\mu_1)\geq\alpha_0,\quad \min(2\lambda_0+2\mu_0,2\lambda_1+2\mu_1)\geq\beta_0.$$

We note that these last conditions ensure that $\mathbb{C}_\epsilon$ is strongly convex in Ω.

(H4) $g\in H^{-1/2}(\partial\Omega)$ satisfies the compatibility condition

$$\int_{\partial\Omega}g\cdot r=0,$$

for every infinitesimal rigid displacement $r(x) = c + Wx$ where c is a constant and W is a skew 2×2 matrix.

Let us introduce the Neumann function N related to Ω and to the tensor $\mathbb{C}_0$. For $y \in \Omega$, we will denote by $N(\cdot, y)$ the weak solution to the problem

$$\begin{cases} \operatorname{div}\left(\mathbb{C}_0 \widehat{\nabla} N(\cdot, y)\right) = -\delta_y \, \mathrm{I}_d & \text{in } \Omega, \\ \left(\mathbb{C}_0 \widehat{\nabla} N(\cdot, y)\right) \cdot \nu = -\dfrac{1}{|\partial\Omega|} \, \mathrm{I}_d & \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} N(\cdot, y) = 0, \quad \int_{\Omega} \left(\nabla N(\cdot, y) - (\nabla N(\cdot, y))^T\right) = 0, & \end{cases}$$

where I_d is the identity matrix in $\mathbb{R}^2$.

We are now ready to state our main result:

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $\sigma_0 \subset\subset \Omega$ be a line segment satisfying* (H1). *Assume* (H2), (H3) *and* (H4) *and let u_ϵ and u_0 be the solutions to (1) and (2) respectively. For every $x \in \sigma_0$, there exists a fourth order symmetric tensor field $\mathcal{M}(x)$ such that, for $y \in \partial\Omega$ and $\epsilon \to 0$*

$$(u_\epsilon - u_0)(y) = 2\epsilon \int_{\sigma_0} \mathcal{M}(x) \widehat{\nabla} u_0(x) \cdot \widehat{\nabla} N(x, y) \, d\sigma_0(x) + o(\epsilon).$$

The term $o(\epsilon)$ is bounded by $C\epsilon^{1+\theta}\|g\|_{H^{-1/2}(\partial\Omega)}$, with $0 < \theta < 1$ and C depending only on θ, Ω, α_0, β_0 and d_0.

Furthermore, for $x \in \sigma_0$ we can write

$$\mathcal{M}\widehat{\nabla} u_0 = a \, \mathit{div}\, u_0 \, \mathrm{I}_d + b \widehat{\nabla} u_0 + c \left(\frac{\partial(u_0 \cdot \tau)}{\partial \tau}\right) \tau \otimes \tau + d \left(\frac{\partial(u_0 \cdot n)}{\partial n}\right) n \otimes n,$$

where τ and n are the tangential and normal directions on σ_0 and

$$a = (\lambda_1 - \lambda_0)\frac{\lambda_0 + 2\mu_0}{\lambda_1 + 2\mu_1}, \quad b = 2(\mu_1 - \mu_0)\frac{\mu_0}{\mu_1},$$

$$c = 2(\mu_1 - \mu_0)\left[\left(\frac{2\lambda_1 + 2\mu_1 - \lambda_0}{\lambda_1 + 2\mu_1} - \frac{\mu_0}{\mu_1}\right)\right],$$

and

$$d = 2(\mu_1 - \mu_0)\frac{\mu_1\lambda_0 - \mu_0\lambda_1}{\mu_1(\lambda_1 + 2\mu_1)}.$$

An analogous result holds in the more general case of an inclusion that is an ϵ-neighborhood of a simple regular curve. The proof of this result can be found in [5].

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EINGENVALUE PROBLEMS AND GAUSS MEASURE

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We find some precise estimates for the first eigenvalue and for the corresponding eigenfunction of a class of elliptic equations whose prototype is $-\left(\gamma u_{x_i}\right)_{x_i} = \lambda\gamma u$ in $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary condition, where γ is the normalized Gaussian function in $\mathbb{R}^n$. To this aim we make use of the notion of rearrangement with respect to Gaussian measure.

We present some results contained in [4] where we find some optimal estimates for the first eigenfunction of a class of elliptic equations whose prototype is $-\left(\gamma u_{x_i}\right)_{x_i} = \lambda\gamma u$ in $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary condition, where γ is the normalized Gaussian function in $\mathbb{R}^n$. Before to illustrate such results we recall some known results for the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$. It is well known that the first eigenvalue of the problem (1) is positive, simple and minimizes the Rayleigh quotient, moreover the following Faber Krahn inequality holds: $\lambda_1 \geq \lambda_1^\sharp$ where $\lambda_1^\sharp$ is the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega^\sharp \\ u = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{2}$$

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where $\Omega^\sharp$ is the ball centered at the origin having the same measure as Ω. Various comparison results have been proved for the eigenfunction associated to the the first eigenvalue. For example if u is any eigenfunction of (1) corresponding to the first eigenvalue, then the following Payne Rayner inequality holds (see [11] for the case $n = 2$, [8]):

$$\|u\|_{L^r(\Omega)} \le K(r, q, n, \lambda)\, \|u\|_{L^q(\Omega)}$$

for any $0 < q < r < \infty$ where K is a suitable constant. The inequalities recalled are isoperimetric in the sense that equalities holds if and only if Ω is a ball. The previous results have been extended to more general linear and non linear problems in bounded domains making use of Schwarz symmetrization (see for instance [13], [8], [10], [2], and [1]).

In [4] we show that the properties we have recalled about the first eigenvalue and the corresponding eigenfunction of the problem (1), can be proved for the following eigenvalue problem

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a_{ij}(x)\dfrac{\partial u}{\partial x_i}\right) = \lambda\gamma(x)u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\gamma(x)$ is the normalized Gaussian function of $\mathbb{R}^n$ defined by

$$\gamma(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{|x|^2}{2}\right),$$

Ω is a domain of $\mathbb{R}^n$ ($n \ge 2$) such that

$$|\Omega|_\gamma = \int_\Omega \gamma(x)\, dx < 1, \tag{4}$$

$\left(a_{ij}(x)\right)_{ij}$ is an $n \times n$ symmetric matrix with measurable coefficients satisfying

$$\gamma(x)\,|\zeta|^2 \le a_{ij}(x)\zeta_i\zeta_j \le C\gamma(x)\,|\zeta|^2, \tag{5}$$

for some $C \ge 1$, for a.e. $x \in \Omega$ and for all $\zeta \in \mathbb{R}^n$, where, here and in the following, we adopt the summation convention.

We consider nontrivial solution u of (3) from the weighted Sobolev space $H_0^1(\Omega, \gamma)$ endowed with the norm

$$\|u\|_{H_0^1(\Omega,\gamma)} = \left(\int_\Omega |Du|^2\, \gamma(x)\, dx\right)^{\frac{1}{2}}, \tag{6}$$

The assumption (4) guarantees that the embedding of $H_0^1(\Omega, \gamma)$ into $L^2(\Omega, \gamma)$ is compact. This allows us to apply standard spectral theory and to find a sequence of eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ strictly increasing and unbounded. Furthermore, if λ_1 is the first eigenvalue of (3), then it minimizes the Rayleigh quotient

$$\lambda_1 = \min_{\substack{u \in H_0^1(\Omega,\gamma) \\ u \neq 0}} \frac{\int_\Omega a_{ij}(x) u_{x_i} u_{x_j}\, dx}{\int_\Omega u^2 \gamma(x)\, dx}.$$

Moreover, by adapting classical techniques, one can easily verify that λ_1 is simple and u does not change sign.

To get sharp estimates for the first eigenvalue λ_1 and for the corresponding eigenfunction u of (3), by the structure of the differential operator, and since Ω is allowed to be unbounded, we use Gaussian symmetrization. This kind of symmetrization transforms a given set Ω into an half space

$$\Omega^\sharp \equiv \left\{ x = (x_1, ..., x_n) \in \mathbb{R}^n : x_1 > a \right\} ,$$

where a is taken such that Ω and $\Omega^\sharp$ have the same Gaussian measure i.e.

$$|\Omega^\sharp|_\gamma = |\Omega|_\gamma \ .$$

The isoperimetric inequality for Gaussian measure states that (see [12], [5] and [9])

$$P_\gamma(\Omega) \geq P_\gamma(\Omega^\sharp)$$

and equality holds if and only if $\Omega = \Omega^\sharp$ modulo a rotation. Here $P_\gamma(\Omega)$ denotes the Gaussian perimeter of Ω that is, when Ω is sufficiently "nice",

$$P_\gamma(\Omega) = \int_{\partial\Omega} \gamma(x)\, dH^{n-1},$$

where H^{n-1} denotes the $(n-1)-$dimensional Hausdorff measure.

In order to describe our results let us introduce the rearrangement, with respect to Gaussian measure, of any measurable function u. To this end let μ be the distribution function of u defined by

$$\mu(t) \equiv |\{x \in \Omega : u(x) > t\}|_\gamma \,, \quad t \in \mathbb{R} \,,$$

let u^* be its decreasing rearrangement defined by

$$u^*(s) \equiv \inf\{t \in \mathbb{R} : \mu(t) \leq s\} \,, \quad s \in \left]0, |\Omega|_\gamma\right[\ ,$$

and finally let $u^\sharp$ be its Gaussian rearrangement which is the function having the same distribution function of u whose level sets are half-spaces. More precisely

$$u^\sharp(x) = u^*(k(x_1)), \quad x \in \Omega^\#,$$

where $k(\sigma)$ is the function

$$k(t) \equiv \frac{1}{\sqrt{2\pi}} \int_t^\infty \exp(-\sigma^2/2)\, d\sigma\,. \tag{8}$$

The variational characterization together with the Pólya-Szegö principle with respect to Gaussian measure (see [9], [14] and [6]) allow to get a "Faber-Krahn type inequality"

Theorem 1. *If λ_1 is the first eigenvalue of problem* (3) *then*

$$\lambda_1 \geq \lambda_1^\sharp\,, \tag{9}$$

where $\lambda_1^\sharp$ is the first eigenvalue of the problem

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left(\gamma(x)\dfrac{\partial u}{\partial x_i}\right) = \lambda\gamma(x)u & \text{in } \Omega^\sharp \\ u = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{10}$$

Furthermore $\lambda_1 = \lambda_1^\sharp$ if and only if, modulo a rotation, $\Omega = \Omega^\sharp$ and

$$a_{i1}(x) = \gamma(x)\delta_{1i}, \quad \text{a.e. } x \in \Omega\,.$$

In [4] we prove that any eigenfunction u of (3) corresponding to λ_1 satisfies the following "Payne Rayner type inequality"

Theorem 2. *Let λ_1 be the first eigenvalue of problem* (3) *and let u be any eigenfunction associated with it. Then for $0 < q < r < \infty$, we have*

$$\|u\|_{L^r(\Omega,\gamma)} \leq \beta(r, q, \lambda_1)\, \|u\|_{L^q(\Omega,\gamma)}\,, \tag{11}$$

where $\beta(r, q, \lambda_1)$ is a constant depending only on r, q and λ_1. Furthermore, equality in (11) *occurs if and only if, modulo a rotation, $\Omega = \Omega^\#$, $u = u^\#$ and $a_{i1}(x) = \gamma(x)\delta_{1i}$, up to a set of measure zero.*

The first step in the proof of Theorem 2 (see [4]) is a comparison result between the first eigenfunction of problem (3) and the first eigenfunction of a suitable "symmetrized problem". Let u be a nonnegative eigenfunction of (3) corresponding to λ_1 and let $S_a = \{x \in \mathbb{R}^n : x_1 > a\}$ be the half-space such that λ_1 is also the first eigenvalue of the problem

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left(\gamma(x)\dfrac{\partial w}{\partial x_i}\right) = \lambda\gamma(x)w & \text{in } S_a \\ w = 0 & \text{on } \partial S_a. \end{cases} \tag{12}$$

Observe that such an half-space always exists since the function $\lambda_1 : a \in \mathbb{R} \to \lambda_1(a) \in \,]0, \infty[\,$, where $\lambda_1(a)$ is the first eigenvalue of problem (12), is a bijection.

Then the following comparison result holds:

Theorem 3. *Let u be an eigenfunction of problem* (3) *corresponding to the first eigenvalue λ_1 and let w be the eigenfunction of* (12) *associated with λ_1 such that*

$$\|u\|_{L^1(\Omega,\gamma)} = \|w\|_{L^1(S_a,\gamma)} \ , \tag{13}$$

then

$$\int_0^s u^*(\sigma)\, d\sigma \leq \int_0^s w^*(\sigma)\, d\sigma, \quad \forall s \in \big[0, |S_a|_\gamma\big] \ . \tag{14}$$

Moreover the above inequality reduces to equality if and only if, modulo a rotation, $\Omega = S_a$, $u(x) = u^\sharp(x) = w(x)$ and $a_{i1}(x) = \gamma(x)\delta_{1i}$, up to a set of measure zero.

Let us observe that as a consequence of the simplicity of λ_1 we have that (12) is actually a one-dimensional problem. Moreover, by standard theory on hypergeometric functions (see [15] for instance) it follows that w has the following asymptotic behavior

$$w(x) \propto x_1^{\lambda_1}\left(1 + O(x_1^{-2})\right) \ . \tag{15}$$

Therefore by (14) we can deduce information on the summability of u, comparing the L^p norms of u and w, with $1 \leq p < \infty$. More precisely ther following result holds

Corollary 4. *Let u be an eigenfunction of* (3) *corresponding to the first eigenvalue λ_1. Then $u \in L^q(\Omega, \gamma)$ for $1 \leq q < \infty$ and*

$$\|u\|_{L^q(\Omega,\gamma)} \leq \|w\|_{L^q(S_a,\gamma)}\,, \quad 1 \leq q < \infty\,. \tag{16}$$

Corollary 4 allows us to consider the eigenfunction w_q of (12), with $1 \le q < \infty$, such that

$$\|u\|_{L^q(\Omega,\gamma)} = \left\|w_q\right\|_{L^q(S_a,\gamma)}\,, \tag{17}$$

and to prove the following result.

Theorem 5. *Let u and w_q be defined as above, with $1 \le q < \infty$. Then*

$$\int_0^s \left[u^*(\sigma)\right]^q \, d\sigma \le \int_0^s \left[w_q^*(\sigma)\right]^q \, d\sigma, \quad \forall s \in \left[0, |S_a|_\gamma\right]. \tag{18}$$

Moreover, if any of above inequalities reduces to equalities, then, modulo a rotation and up to a set of measure zero, $\Omega = S$, $u(x) = u^\sharp(x) = w_q(x)$ and $a_{i1}(x) = \gamma(x)\delta_{1i}$.

Let us observe that the reverse Hölder inequality (11), using Theorem 5, follows with

$$\beta(r, q, \lambda_1) = \frac{\left\|w_q\right\|_{L^r(S_a,\gamma)}}{\left\|w_q\right\|_{L^q(S_a,\gamma)}}.$$

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REGULARITY FOR DEGENERATE ELLIPTIC EQUATIONS UNDER MINIMAL ASSUMPTIONS

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1. Introduction.

Let $X_1, \dots, X_m$ be a system of non-commuting Hörmander vector fields in $\mathbb{R}^n$ $(m \leq n)$ and let X_j^* be the formal adjoint of the vector field X_j. We study the local regularity of the generalized solution to the Dirichlet problem associated to the equation

$$Lu \equiv X_i^*(a_{ij} X_j u) = f\,, \tag{1.1}$$

under minimal assumptions on the function f. The case when f is a measure follows with minor changes from our statements. Under our assumptions, weak solutions do not always exist so we need to define a "very weak solution" (see e.g. [6] for the Euclidean case). The case of lower order terms is discussed in [2]. Our results generalizes the case of uniformly elliptic equations considered in [4] (see also [5]).

2. Preliminaries.

In this section we collect all the relevant definitions in order to formulate our results. For precise definitions and proofs see [1]. Let $\mathrm{X} = (X_1, X_2, \dots, X_m)$ be a given system of C^∞ vector fields on $\mathbb{R}^n$ satisfying

Hörmander condition in a bounded domain Ω, i.e. $rank\, Lie\{X_1, X_2, \ldots, X_m\} = n$, $\forall x \in \Omega$. We denote by $\rho(x, y)$ the Carnot–Caratheodory distance generated by the system X and $B_r \equiv B(x, r) = \{y \in \mathbb{R}^n \,:\, \rho(x, y) < r\}$ the metric ball centered at x of radius r. Let Q be the homogenous dimension of Ω. We briefly recall the function spaces we need to formulate our results.

Definition 2.1. (Sobolev spaces) Let $1 \leq p < +\infty$. We say that u belongs to $W^{1,p}(\Omega, \mathrm{X})$ if u and $X_j u$ belong to $L^p(\Omega)$, $j = 1, 2, \ldots, m$. We set

$$\|u\|_{W^{1,p}(\Omega,\mathrm{X})} \equiv \|u\|_{L^p(\Omega)} + \sum_{j=1}^{m} \|X_j u\|_{L^p(\Omega)}. \tag{2.2}$$

We denote by $W_0^{1,p}(\Omega, \mathrm{X})$ the completion of $C_0^\infty(\Omega)$ with respect to the above norm.

Remark 2.2. $X_j u$ denotes the distributional derivative of u defined by

$$< X_j u, \phi >= \int_\Omega u X_j^* \phi \, dx, \qquad \forall\ \phi \in C_0^\infty(\Omega)$$

where $X_j^* = -\sum_{i=1}^n \partial_i (c_{ij}\ \cdot)$ is the formal adjoint of $X_j = \sum_{i=1}^n c_{ij} \partial_i$.

Definition 2.3. (Schechter classes) Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ and let $1 \leq p < \infty$. We say that $u \in L^1(\Omega)$ belongs to the Schechter class $M_p(\Omega, \mathrm{X})$ if

$$M_p(u) \equiv \left(\int_\Omega \left(\int_{B_\delta(x)\cap\Omega} |u(y)| \frac{\rho^2(x, y)}{|B(x, \rho(x, y))|} \, dy \right)^p dx \right)^{\frac{1}{p}} < \infty$$

for some $\delta > 0$.

Definition 2.3. (Stummel-Kato class) Let $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$. If

$$\eta(r) \equiv \sup_{x\in\Omega} \int_{\{y\in\Omega | \rho(x,y)<r\}} |u(y)| \frac{\rho^2(x, y)}{|B(x, \rho(x, y))|} \, dy < \infty\,,$$

we say that $u \in \tilde{S}(\Omega, \mathrm{X})$. If $\lim_{r\to 0^+} \eta(r) = 0$ we say that $u \in S(\Omega, \mathrm{X})$.

Definition 2.5. (Morrey classes) Let Ω be a bounded domain in $\mathbb{R}^n$, $1 \leq p < \infty$ and $\lambda > 0$. We say that $f \in L^p_{loc}(\Omega)$ belongs to the Morrey class $L^{p,\lambda}(\Omega, X)$ if

$$\|f\|_{p,\lambda} \equiv \sup_B \left(\frac{r_B^\lambda}{|B|} \int_B |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty \, .$$

We can compare Morrey and Lebesgue classes as follows.

Proposition 2.6. *Let $q \geq p$ and $\frac{\mu}{q} \leq \frac{\lambda}{p}$. Then*

$$L^{q,\mu}(\Omega, X) \subseteq L^{p,\lambda}(\Omega, X) \, .$$

Definition 2.7. (Weak Morrey classes) We say that $f \in L^{p,\lambda}_w(\Omega, X)$ if there exists $C > 0$, independent on r and x_0, such that

$$\sup_{t>0} t^p |\{x \in \Omega \cap B_r(x_0) : |f(x)| > t\}| \leq C \, \frac{|B_\rho|}{\rho^\lambda} \, .$$

Proposition 2.8. *Let $1 \leq q < p < \infty$ and $0 < \lambda < Q$, then*

$$L^{p,\lambda}_w(\Omega, X) \subseteq L^{q,\mu}(\Omega, X)$$

where $\mu = \frac{Q-\lambda}{p} \, q$.

Proposition 2.9. *Let $0 < \lambda < 2 \leq \mu < Q$. We have*

$$L^{1,\lambda}(\Omega, X) \subseteq S(\Omega, X) \subseteq \tilde{S}(\Omega, X) \subseteq L^{1,\mu}(\Omega, X).$$

Proposition 2.10. *Let $\Omega \subseteq \mathbb{R}^n 4$, $n \geq 3$ be a bounded domain and $1 \leq p < q \leq \infty$.Then*

$$\tilde{S}(\Omega, X) \subset M_q(\Omega, X) \subset M_p(\Omega, X) \subset M_1(\Omega, X).$$

We have

Proposition 2.11. *Let $0 < \lambda < 2 < Q$. Then*

$$L^{1,\lambda}(\Omega, X) \subseteq \tilde{S}(\Omega, X) \subseteq L^{1,2}(\Omega, X) \subseteq \bigcap_{1 \leq p < \infty} M_p(\Omega, X).$$

Definition 2.12. We say that $f \in L^1_{loc}(\Omega)$ belongs to the space $BMO(\Omega, X)$ if

$$\sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty \, ,$$

where B ranges over the set of metric balls contained in Ω.

3. Generalized solutions.

We consider the equation (1.1) under the assumptions in the previous section. We also assume that $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$ for $i, j = 1, 2, \ldots, m$ and there exist $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^m,\ \text{a.e. } x \in \Omega\,.$$

Definition 3.1. (see e.g. [6]) Let μ be a measure of bounded variation in Ω. We say that $u \in L^1(\Omega)$ is a very weak solution of $Lu = \mu$ (vanishing on $\partial\Omega$), if

$$< L^*v, u >= \int_\Omega v\, d\mu,\ \forall\ v \in H_0^1(\Omega, \mathrm{X}) \cap C_b^0(\Omega)\,,\ L^*v \in C_b^0(\Omega), \tag{3.3}$$

where $C_b^0(\Omega)$ is the set of all continuous functions, bounded on Ω.

We have

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then there exists the very weak solution of $Lu = \mu$ and it is unique. Moreover if $1 < p < \frac{Q}{Q-1}$ then $u \in {W_0}^{1,p}(\Omega, \mathrm{X})$ and there exists $C = C(\Omega, \lambda, Q)$ such that $\|u\|_{W_0^{1,p}} \leq C\|\mu\|$.*

In general if μ is not in $(H_0^1(\Omega, \mathrm{X}))^*$ then weak solutions do not exist. However, when weak and very weak solutions of the Dirichlet problem exist, they coincide.

We define the Green function for the operator L and the domain Ω with pole at $y \in \Omega$, as the very weak solution of $LG^y = \delta_y$.

4. Regularity of generalized solutions.

Let Ω be a bounded domain with smooth boundary without characteristic points. We stress that the existence of such domains is not trivial to show (see e.g. [7]).

Theorem 4.1. *Let $f \in L^{1,\lambda}(\Omega, \mathrm{X})$ and let u be the very weak solution of $Lu = f$.*

– *if $2 < \lambda \leq Q$, then $u \in L_w^{p_\lambda,\lambda}(\Omega, \mathrm{X})$, where $\frac{1}{p_\lambda} = 1 - \frac{2}{\lambda}$. Moreover, there exists $C > 0$, independent of u and f, such that $\|u\|_q \leq C\|f\|_{1,\lambda}$;*

– *if* $\lambda = 2$*, then* $u \in \mathrm{BMO}_{loc}(\Omega, \mathrm{X})$ *i.e. there exists* $r_0 > 0$ *such that* $\forall\ \Omega' \Subset \Omega$ *with* $d := \rho(\Omega', \partial\Omega) < r_0$*,* $x_0 \in \Omega'$ $\forall\ 0 < r < d/2$ *we have*

$$\fint_{B_r(x)} |u(y) - u_{B_r(x)}|\, dy \leq C\,,$$

where C *is independent on* u *and on the ball.*
– *if* $0 < \lambda < 2$*, then* $u \in C^{0,\beta}(\Omega, \mathrm{X})$.

Theorem 4.2. *Let* u *be the weak solution of* $Lu = f$.

– *if* $f \in \tilde{S}(\Omega)$ *then* u *is bounded in* Ω*;*
– *if* $f \in S(\Omega, X)$*, then* u *is continuous in* Ω.

We stress that, because of the inclusion $\tilde{S}(\Omega, \mathrm{X}) \subset (H_0^1(\Omega, \mathrm{X})^*$, the solution we consider is the weak one.

If we add a signum restriction on the function f we have

Theorem 4.3. *Let* $u \in L^1(\Omega)$ *the very weak solution of* $Lu = f$ *and let* $f \geq 0$.

– $u \in L^q_{loc}(\Omega)$ *iff* $f \in M^q_{loc}(\Omega, \mathrm{X})$*,* $1 < q < \infty$*;*
– $u \in L^\infty(\Omega)$ *iff* $f \in \tilde{S}_{loc}(\Omega, \mathrm{X})$*;*
– $u \in C^0(\Omega)$ *iff* $f \in S_{loc}(\Omega, \mathrm{X})$.

Theorem 4.4. *Let* $f \in L^1(\Omega)$ $f \geq 0$ *and let* u *be the very weak solution of* $Lu = f$.
If $u \in C^{0,\alpha}(\Omega, \mathrm{X})$ *with* $0 < \alpha < 2$ *then* $f \in L^{1,\alpha}_{loc}(\Omega, \mathrm{X})$.

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GAUSSIAN BOUNDS FOR HEAT KERNELS IN THE SETTING OF HÖRMANDER VECTOR FIELDS

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Let $X_1, X_2, \dots, X_q$ be a family of Hörmander's vector fields in $\mathbb{R}^n$. A systematic study of the properties of nonvariational operators of the kinds

$$L = \sum_{i,j=1}^{q} a_{ij}(x)\, X_i X_j$$

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x)\, X_i X_j$$

has begun in recent years. Here the matrix $\{a_{ij}\}$ is symmetric positive definite, and its entries are functions satisfying minimal smoothness assumptions. In this context, we will discuss and announce some recent results regarding the existence of a fundamental solution h of H and sharp Gaussian bounds for h.

Let us consider a system of smooth real vector fields, defined in a domain $\Omega \subseteq \mathbb{R}^n$

$$X_i = \sum_{j=1}^{n} b_{ij}(x)\, \partial_{x_j} \quad i = 1, 2, \dots, q \quad (q \leq n)$$

and assume they satisfy Hörmander's condition (of step s) in Ω: the vector space spanned at every point of Ω by: the fields X_i; their commutators

$[X_i, X_j] = X_iX_j - X_jX_i$; the commutators of the X_k's with the commutators $[X_i, X_j]$;...and so on, up to some step s, is the whole $\mathbb{R}^n$.

Under these assumptions, it is known (Hörmander [11]) that the second order differential operator "Hörmander' sum of squares"

$$L = \sum_{i=1}^{q} X_i^2$$

is hypoelliptic in Ω, that is: if $Lu = f$ in Ω in distributional sense, and $f \in C^\infty(A)$ with $A \subset \Omega$, then $u \in C^\infty(A)$. Analogously, the evolution operator

$$H = \partial_t - \sum_{i=1}^{q} X_i^2 \tag{1}$$

is hypoelliptic in $\mathbb{R} \times \Omega$. Roughly speaking, Hörmander's theorem says that an operator with nonnegative characteristic form, even though degenerate, still shares some good properties of nondegenerate elliptic or parabolic operators, whenever the "missing directions" in the derivatives involved in the equation are recovered by the commutators of the vector fields. The most famous and simple instance of this situation is the following:

Example 1. Kohn's Laplacian in 3 variables (x, y, z):

$$L = X_1^2 + X_2^2$$

$$X_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}; \quad X_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}; \quad [X_1, X_2] = -4\frac{\partial}{\partial z}.$$

The vector fields X_1, X_2, $[X_1, X_2]$ span $\mathbb{R}^3$ at any point: Hörmander's condition holds; the operators L in $\mathbb{R}^3$ and $\partial_t - L$ in $\mathbb{R}^4$ are hypoelliptic.

There are two kinds of structures typically associated to a set of Hörmander's vector fields. The first is a metric structure: whenever Hörmander's condition holds, it is possible to join any two points of the space by arcs of integral curves of the vector fields (Rashevski-Chow's Theorem, 1938, 1939). Then, the minimal length of these "piecewise integral curves" defines a distance between the two points, called Carnot-Carathéodory distance, or "distance induced by the vector fields". A relevant fact proved by Fefferman-Phong [7] is that Lebesgue's measure is locally doubling with respect to this distance:

$$|B(x, 2r)| \leq c\,|B(x, r)|$$

at least for x ranging in a compact set and $r \leq r_0$. This fact allows to adapt many typical arguments from real analysis to this context, in the spirit of Coifman-Weiss' theory of spaces of homogeneous type [6].

A second structure is of algebraic nature. In several important instances of systems of Hörmander vector fields in $\mathbb{R}^n$ (but not always!) the space $\mathbb{R}^n$ happens to be endowed with a "Carnot group" structure, that is: a Lie group operation ("translation"):

$$(x, y) \mapsto x \circ y$$

and a family of group automorphisms ("dilations"), of the kind:

$$x \mapsto D(\lambda)x = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n) \tag{2}$$

(α_i positive integers) such that the vector fields X_i are translation left invariant

$$X_i^x [f(y \circ x)] = \left(X_i^x f\right)(y \circ x)$$

and homogeneous of degree 1:

$$X_i^x [f(D(\lambda)x)] = \lambda \left(X_i^x f\right)(D(\lambda)x).$$

Then, Folland [9] has proved that L has a fundamental solution of kind $\Gamma(x, y) = \Gamma(y^{-1} \circ x)$, homogeneous of degree $2 - Q$, where $Q = \sum_{i=1}^n \alpha_i$ (see (2)) is the "homogeneous dimension" of the group:

$$\Gamma(D(\lambda)x) = \lambda^{2-Q} \Gamma(x)$$

This is the starting point in order to apply to this context results from the theory of singular and fractional integrals in spaces of homogeneous type. For the "parabolic" operator H one has the analogous behavior, where the vector field ∂_t is homogeneous of degree 2, like in the classical parabolic case.

We now come to our main topic, that is Gaussian bounds for the fundamental solution of "heat-type" operators. For operators of the kind (1) with left invariant homogeneous vector fields on a Carnot group in $\mathbb{R}^n$, Varopoulos ([15], [16], see also [17]) has proved the following Gaussian bounds for the fundamental solution h:

$$\frac{c_1}{t^{Q/2}} e^{-\|x^{-1} \circ y\|^2 / c_2 t} \leq h(t, x, y) \leq \frac{c_3}{t^{Q/2}} e^{-\|x^{-1} \circ y\|^2 / c_4 t}$$

$\forall x, y \in \mathbb{R}^n, t > 0$, where $\|\cdot\|$ is the "homogeneous norm" (with respect to dilations (2)), so that $\left\|x^{-1} \circ y\right\|$ is the distance in the group, equivalent to the distance induced by the vector fields. This bound is perfectly analogous to the classical one which holds for standard parabolic operators.

For the operators of the same form, but without an underlying group structure (that is, assuming that $\{X_1, X_2, ..., X_q\}$ is *any* system of Hörmander's vector fields), the analogous result is:

$$c_1 \frac{e^{-d(x,y)^2/c_2 t}}{\left|B\left(x, \sqrt{t}\right)\right|} \leq h\left(t, x, y\right) \leq c_3 \frac{e^{-d(x,y)^2/c_4 t}}{\left|B\left(x, \sqrt{t}\right)\right|}$$

$\forall x, y \in \mathbb{R}^n$, $t \in (0, \infty)$, where $d(x, y)$ is the distance induced by the vector fields, and $B(x, r)$ the metric ball. This result has been accomplished in several steps, by several authors, in two group of papers: Sanchez-Calle [14], Fefferman, Sanchez-Calle [8], and Jerison, Sanchez-Calle [10], with analytical techniques; Kusuoka-Stroock [12], [13], by means of stochastic techniques and Malliavin calculus.

In recent years, nonvariational operators of the kind

$$H_A = \partial_t - \sum_{i,j=1}^{q} a_{ij}\left(t, x\right) X_i X_j \tag{3}$$

with X_i left invariant homogeneous Hörmander's vector fields on a Carnot group, and

$$\lambda\left|\xi\right|^2 \leq \sum_{i,j=1}^{q} a_{ij}(t, x)\,\xi_i \xi_j \leq \lambda^{-1}\left|\xi\right|^2 \quad \forall \xi \in \mathbb{R}^q \tag{4}$$

with coefficients $a_{ij} = a_{ji}$, Hölder continuous with respect to the "parabolic distance" induced by the group structure, have been studied by Bonfiglioli-Lanconelli-Uguzzoni, who have proved the existence of a fundamental solution h_A satisfying the following bounds:

$$\frac{c_1}{t^{Q/2}} e^{-\left\|x^{-1} \circ y\right\|^2/c_2 t} \leq h_A\left(t, x, y\right) \leq \frac{c_3}{t^{Q/2}} e^{-\left\|x^{-1} \circ y\right\|^2/c_4 t}$$

$\forall x, y \in \mathbb{R}^n, t > 0$. (When $A = I$, this is Varoupulos' estimate). The result has been accomplished in several steps, see [1], [2], [3]. An application of these Gaussian estimates is the proof of an invariant Harnack inequality for the operator H_A, which is carried out by Bonfiglioli-Uguzzoni in [4]. Note that operators of kind (3) do not have smooth coefficients; hence they are not hypoelliptic, and the mere existence of a fundamental solution is not trivial.

We are therefore led to consider the more general case of operators of type (3), when $\{X_1, X_2, ..., X_q\}$ is *any* system of Hörmander's vector fields and the coefficients $a_{ij} = a_{ji}$, are Hölder continuous with respect to the parabolic distance induced by vector fields, and satisfy the ellipticity condition (4). These operators have been recently studied by Bramanti, Brandolini, Lanconelli, Uguzzoni [5]; our main result consists in showing the existence of a fundamental solution h_A, satisfying Gaussian bounds of the kind:

$$c_1 \frac{e^{-d(x,y)^2/c_2 t}}{\left|B\left(x, \sqrt{t}\right)\right|} \leq h_A(t, x, y) \leq c_3 \frac{e^{-d(x,y)^2/c_4 t}}{\left|B\left(x, \sqrt{t}\right)\right|}$$

for any $x, y \in \mathbb{R}^n$, $t \in (0, T)$. (When $A = I$, this is the result of Jerison-Sanchez-Calle or Kusuoka-Stroock). We assume that H_A coincides with the heat operator outside a large compact set, so in some sense our result is of local nature.

Following the general strategy adopted by Bonfiglioli-Lanconelli-Uguzzoni in the case of Carnot groups, to get our results we first consider the operator H_A with a constant matrix $\{a_{ij}\}$, in a fixed ellipticity class. This is an operator with smooth coefficients which is hypoelliptic, and by known results possesses a global fundamental solution h_A; our first, and more difficult, task is to prove a number of uniform estimates on h_A, depending on the constant coefficients a_{ij} only through the ellipticity constant λ in (4). More precisely, we prove the following uniform bounds:

1. Upper and lower bounds on h_A:

$$\frac{c_3}{\left|B\left(x, \sqrt{t}\right)\right|} e^{-d(x,y)^2/c_4 t} \leq h_A(t, x, y) \leq \frac{c_1}{\left|B\left(x, \sqrt{t}\right)\right|} e^{-d(x,y)^2/c_2 t}$$

2. Upper bounds on the derivatives of h_A:

$$\left|X_x^I X_y^J \partial_t^i h_A(t, x, y)\right| \leq \frac{c_1}{t^{i+\frac{|I|+|j|}{2}} \left|B\left(x, \sqrt{t}\right)\right|} e^{-d(x,y)^2/c_2 t}$$

3. Estimate on the difference of the fundamental solutions of two operators (and their derivatives):

$$\left|X_x^I X_y^J \partial_t^i h_A(t, x, y) - X_x^I X_y^J \partial_t^i h_B(t, x, y)\right| \leq$$

$$\leq \|A - B\| \frac{c_1}{t^{i+\frac{|I|+|j|}{2}} \left|B\left(x, \sqrt{t}\right)\right|} e^{-d(x,y)^2/c_2 t}$$

for any multiindeces I, J (where $X^I = X_{i_1} X_{i_2} ... X_{i_r}$ if $I = (i_1, i_2, ..., i_r)$ and $|I| = r$).

These estimates, by a suitable adaptation of the classical Levi's parametrix method, enable us to prove the existence of the fundamental solution for the operator H_A with variable (Hölder continuous) coefficients, and to deduce the desired Gaussian bounds for it.

The line of the proofs of these uniform estimates is complex and cannot be summarized here. The reader is referred to [5].

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SOME REMARKS ON THE EXTINCTION TIME FOR THE MEAN CURVATURE FLOW

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1. Introduction.

Let us consider a family of bounded open sets $(\Omega_t)_{t\geq 0}$ in $\mathbb{R}^n$ $(n \geq 2)$ and set $\Gamma_t = \partial\Omega_t$. If Γ_t is a smooth $(n-1)$-dimensional hypersurface it is said to be moving by mean curvature if the following initial value problem is satisfied

$$\begin{cases} V = H & \text{on } \Gamma_t \\ (\Gamma_t)_{t=0} = \Gamma_0\,, \end{cases} \tag{1.1}$$

where $V(x,t)$ and $H(x,t)$ denotes respectively the inward normal velocity and $(n-1)$ times the mean curvature of Γ_t at a point $x \in \Gamma_t$.

It is well known (see [9] for smooth convex, and [4] for general continuous hypersurfaces) that Γ_t shrinks to a point in a finite time t^* defined as

$$t^* = t^*(\Gamma_0) = \inf\{t : \Gamma_t \neq \emptyset\}$$

and called extinction time. The simplest upper bound estimate for t^* relies on a monotonicity property of the mean curvature equation according to which, given two sets Ω_0 and D_0 in $\mathbb{R}^n$ such that $\overline{\Omega_0} \subset D_0$, the inclusion remains true during the whole evolution of their boundaries: $\overline{\Omega_t} \subset D_t$. Therefore, denoting by d_0 the diameter of Ω_0, since Ω_0 lies in a ball of radius $R = \left(\frac{n}{2(n+1)}\right)^{1/2} d_0$, by the monotonicity it follows that t^* can be estimated with the extinction time of a ball of radius R, that is

$$0 \le t^* \le \frac{n}{4(n^2-1)} d_0^2. \tag{1.2}$$

This estimate is not sharp and it has been refined in [6], where the authors have proved that

$$0 \le t^* \le C \left(\mathcal{H}^{n-1}(\Gamma_0)\right)^{2/n-1}. \tag{1.3}$$

Here $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure and the constant $C = C(n)$ comes from a Sobolev type inequality on manifolds whose best constant is still unknown (see [10]). In this paper we will prove a sharp upper bound for t^* involving the $n-$dimensional measure of Ω_0 rather than the $(n-1)$-dimensional measure of its surface. More precisely we will show that the extinction time of Γ_0 can be estimated from above by the extinction time of the ball having the same volume as Ω_0. The sharpness of our estimate relies on an isoperimetric inequality involving the total mean curvature of mean convex sets (see Section 2 for definitions). For this reason our upper bound holds true in the case of general bounded convex sets and smooth mean convex sets.

2. Notation and Preliminaries.

We begin by recalling some definitions and properties of rearrangements of functions. Let Ω be a bounded open set of $\mathbb{R}^n$ and let $u : \Omega \to]-\infty, 0]$ be a measurable function. We denote by

$$\mu(\theta) = \mathcal{L}^n(\{x \in \Omega : u(x) < \theta\}), \quad \theta \le 0,$$

the distribution function of u, where $\mathcal{L}^n$ will denote here and in what follows the Lebesgue measure in $\mathbb{R}^n$, and by

$$u^*(s) = \sup\{\theta \le 0 : \mu(\theta) < s\}, \quad s \in (0, |\Omega|)$$

the increasing rearrangement of u. In the following we will denote by $\Omega^\#$ the ball centered at the origin having the same measure as Ω and by $u^\#$ the negative spherically symmetric increasing function whose level sets are balls having the same measure as the corresponding level sets of u. This means

$$u^\#(x) = u^*(\omega_n |x|^n), \quad x \in \Omega^\#,$$

where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. If Ω has a C^2 boundary, then the principal curvatures (oriented so that convex sets have non-negative curvatures) will be denoted by $k_1, ..., k_{n-1}$ and $(n-1)$ times the mean curvature will be denoted by $H[\partial\Omega]$, that is

$$H[\partial\Omega] = k_1 + k_2 + ... + k_{n-1}.$$

According to [13] we will say that a domain Ω is mean convex or $1-$convex if and only if $H[\partial\Omega] \geq 0$.

If $u : \Omega \to]-\infty, 0]$ is a function whose level sets $\{x \in \Omega : u(x) = \theta\}$ have finite perimeter and are 1-conex, then we denote by

$$\lambda(\theta) = \mathcal{H}^{n-1}(\{x \in \Omega : u(x) = \theta\}), \quad \theta \leq 0,$$

and we define the rearrangement of u with respect to the perimeter of its level sets as

$$u_1^*(s) = \sup\{\theta \leq 0 : \lambda(\theta) < s\}, \quad s \in (0, \mathcal{H}^{n-1}(\partial\Omega)).$$

We will denote by $\Omega^\bigstar$ the ball centered at the origin having the same perimeter as Ω and set

$$u^\bigstar(x) = u_1^*(n\omega_n|x|^{n-1}), \quad x \in \Omega^\bigstar.$$

We explicitly remark that $u^\bigstar$ is a negative spherically symmetric increasing function whose level sets are balls having the same perimeter as the corresponding level sets of u.

It easily follows from the above definitions and the classical isoperimetric inequality that

$$H[\{u^\bigstar = \theta\}] \geq H[\{u^\# = \theta\}]. \tag{2.1}$$

Finally we recall the following Alexandrov-Fenchel inequality involving the total mean curvature of level sets of a function u (see [2], [12]).

Theorem 2.1. *Let u be a nonpositive measurable function having mean convex level sets; then*

$$\int_{u=\theta} H[\{u = \theta\}]\, d\mathcal{H}^{n-1} \geq \int_{u^\bigstar=\theta} H[\{u^\bigstar = \theta\}]\, d\mathcal{H}^{n-1}, \quad \theta < 0. \tag{2.2}$$

3. A sharp estimate of the extinction time.

We first recall a different approach to motion by mean curvature first proposed by Osher and Sethian in a numerical framework (see [11]) and then studied by Evans and Spruck in [4].
Let Ω be a bounded open set in $\mathbb{R}^n$ and let us choose a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Gamma_0 = \partial\Omega = \{x \in \mathbb{R}^n : \ f(x) = 0\}.$$

In the following parabolic problem

$$\begin{cases} w_t = |Dw|\mathrm{div}\left(\dfrac{Dw}{|Dw|}\right) & \text{in } \mathbb{R}^n \times (0, T) \\ w(x, 0) = f(x); \end{cases} \tag{3.1}$$

the equation states that each level set of w evolves according to its mean curvature. Consequently, the evolution of Γ_0 is given by $\Gamma_t = \{x \in \mathbb{R}^n : w(x, t) = 0\}$, for each time $t > 0$. In particular, if Ω is a mean convex open set, we can set

$$w(x, t) = u(x) + t$$

and problem (3.1) becomes

$$\begin{cases} |Du|\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Note that, in general, neither a smooth solution to (3.1) nor a smooth solution to (3.2) exists, but it has been proved in [4] (see also [3], [5], [6] and [7]), that problems (3.1) and (3.2) admit a unique viscosity solution which provides a possible generalization of the classical mean curvature motion (1.1).

Proposition 3.1. *Let u be a smooth solution to problem* (3.2) *and let v be the solution of the following symmetrized problem*

$$\begin{cases} |Dv|\,\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) = 1 & \textit{in } \Omega^\star \\ v = 0 & \textit{on } \partial\Omega^\star. \end{cases} \tag{3.3}$$

Then

$$0 \geq u^\#(x) \geq v(x), \quad x \in \Omega^\star. \tag{3.4}$$

Proof. Let $\theta \leq 0$; by integrating on the set $\{u < \theta\}$ the equation in (3.2), using the coarea-formula, and the fact that $\operatorname{div}\left(\frac{Du}{|Du|}\right)\Big|_{\{u=\sigma\}} = H[\{u=\sigma\}]$ we get

$$\mu(\theta) = \int_{-\infty}^{\theta}\left(\int_{u=\sigma} H[\{u=\sigma\}]\, d\mathcal{H}^{n-1}\right) d\sigma.$$

Differentiating with respect to θ, using inequalities (2.1) and (2.2) we have

$$\begin{aligned}\mu'(\theta) &= \int_{u=\theta} H[\{u=\theta\}] d\mathcal{H}^{n-1} \geq \int_{u^\star=\theta} H[\{u^\star=\theta\}] d\mathcal{H}^{n-1}\\ &\geq \int_{u^\#=\theta} H[\{u^\#=\theta\}] d\mathcal{H}^{n-1} = C_n \mu(\theta)^{(n-2)/n},\end{aligned}$$

where $C_n = n(n-1)\omega_n^{2/n}$. Thus μ solves the following problem

$$\begin{cases}\mu'(\theta) \geq C_n\mu(\theta)^{(n-2)/n}, & \theta \leq 0\\ \mu(0) = |\Omega|.\end{cases}$$

Arguing for v in an analogous way, all the inequalities become equalities and then the distribution function ν of v solves the problem

$$\begin{cases}\nu'(\theta) = C_n\nu(\theta)^{(n-2)/n}, & \theta \leq 0\\ \nu(0) = |\Omega|.\end{cases}$$

Then $\mu(\theta) \leq \nu(\theta)$ and the claim immediately follows. □

From (3.4) straightly follows our main theorem.

Theorem 3.1. *Let Ω be a smooth mean convex bounded open set in $\mathbb{R}^n$ and let $\Gamma_0 = \partial\Omega$. If Γ_t denotes the evolution of Γ_0 by mean curvature and $\Gamma_t = \partial\Omega_t$, where Ω_t is a smooth mean convex bounded open set in $\mathbb{R}^n$, then the following estimate holds*

$$0 \leq t^* \leq \frac{1}{2(n-1)}\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{2}{n}}. \tag{3.5}$$

Actually, we can prove a more precise pointwise comparison result as stated in the following

Proposition 3.2. *Under the assumptions of Proposition 3.1 we get*

$$0 \geq u^\star(x) \geq v(x), \quad x \in \Omega^\star. \tag{3.6}$$

Proof. Let $\theta \leq 0$. It is well known (see [9]) that

$$\lambda'(\theta) = \int_{u=\theta} H^2[\{u = \theta\}]\, d\mathcal{H}^{n-1}.$$

By Hölder inequality we get

$$\mu'(\theta) = \int_{u=\theta} H[\{u = \theta\}] d\mathcal{H}^{n-1} \leq (\lambda(\theta))^{1/2} \left(\int_{u=\theta} H^2[\{u = \theta\}]\, d\mathcal{H}^{n-1} \right)^{1/2}$$
$$= \lambda(\theta)^{1/2} \left(\lambda'(\theta)\right)^{1/2}.$$

On the other hand

$$\mu'(\theta) \geq \int_{u_1^\star=\theta} H[\{u^\star = \theta\}]\, d\mathcal{H}^{n-1} = \tilde{c}(n)\lambda(\theta)^{(n-2)/(n-1)},$$

where $\tilde{c}(n) = (n-1)(n\omega_n)^{1/n-1}$. Hence we can say that λ satisfies

$$\begin{cases} \lambda'(\theta) \geq \tilde{c}(n)(\lambda(\theta))^{(n-2)/(n-1)} \\ \lambda(0) = \mathcal{H}^{n-1}(\partial\Omega). \end{cases}$$

In a similar way we find that the function σ, which denotes the perimeter of the level sets of v, is the solution of the following problem

$$\begin{cases} \sigma'(\theta) = \tilde{c}(n)(\sigma(\theta))^{(n-2)/(n-1)} \\ \sigma(0) = \mathcal{H}^{n-1}(\partial\Omega). \end{cases}$$

Then $\lambda(\theta) \leq \sigma(\theta)$,for all $\theta < 0$ and the claim follows. □

For the non-smooth case and for some numerical example we refer to [1].

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AN ISOPERIMETRIC INEQUALITY RELATED TO GAUSSIAN MEASURE AND APPLICATIONS

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We present a relative isoperimetric inequality with respect to the measure $d\mu = (2\pi)^{-N/2} x_N^k \exp(-|x|^2/2)dx$, $k > 0$. As an application, we consider a class of linear elliptic problems whose prototype is

$$\begin{cases} -\text{div}\left(x_N^k \exp(-|x|^2/2)\nabla u\right) = x_N^k \exp(-|x|^2/2) f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \cap \mathbb{R}_+^N, \end{cases}$$

where $\Omega \subset \mathbb{R}_+^N = \left\{x = (x_1, ..., x_N) \in \mathbb{R}^N : x_N > 0\right\}$ is a domain (possibly unbounded) and f belongs to a suitable weighted Lebesgue space. We estimate the solution to such a problem in terms of the solution to a symmetric one-dimensional problem, belonging to the same class. For the proofs of the results announced here we refer the reader to [1].

Let Ω be a Lebesgue measurable subset of $\mathbb{R}_+^N$ (possibly unbounded), where

$$\mathbb{R}_+^N = \left\{x = (x_1, ..., x_N) \in \mathbb{R}^N : x_N > 0\right\}.$$

The measure and perimeter of Ω, with respect to the measure $d\mu$, are defined respectively by

$$\mu(\Omega) = \frac{1}{(2\pi)^{N/2}} \int_\Omega x_N^k \exp(-|x|^2/2)\, dx, \quad \text{with } k > 0,$$

and

$$P_\mu(\Omega) = \begin{cases} \dfrac{1}{(2\pi)^{N/2}} \displaystyle\int_{\partial\Omega} x_N^k \exp(-|x|^2/2)\, d\mathcal{H}^{N-1}(x)\,, & \\ \qquad\qquad\qquad\qquad \text{if } \partial\Omega \text{ is } (N-1)\text{-rectifiable,} & \\ +\infty \qquad\qquad\qquad\quad \text{otherwise.} & \end{cases}$$

For any number $m \in \left(0, \mu\left(\mathbb{R}^N_+\right)\right)$, the isoperimetric problem with respect to $d\mu$ reads as

$$I_\mu(\Omega) = \min\left\{P_\mu(\Omega)\,, \text{ with } \mu(\Omega) = m\right\}. \tag{1}$$

Let $\varphi(t)$ be the function defined by

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp(-\sigma^2/2)\, d\sigma\,, \quad t \in \mathbb{R}\,,$$

and denote by φ^{-1} its inverse. Then the set

$$\Omega^\star = \left\{ x \in \mathbb{R}^N_+ : x_1 < \varphi^{-1}\left(\frac{\mu(\Omega)}{\mu\left(\mathbb{R}^N_+\right)}\right)\right\}, \tag{2}$$

verifies

$$\mu(\Omega) = \mu\left(\Omega^\star\right).$$

The following result claims that $\Omega^\star$ realizes the minimum in (1).

Theorem1. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N_+$, then*

$$P_\mu(\Omega) \geq P_\mu\left(\Omega^\star\right),$$

or equivalently

$$P_\mu(\Omega) \geq I(\mu(\Omega)) \equiv \frac{\mu\left(\mathbb{R}^N_+\right)}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\varphi^{-1}\left(\frac{\mu(\Omega)}{\mu\left(\mathbb{R}^N_+\right)}\right)\right]^2\right\}.$$

Now, in view of the applications of the previous theorem to degenerate elliptic equations, we introduce the notion of rearrangement with respect to $d\mu$.

Let u be a Lebesgue measurable function defined in Ω. Then its distribution function is the function defined by

$$m_u(t) = \mu\left(\{x \in \Omega : |u(x)| > t\}\right), \quad \forall t \in [0, \infty[\,.$$

The decreasing rearrangement of u is the function given by

$$u^*(s) = \inf\{t \in \mathbb{R} : m_u(t) \leq s\}, \quad \forall s \in \,]0, \mu(\Omega)]\,.$$

We will say that two functions u and v are equimeasurable or, equivalently, that v is a rearrangement of u, if they have the same distribution function. Finally the rearrangement $u^\star$ of u, with respect to the measure $d\mu$, is given by

$$u^\star(x) = u^*\left[\mu\left(\mathbb{R}^N_+\right)\varphi(x_1)\right], \quad \forall x \in \Omega^\star.$$

Observe that by definition $u^\star$ depends on one variable only, it is an increasing function and moreover u and $u^\star$ are equimeasurable. Therefore by Cavalieri's principle, one has

$$\|u\|_{L^p(\Omega, d\mu)} = \left\|u^\star\right\|_{L^p(\Omega^\star, d\mu)}, \quad \forall p \in [1, \infty]\,. \tag{3}$$

Now we introduce the weighted Sobolev space $W_k(\Omega, d\mu)$ of the functions u satisfying the following two conditions

i) $\displaystyle\int_\Omega \left(|Du|^2 + u^2\right) d\mu < \infty,$

ii) there exists a sequence of functions u_n in $C^1(\overline{\Omega})$ with $u_n(x) = 0$ on $\partial\Omega \setminus \{x_N = 0\}$, such that

$$\lim_{n\to\infty} \int_\Omega \left(|D(u_n - u)|^2 + (u_n - u)^2\right) d\mu = 0\,.$$

In this setting a Pólya-Szegö-type inequality holds (see [3]).

Theorem 2. *Let u be a nonnegative function in $W_k(\Omega, d\mu)$, then the following inequality holds true*

$$\int_\Omega |Du|^2 \, d\mu \geq \int_{\Omega^\star} \left|Du^\star\right|^2 d\mu. \tag{4}$$

Combining the previous inequality together with (3) one gets that $W_k(\Omega, d\mu)$ in continuously embedded in $L^2(\Omega, d\mu)$.

Corollary 3. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N_+$ such that $\mu(\Omega) < \mu(\mathbb{R}^N_+)$. The there exists a constant $C = C(\mu(\Omega))$ such that for each function u in $W_k(\Omega, d\mu)$ it holds*

$$\int_\Omega |u|^2 \, d\mu \leq C \int_\Omega |Du|^2 \, d\mu \, .$$

By the Corollary 3 we can equip $W_k(\Omega, d\mu)$ with the norm

$$\|u\|_{W_k(\Omega, d\mu)} \equiv \left(\int_\Omega |Du|^2 \, d\mu \right)^{1/2} .$$

Now we are in position to state our comparison result

Theorem 4. *Let u be the solution to the problem*

$$\begin{cases} -div\,(A(x)\nabla u) = x_N^k \exp(-|x|^2/2) f(x) & in\ \Omega \\ u \in W_k(\Omega, d\mu), \end{cases} \tag{5}$$

where Ω is a connected open subset of $\mathbb{R}^N_+$ such that $\mu(\Omega) < \mu(\mathbb{R}^N_+)$, $A(x) = \big(a_{ij}(x)\big)_{ij}$ is an $N \times N$ symmetric matrix with measurable coefficients satisfying

$$x_N^k \exp(-|x|^2/2)\,|\zeta|^2 \leq a_{ij}(x)\zeta_i\zeta_j \leq C x_N^k \exp(-|x|^2/2)\,|\zeta|^2\,, \quad C \geq 1$$

for a.e. $x \in \Omega$ and for all $\zeta \in \mathbb{R}^n$. Moreover we assume that

$$f \in L^2(\Omega, d\mu).$$

Let $v \in W_k(\Omega^\star, d\mu)$ be the solution of the following "symmetrized" problem

$$\begin{cases} -div\,\big(x_N^k \exp(-|x|^2/2)\nabla v\big) = x_N^k \exp(-|x|^2/2) f^\star(x) & in\ \Omega^\star \\ v \in W_k(\Omega^\star, d\mu). \end{cases} \tag{6}$$

Then

$$u^\star \leq v = v^\star \quad a.e.\ in\ \Omega^\star,$$

and

$$\int_\Omega |Du|^q \, d\mu \leq \int_{\Omega^\star} |Dv|^q \, d\mu \,, \quad for\ all\ 0 < q \leq 2\, .$$

Observe that Lax-Milgram Theorem ensures the existence and the uniqueness of the solutions to problems (5) and (6). Moreover an easy computation gives

$$v(x) = \int_{x_1}^{\frac{\mu(\Omega)}{\mu(\mathbb{R}^N_+)}} \left(\int_{-\infty}^{\rho} f^{\bigstar}(\sigma) \exp(-\sigma^2/2)\, d\sigma \right) \exp(\rho^2/2)\, d\rho\,.$$

We finally remark that results of the same type are contained in [2] where it is studied the isoperimetric problem with respect to the measure $y^k\, dx dy$ defined in $\mathbb{R}^2_+$. In that case the "optimal" set turns out to be an half-circle.

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THREE-DIMENSIONAL BONNESEN TYPE INEQUALITIES

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A well-known result in convex geometry proved by Favard states that among all convex plane sets of given perimeter and area, the symmetric lens is the unique element of maximum circumradius. In this note a new proof of Favard's theorem is exhibited and possible extensions in higher dimensions are discussed.

1. Bonnesen type inequalities.

Let K denote a convex body in $\mathbb{R}^2$, i.e. a compact convex subset of the plane with non-empty interior. A Bonnesen type inequality is a geometric inequality that involves the perimeter L, the area A, the inradius r and/or the circumradius R of the body K. We recall that the inradius and the circumradius of K are the radius of the largest disc contained in K and the radius of the smallest disc containing K, respectively. The original Bonnesen inequalities, contained in [2], are the following:

$$\frac{L^2}{4\pi} - A \geq \pi \left(\frac{L}{2\pi} - r \right)^2 \tag{1}$$

$$\frac{L^2}{4\pi} - A \geq \pi \left(R - \frac{L}{2\pi} \right)^2 . \tag{2}$$

In (1) equality holds if and only if K is a “stadium” (or a baby biscuit, if one prefers), namely a set obtained from a rectangle by gluing two semidiscs of radius t to opposite sides of length $2t$. In (2) equality holds if and only if K is a disc. The quantity on the left-hand side of both inequalities is the isoperimetric deficit of K. In fact, a common feature of Bonnesen style inequalities is estimating from below the isoperimetric deficit of K in terms of quantities involving the inradius and/or the circumradius. Thus all these inequalities are sharper versions of the classical isoperimetric inequality. Inequalities (1) and (2) imply

$$\frac{L^2}{4\pi} - A \geq \frac{\pi}{4}(R-r)^2 \,, \tag{3}$$

the well-known estimate of the isoperimetric deficit in terms of R and r. In the literature many variants of the original Bonnesen inequalities are known. The main sources are the books by Bonnesen [2], Bonnesen and Fenchel [3], Schneider [9] and the survey by Osserman [6], which is an excellent guide in the world of these inequalities. However, in this note, we shall focus our attention on the original Bonnesen inequalities only. Inequality (1) is sharp, since for every value of the isoperimetric deficit there exists a set for which equality holds. This fact can be rephrased as follows: Among all convex bodies of given L and A, the stadium is the one with minimum inradius. On the other hand, inequality (2) is not sharp in the following sense. If the isoperimetric deficit is strictly positive, then (2) is strict also and it does not provide the maximum possible circumradius, for fixed L and A. Favard showed in 1929 that under these constraints the symmetric lens is the only maximizing set. We recall that a symmetric lens is the intersection of two discs with the same radius. It is natural to ask which are in higher dimensions the convex sets corresponding to lenses in the plane case, that is the sets with maximum circumradius under suitable restrictions on the volume, the surface area and so on. Surprisingly, the problems which naturally correspond in higher dimensions to the one solved by Favard in the plane are unsolved. In the first part of this note (Sections 2-4) we deal with Favard's result and we give a new proof of it. In the second (Section 5) we discuss 3-dimensional extensions of Favard's theorem and we present a couple of results, by Zalgaller and by Campi and Gronchi respectively, concerning convex sets of maximum diameter under suitable restrictions.

2. Favard's problem.

Let L_0 and A_0 be two positive numbers such that

$$L_0^2 - 4\pi A_0 \geq 0$$

and denote by $\Lambda(L_0, A_0)$ the class of all plane convex bodies of perimeter L and A such that $L \leq L_0$ and $A \geq A_0$.

Problem 2.1. Which is the element from $\Lambda(L_0, A_0)$ with maximum circumradius?

In the literature this problem is called Favard's problem. Here is the solution.

Theorem 2.2. (Favard [5], 1929) *The symmetric lens of perimeter L_0 and area A_0 is the unique solution of Problem 2.1.*

Notice that, if D denotes the diameter of the convex body K, then

$$\frac{1}{2}D \leq R\,.$$

Since for the symmetric lens $D/2 = R$, Favard's theorem 2.2 implies also the following result:

The symmetric lens of perimeter L_0 and area A_0 is the unique element from $\Lambda(L_0, A_0)$ of maximum diameter.

Such a result can be also obtained directly. Indeed, it is easy to check that the area between a chord and an arc of given length is maximal for the arc of circle.

Favard's original proof consists in finding an upper bound for the circumradius and in showing that the circumradius of the lens attains just that value. Besicovitch [1] and, more recently, Zalgaller [10] provided two new and independent proofs of Theorem 2.2. Besicovitch's proof makes use of local variations of the set assumed to be the maximizer. In such a way it turns out that the candidates are reduced to the lens and the Reuleaux triangle. A direct computation leads to the conclusion. Zalgaller's proof is based on Pólya symmetrization, a circular version of Steiner symmetrization. A similar symmetrization to Pólya's was introduced earlier by Bonnesen. We shall describe both symmetrizations in the next section.

3. Pólya symmetrization and Bonnesen symmetrization.

Let K be a planar convex body and $\Gamma(\rho, \tau)$ a circular annulus of radii $\rho < \tau$ containing the boundary of K. We can assume that the annulus is centered at the origin o. Denote by (r, θ) the polar coordinates in the plane and let s be the half-line from the origin corresponding to $\theta = 0$. For $\rho < r < \tau$, let $4\theta^*$ be the linear measure of $K \cap \partial C_r$, where C_r is the disk with center at o and radius r. Moreover, let $n(r)$ be the number of components of $K \cap \partial C_r$ with positive linear measure.

3.1. Pólya symmetrization. (see [2], p.194; see also [8] and [3], p.77)

The Pólya symmetral K_P of K is defined as the set such that the points $(r, 2\theta^*)$, $(r, 2\pi - 2\theta^*)$ belong to its boundary, for every r.

3.1. Bonnesen symmetrization. (see [2], p.67)

The Bonnesen symmetral K_B of K is defined as the set such that the points (r, θ^*), $(r, 2\pi - \theta^*)$, $(r, \pi - \theta^*)$, $(r, \pi + \theta^*)$ belong to its boundary, for every r.

For Pólya and Bonnesen symmetrals the following properties hold:

(i) $A(K) = A(K_P) = A(K_B)$;
(ii) $L(K) \geq L(K_P)$;
(iii) if $n \geq 2$, when $\theta^*(r) < \pi/2$, then $L(K) \geq L(K_B)$.

The functions $A(K \cap C_r)$, $A(K_P \cap C_r)$, $A(K_B \cap C_r)$ have the same derivative with respect to r; hence property (i) follows. Properties (ii) and (iii) can be deduced from Jensen's inequality, applied to the function $\sqrt{1 + x^2}$.

In (i) equality holds if and only if $n(r) \equiv 1$ and K has an axis of symmetry (through o).

In (ii) equality holds if and only if $n(r) \equiv 2$ and each of the two components of $K \setminus C_\rho$ has an axis of symmetry (through o).

It is worth noticing that K_P and K_B need not be convex.

4. A new proof of Favard's theorem.

This proof consists of showing that a solution of Problem 2.1 has two antipodal points on the smallest circle containing it. Thus such a solution has the largest possible diameter; therefore it has to be the lens.

Assume that K is a solution of Favard's problem and let $\Gamma(\rho, \tau)$ be the minimal annulus of K, i.e. the unique annulus containing ∂K such that $\tau - \rho$ is minimum (see [2], p.45 and p.67). Such an annulus has the property that every circle with center at o and radius between ρ and τ intersects K at least in two arcs.

If K has two antipodal points on the largest circle of $\Gamma(\rho, \tau)$, then τ is just the circumradius of K. Assume that it is not so. Let K_B be the Bonnesen symmetral of K with respect to $\Gamma(\rho, \tau)$ and K^* the convex hull of K_B. Since K is assumed to be a solution, we have to exclude that $L(K^*) < L(K)$. The equality $L(K^*) = L(K)$ implies that in the process of symmetrization every circle with radius between ρ and τ has just two arcs in common with K. Therefore, on the circle of radius τ the set K has only two components, not containing two antipodal points and so contained in an arc strictly smaller than half circle. Thus the circumradius of K would be less than τ, while the one of K^* is just τ. The conclusion is that a solution of Favard's problem must have two antipodal points on its circumcircle.

5. Favard type problems in three dimensions.

It is natural to ask whether in higher dimensions it is possible to find estimates of the circumradius of a convex set in terms of quantities like volume, surface area and so on. Let us focus our attention on possible extensions of Favard's theorem in three dimensions. Let K be a convex body of $\mathbb{R}^3$. How to state a three-dimensional version of Problem 2.1? How to replace the constraints on perimeter and area?

While the volume V is a natural substitute of the area, there are at least two possibilities for the other quantity. According to Kubota's integral formula (see [9], p. 295), the perimeter of a planar convex body is an average of the lengths of its orthogonal projections. Thus the same formula suggests that the role played by the perimeter L in the plane can be interpreted in the 3-space by the *surface area* S, that is the average of the areas of its two-dimensional projections, or by the *mean width* B, the average of the lengths of its one-dimensional projections. For a smooth K, the *total mean curvature* is defined by

$$M = \frac{1}{2}\int_{\partial K}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) d\sigma \, ,$$

where R_1, R_2 are the principal radii of curvature of ∂K and σ is the $(n-1)$-dimensional Hausdorff measure. We have that

$$M = 2\pi B \, .$$

On the other hand, the surface area S of K is given by

$$S = \int_{\partial K}\left(\frac{1}{R_1}\frac{1}{R_2}\right) d\sigma \, .$$

The quantities V, S, B satisfy the following inequalities of isoperimetric type (see [4], p. 145):

$$S^3 \geq 36\pi V^2 \, ,$$

$$\pi B^3 \geq 6V \, .$$

Let $\Omega(S_0, V_0)$ be the class of all convex bodies in $\mathbb{R}^3$ such that $S \leq S_0$, $V \geq V_0$. Unfortunately a three-dimensional result analogous to Favard's theorem is not available. The following weaker result concerning the maximum of the diameter holds.

Theorem 5.1. (Zalgaller [10], 1994) *The unique body in* $\Omega(S_0, V_0)$ *having maximum diameter is a mean curvature spindle-shaped body of surface area* S_0 *and volume* V_0.

According to Definition 16 in [10], a mean curvature spindle-shaped body is a centrally symmetric convex body of revolution whose surface has constant mean curvature in the central part and consists of two cones in the parts adjacent to the axis of revolution.

Let $\Psi(B_0, V_0)$ be the class of all convex bodies in $\mathbb{R}^3$ such that $B \leq B_0$, $V \geq V_0$. Recently Campi and Gronchi obtained a result analogous to Zalgaller's theorem. Precisely they showed that *in* $\Psi(B_0, V_0)$ *the unique body of maximum diameter is a Gaussian curvature spindle-shaped body of mean width* B_0 *and volume* V_0.

The above result is the object of a forthcoming paper. The strategy of the proof is analogous to that used by Zalgaller for Theorem 5.1 and it is based mainly on local variations of the maximizer.

In conclusion, the problems of finding in $\Omega(S_0, V_0)$ or in $\Psi(B_0, V_0)$ the element of maximum circumradius remain open. It is reasonable to conjecture that the solutions are the same of the corresponding problems for the diameter. For solving such a conjecture it would be sufficient to show that the bodies with largest circumradius have two antipodal points on the circumscribed sphere.

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ESISTENZA DI SOLUZIONI PER UN PROBLEMA DI NEUMANN

PASQUALE CANDITO

In this note we present some multiplicity results for a nonlinear Neumann problem with discontinuous nonlinearities. Our approach is based on critical point theory for non differentiable functionals (see [2]).

In questa nota illustriamo alcuni recenti risultati di esistenza di soluzioni deboli per un problema di Neumann con termine non lineare discontinuo. La metodologia adottata si basa prevalentemente su dei teoremi di punto critico contenuti nei lavori [1] e [6]. Tali risultati estendono, sotto diversi aspetti, un teorema di tre punti critici contenuto in [8]. Inoltre sfruttiamo le idee provenienti dallo studio di alcuni problemi differenziali nell'ambito delle inclusioni differenziali (vedi [3], [6]). Per ulteriori approfondimenti, rimandiamo al lavoro [2].

Consideriamo il seguente problema di Neumann

$$(N) \qquad \begin{cases} \Delta_p u - a(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{su } \partial\Omega, \end{cases}$$

dove, $\Omega \subset \mathbb{R}^n$ è un aperto limitato e non vuoto con frontiera regolare, $\lambda \in (0, +\infty)$, $p \in (n, +\infty)$, $\Delta_p := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ è il p-laplaciano, $a \in L^\infty(\Omega)$, con $\operatorname{ess\,inf}_\Omega a > 0$, e $f : \Omega \times \mathbb{R} \to \mathbb{R}$ tale che

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(m_1) $x \to f(x,u)$ è misurabile per ogni $u \in \mathbb{R}$;
(m_2) esiste un insieme $\Omega_f \subseteq \Omega$ con $m(\Omega_f) = 0$ tale che

$$D_f := \cup_{x \in \Omega \setminus \Omega_f} \{u \in \mathbb{R} : f(x,\cdot) \text{ è discontinua in } u\}$$

ha misura nulla secondo Lebesgue;
(h_1) per ogni $\rho > 0$ esiste $\mu_\rho \in L^1(\Omega)$ tale che per q.o. $x \in \Omega$, si ha

$$\sup_{|u| \le \rho} |f(x,u)| \le \mu_\rho(x).$$

Inoltre, posto

$$F(x,u) = \int_0^u f(x,t)\,dt \quad \forall (x,u) \in \Omega \times \mathbb{R},$$

associamo al problema (N) la seguente disequazione variazionale-emivariazionale

$$(H) \qquad \begin{cases} -\int_\Omega |\nabla u\,(x)|^{p-2}\,\nabla u\,(x)\,\nabla v\,(x)\;dx\; - \\ -\int_\Omega a\,(x)\,|u\,(x)|^{p-2}\,u\,(x)\,v\,(x)\;dx \\ \le \lambda \int_\Omega F_u^0(u(x); v(x) - u(x))dx \qquad \text{for all} \quad v \in K, \end{cases}$$

dove $K \subseteq W^{1,p}(\Omega)$ è un chiuso e convesso contenente le funzioni costanti e $F_u^0(x,\cdot)$ indica la derivata direzionale generalizzata di F rispetto a u. Chiaramente, se f è continua e K coincide con l'intero spazio, le soluzioni di (H) sono soluzioni deboli per (N). Invece, se f è discontinua rispetto alla seconda variabile, le soluzioni di (H) sono soluzioni di un'opportuna inclusione differenziale associata a (N). Quindi, il primo passo per ottenere i nostri risultati è stato quello di stabilire l'esistenza di soluzioni per (H) e, successivamente, imporre delle condizioni aggiuntive per garantire che tali soluzioni siano anche soluzioni per (N). In questo ordine di idee presentiamo il seguente risultato.

Teorema 1. *In aggiunta a* $(m_1) - -(m_3)$ *e* (h_1) *supponiamo che*

(h_2) *esistono due costanti* γ *e* δ*, con* $0 < \gamma < \delta$*, tali che*

$$\int_\Omega \inf_{|u| \le \gamma} F(x,u)dx > \frac{1}{c^p \|a\|_1} \left(\frac{\gamma}{\delta}\right)^p \int_\Omega F(x,\delta)\,dx,$$

dove $c := \sup\left\{\|u\|_{C^0} \,/\, \|u\| : u \in W^{1,p}(\Omega),\, u \neq 0\right\}$*;*

(h_3) *esistono due funzioni* $\eta, \mu \in L^1(\Omega)$, *con* $\|\eta\|_1 < \frac{1}{pc^p\lambda_h}$, *dove*

$$\lambda_h = \frac{\frac{h}{p}\left(\frac{\gamma}{c}\right)^p}{\int_\Omega \inf_{|u|\leq\gamma} F(x,u)\,dx - \frac{1}{c^p\|a\|_1}\left(\frac{\gamma}{\delta}\right)^p \int_\Omega F(x,\delta)\,dx},$$

con $h > 1$, *tali che*

$$F(x,u) \geq -\mu(x) - \eta(x)|u|^p,$$

per quasi ogni $x \in \Omega$ *e per ogni* $u \in \mathbb{R}$.

Allora esistono un intervallo aperto $\Lambda_1 \subseteq [0, \lambda_h]$ *e* $\sigma \in]0, +\infty[$ *tali che, per ogni* $\lambda \in \Lambda_1$, (H) *ha almeno tre soluzioni deboli le cui norme in* $W^{1,p}(\Omega)$ *sono minori di* σ.

Osserviamo che una condizione meno generale che implica la (h_3) è la seguente:

(h_3') *Esistono una funzione non negativa* $\tau \in L^1(\Omega)$ *e una costante positiva* s *con* $s < p$, *tali che*

$$F(x,u) \geq -\tau(x)(1 + |u|^s)$$

per quasi ogni $x \in \Omega$ *e per ogni* $u \in \mathbb{R}$.

Enunciamo ora il risultato principale

Teorema 2. *Nelle ipotesi del Teorema 1, per quasi ogni* $x \in \Omega$, *poniamo*

$$f^-(x,z) := \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|z-t|<\delta} f(x,t),$$
$$f^+(x,z) := \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|z-t|<\delta} f(x,t)$$

e supponiamo che

(m_4) $f^-(x,z)$ *e* $f^+(x,z)$ *siano sup-misurabili;*

(m_5) *per quasi ogni* $x \in \Omega$, *per ogni* $z \in D_f$ *e per ogni* $\lambda \in [0, \lambda_h]$, *la condizione*

$$a(x)|z|^{p-2}z + \lambda f^-(x,z) \leq 0 \leq a(x)|z|^{p-2}z + \lambda f^+(x,z)$$

implica $a(x)|z|^{p-2}z + \lambda f(x,z) = 0$.

Allora esistono un intervallo aperto $\Lambda_1 \subseteq [0, \lambda_h]$ *e* $\sigma \in]0, +\infty[$ *tali che, per ogni* $\lambda \in \Lambda_1$, (N) *ammette almeno tre soluzioni deboli le cui norme in* $W^{1,p}(\Omega)$ *sono minori di* σ.

Chiaramente l'ipotesi (m_4) è di natura tecnica, mentre la condizione (m_5) è stata introdotta nell'ambito dell'equazioni differenziali a derivate parziali da S.A. Marano e sviluppata successivamente anche in collaborazione con G. Bonanno in una serie di lavori della metà degli anni novanta nell'ambito dell'Analisi Multivoca. Combinando tale condizione con un noto lemma contenuto in [5], si garantisce che le soluzioni della disequazione (H) siano effettive soluzioni del problema (N). A tal proposito si veda [2] e le referenze lì riportate.

Il prossimo esempio mostra un'applicazione del Teorema 2 nel caso in cui

$$f(x, u) = \alpha(x)h(u) + \beta(x)g(u), \quad \forall (x, u) \in \Omega \times \mathbb{R}$$

dove, $\alpha, \beta \in L^1(\Omega)$ con $\min(\alpha(x), \beta(x)) \geq 0$ q.o in Ω e $h, g : \mathbb{R} \to \mathbb{R}$ essenzialmente localmente limitate.

Esempio 1. Sia $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Posto $p = 4$, $a(x, y) = 1$, $\alpha(x, y) = x^2 + y^2$, $\beta(x, y) = 1$,

$$h(\xi) = \begin{cases} -e^{e^{\xi}} e^{\xi} & \text{se } \xi \leq 3 \\ -\xi^2 & \text{se } \xi > 3 \end{cases} \quad \text{e} \quad g(\xi) = \left(e^{e^3} e^3/3\right) \xi^5 \quad \forall \xi \in \mathbb{R},$$

scegliendo $\gamma = 1$, $\delta = 3$, $s = 3$ e $\tau(x) = e^{e^3}$ $\forall x \in \Omega$, grazie al Teorema 2, con (h_3') al posto di (h_3), esiste un intervallo aperto $\Lambda \subseteq [0, 10^{-7}]$ tale che per ogni $\lambda \in \Lambda$, (N) ha almeno tre soluzioni deboli limitate in norma uniformemente rispetto a λ.

In [9], richiedendo h e g continue, è stata stabilita l'esistenza di infinite soluzioni per il problema (N). Successivamente, in [7], la stessa conclusione è stata ottenuta per (H) con h e $g \in L^1_{loc}(\mathbb{R})$ e in [3], per (N), con h e g suscettibili di soddisfare condizioni analoghe a quelle richieste dal Teorema 2. L' ipotesi centrale su cui si basano tali risultati è la seguente:

Esistono $\{\xi_n\}_{n\in\mathbb{N}}$ *e* $\{r_n\}_{n\in\mathbb{N}}$ *con* $r_n > 0$ $\forall n \in \mathbb{N}$*, tali che*

$$(M_1) \qquad \lim_{n\to+\infty} r_n = +\infty, \int_0^{\xi_n} h(t)\,dt = \inf_{|\xi| \leq c(pr_n)^{1/p}} \int_0^{\xi} h(t)\,dt\,;$$

$$(M_2) \qquad \frac{1}{p}\|a\|_1 |\xi_n|^p + \int_0^{\xi_n} g(t)\,dt \|\beta\|_1 < r_n \quad \forall n \in \mathbb{N}\,.$$

Chiaramente nel nostro contesto tale ipotesi non è soddisfatta in quanto si ha $\xi_n = c(pr_n)^{1/p}$ e $\frac{1}{p}\|a\|_1|\xi_n|^p = \|a\|_1 c^p r_n \geq r_n$.

Enunciamo ora un caso particolare del Teorema 2. Sia $h : \mathbb{R} \to \mathbb{R}$ misurabile e limitata tale che $h(0) = 0$ e $m(D_h) = 0$, dove $D_h = \{z \in \mathbb{R} : h(\cdot)$ è discontinua in $z\}$.

Teorema 3. *Supponiamo che esistano γ, δ, con $0 < \gamma < \delta$:*

(1) $h(z)z \geq 0$ *per ogni* $z \in \mathbb{R} \setminus]\gamma, \delta[$ *e* $\int_0^\delta h(z)dz < 0$*;*
(2) $D_h \cap]\gamma, \delta[= \emptyset$ *e* $\lim_{z \to \gamma^+} h(z) = \lim_{z \to \delta^-} h(z) = 0$.

Allora, per ogni $\alpha \in L^1(\Omega)$ non negativa e non nulla esiste un intervallo aperto

$$\Lambda_1 \subseteq \left[0, \frac{\|a\|_1 \delta^p}{-\|\alpha\|_1 \int_0^\delta h(z)\, dz} \right]$$

e un numero positivo σ tali che, per ogni $\lambda \in \Lambda_1$, il problema

$$(N_h) \qquad \begin{cases} \Delta_p u - a(x)|u|^{p-2}u = \lambda \alpha(x) h(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{su } \partial\Omega, \end{cases}$$

ammette almeno due soluzioni deboli non banali le cui norme in $W^{1,p}(\Omega)$ sono minori di σ.

Infine presentiamo un' applicazione dei precedenti risultati nel caso in cui D_h ha la potenza del continuo. Sia $h : \mathbb{R} \to \mathbb{R}$ definita ponendo

$$h(z) = \begin{cases} 1 & \text{se } z \in C \\ z^2 - 4z + 3 & \text{se } z \in]1, 3[\\ 0 & \text{altrimenti,} \end{cases}$$

dove C indica l'insieme di Cantor. Scegliendo $\gamma = 1$ e $\delta = 3$, dalla dimostrazione del Teorema 3, si ricava che per ogni funzione non negativa e non nulla $\alpha \in L^1(\Omega)$ esiste un intervallo aperto $\Lambda_1 \subseteq \left[0, \frac{3^{p+1}}{4} \frac{\|a\|_1}{\|\alpha\|_1}\right]$ tale che, per ogni $\lambda \in \Lambda_1$, il problema (N_h) ammette almeno tre soluzioni non banali e limitate in norma uniformemente rispetto a λ.

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QUALITATIVE PROPERTIES OF FREE BOUNDARIES WITH BERNOULLI TYPE CONDITIONS

SIMONE CECCHINI

We consider the external Bernoulli's free boundary problem and show how the curvature of the inner curve (which is the known part of the boundary) determines the free boundary one. In particular, we define an injective correspondence between arcs where the curvature has a definite sign placed on the free boundary and on the inner curve.

We extend some previous results from A. Acker by showing that any arc of the free boundary where the curvature is positive bends less than the corresponding one on the inner curve.

Main Results.

We begin with recalling the classical exterior Bernoulli's problem in the plane (see also [4]): given a simple closed curve γ, find an external curve Γ such that Γ and γ bound an annular domain Ω, whose capacity potential u, determined by the conditions

$$\Delta u = 0 \quad \text{in} \quad \Omega\,, \tag{1}$$

$$u = 1 \quad \text{on} \quad \gamma\,, \tag{2}$$

$$u = 0 \quad \text{on} \quad \Gamma\,, \tag{3}$$

also satisfies

$$|\nabla u| = 1 \quad \text{on} \quad \Gamma\,, \tag{4}$$

where ∇u denotes the gradient of u.

By adapting A. Acker's method (see [1], [2]) to the modified curvature functions

$$h^* = \frac{h}{|\nabla u|} \quad \text{and} \quad k^* = \frac{k}{|\nabla u|} \tag{5}$$

introduced and studied by G. Talenti [6] (who proved them to be *harmonic conjugate*), we show how the curvature of γ determines that of the free boundary. Here h and k are the curvature of level and steepest descent curves of u respectively (see [6] for definitions and related properties).

Given a solution to the above-mentioned Bernoulli's problem, the main results (whose proofs are presented in [5]) can be stated as follows.

Theorem 1. *Let $\{z_1, z_2, \dots, z_n\}$ an enumeration of distict points of Γ, whose subscripts are given according to their order of occurrence on Γ, counterclockwise.*

Suppose that each z_i satisfies one of the following properties:

(i) the value of h^* at z_i is a local minimum on Γ;
(ii) the value of h^* at z_i is a local maximum on Γ;
(iii) on Γ, the sign of h^* changes at z_i, turning from positive to negative;
(iv) on Γ, the sign of h^* changes at z_i, turning from negative to positive.

Then there exists a collection $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ of subarcs of γ, possibly degenerating to points, such that h^* is constant on ζ_i and satisfies (i)–(iv) on γ, respectively, with z_i replaced by ζ_i.

Moreover, if z_i satisfies (i) or (ii), then $h^* > h^*(z_i)$ or $h^* < h^*(z_i)$ on ζ_i, respectively.

Remark. Roughly speaking, (iii) and (iv) inform us that the curvature of the free boundary Γ changes its sign a number of times that does not exceed that of the given curve γ.

Assertions (i) and (ii) can be hardly exploited to guess anything of the relationship between the shape of Γ and that of γ, because of h^* being different from h on the inner curve. Theorems 2 and 3 below take this remark into account.

Given a curve β, we define a *partition* of β as a collection of subarcs $\{\beta_1, \beta_2, \dots, \beta_m\}$ of β, with pairwise disjoint interiors, such that $\bigcup_{i=1}^m \beta_i = \beta$

Theorem 2. *Let* $\{\Gamma_1, \Gamma_2, \dots, \Gamma_m\}$ *a partition of* Γ *made of arcs which are maximal with respect to the property that for each* $i \in \{1, 2, \dots, m\}$*, one and only one of the following holds*

(i) $h(z) > 0$ *for all* $z \in \Gamma_i$*;*
(ii) $h(z) < 0$ *for all* $z \in \Gamma_i$*;*
(iii) $h(z) = 0$ *for all* $z \in \Gamma_i$*.*

Then for each $i \in \{1, 2, \dots, m\}$ *we can determine a subarc* $\gamma_i \subset \gamma$ *which has the same property of* Γ_i*. Moreover the arcs* γ_j *are mutually disjoint.*

Theorem 2 lets us infer that the geometry of Γ is never more convoluted than that of γ: there are more protuberances/throughs on γ than on Γ.

As a by-product of Theorem 2, we can infer a classical result of D.E. Tepper [7]: if γ is convex, then Γ is strictly convex.

The next theorem takes the analysis of curvature a little deeper: we show that for each positive maximum of h on Γ we can determine a point on γ (not necessarily an extremum point) on which the curvature is larger.

Theorem 3. *Let* $\{z_1, z_2, \dots, z_n\}$ *be given as in Theorem 1. Suppose that each* z_i*,* $1 \leq i \leq n$*, satisfies one of the following properties:*

(i) the value of h *at* z_i *is a non-negative relative maximum on* Γ*;*
(ii) on Γ *the sign of* h *changes at* z_i*, turning from positive to negative;*
(iii) on Γ *the sign of* h *changes at* z_i*, turning from negative to positive.*

Then there exists a collection $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ *of subarcs of* γ*, possibly degenerating to points, such that each* ζ_i *satisfies:*

(a) if (i) holds, then ζ_i *is a point and* $h(\zeta_i) > h(z_i) \cdot e^{h(z_i)}$*; moreover* z_i *is joined to* ζ_i *by a level curve* $\{k = 0\}$*;*
(b) if (ii) or (iii) holds, then (ii) or (iii) holds for h *on* γ*, with* z_i *replaced by* ζ_i*.*

If we consider the decomposition of Γ given by Theorem 2 along with Theorem 3, we can conclude that the positive maximum of h on γ_i is greater than the maximum of h on Γ_i, i.e. Γ_i bends less then γ_i. Thus we can infer not only that the geometry of Γ is simpler than that of γ (as we can do from Acker's papers), but also that the protuberances of γ directed away from the inside are alleviated in passing from γ to Γ.

Remark. We stress that the result of Theorem 3 does not hold in the case of negative minimmum of h. Here we report two numerical examples showing that, corresponding to a negative minimum point P of h on Γ we can find a point Q of γ at which h is either larger or smaller than at P (we assume the curves oriented counterclockwise).

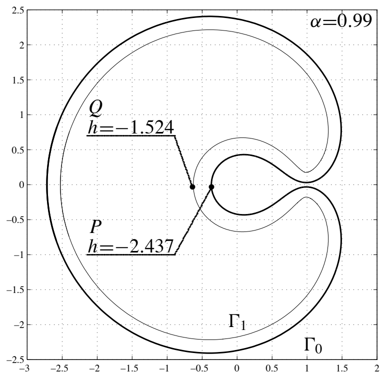

Figure 1: $\Gamma(\theta) = e^{\cos\theta + i\pi(1+\alpha\sin\theta)}$

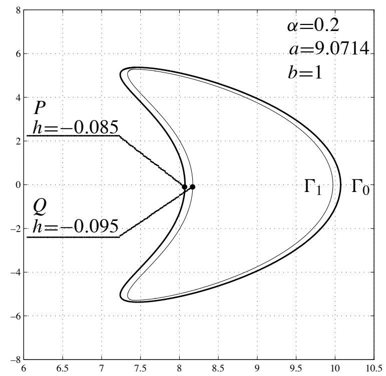

Figure 2: $\Gamma(\theta) = (a - b\sin\theta)e^{i\alpha\pi\cos\theta}$

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WAVE PROPAGATION IN OPTICAL WAVEGUIDES

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We present a mathematical framework for studying the problem of electromagnetic wave propagation in a 2-D or 3-D optical waveguide (optical fiber). We will consider both the case of a rectilinear waveguide and the one of a waveguide presenting imperfections, with applications to phenomenons of physical interest. Numerical examples will be given.

1. Introduction.

A typical optical fiber is made of silica glass or plastic. Its central region is called *core*, and it is surrounded by a *cladding*, which has a slightly lower index of refraction. The cladding is surrounded by a protective *jacket*.

In optical waveguides, most of the electromagnetic radiation propagates without loss as a set of guided modes along the fiber axis. The electromagnetic field intensity of the guided modes in the cladding decays exponentially transversally to the waveguide's axis. That is why the radius of the cladding, which is typically several times larger than the radius of the core, can be considered infinite.

In the model we used, we study the following Helmholtz equation

$$L_0 u := \Delta u + k^2 n(x)^2 u = f, \quad x \in \mathbb{R}^N, \tag{1}$$

$N = 2, 3$, where k is the *wavenumber* and n is a positive function representing the index of refraction.

In Sections 2 and 3 we describe how to construct a Green's function for the case, respectively, of a 2-D and a 3-D rectilinear waveguide.

A mathematical framework for studying 2-D optical waveguides with small imperfections and related numerical experiments are shown in Sections 4 and 5, respectively.

2. 2-D rectilinear waveguides.

A rectilinear waveguide can be described by assuming that

$$n(x) = n_0(x_1) = \begin{cases} n_{co}(x_1), & \text{if } |x_1| \le h, \\ n_{cl}, & \text{if } |x_1| > h, \end{cases}$$

where n_{co} is a bounded even function. Thanks to the symmetry of the problem, we can separate the variables and look for solutions of the homogeneous Helmholtz equation of the form $u(x_1, x_2) = v(x_1, \lambda)e^{ik\beta x_2}$, with $\lambda = k^2(n_*^2 - \beta^2)$ and $n_* = \max n$. This leads to consider the following eigenvalue equation

$$v''(x_1, \lambda) + [\lambda - q(x_1)]v = 0, \quad x_1 \in \mathbb{R}, \tag{2}$$

where $q(x_1) = k^2[n_*^2 - n(x_1)^2]$.

Bounded solutions of (2) are of the form

$$v_j(x_1, \lambda) = \begin{cases} \phi_j(h, \lambda) \cos Q(x_1 - h) + \frac{\phi_j'(h,\lambda)}{Q} \sin Q(x_1 - h), & x_1 > h, \\ \phi_j(x_1, \lambda), & |x_1| \le h, \\ \phi_j(-h, \lambda) \cos Q(x_1 + h) + \frac{\phi_j'(-h,\lambda)}{Q} \sin Q(x_1 + h), & x_1 < -h, \end{cases}$$

$j = s, a$, with $Q = \sqrt{\lambda - d^2}$ and $d^2 = k^2(n_*^2 - n_{cl}^2)$. Here, v_j is symmetric or anti-symmetric in x_1 if $j = s$ or $j = a$, respectively. Solutions can be classified as follows:

- *Guided modes*. For $0 < \lambda < d^2$, only a finite number of eigenvalues λ_m^j are supported by (2). The solutions decay exponentially outside the core and they correspond to solutions of the Helmholtz equation which propagate most of their energy inside the core.
- *Radiation modes*. For $d^2 < \lambda < k^2 n_*^2$, $v_j(x_1, \lambda)$ are bounded and oscillatory. Thus, the corresponding solutions of the Helmholtz equation are bounded and oscillatory both in the x and the z directions.
- *Evanescent modes*. For $\lambda > k^2 n_*^2$, v_j are bounded and oscillatory, but the corresponding solutions of the Helmholtz equation decay exponentially in one direction along the x_2 axis and increase along the other one.

By using the theory of Titchmarsh for the eigenvalues problems of singular differential operators, it is possible to construct a resolution formula for (1):

$$u(x) = \int_{\mathbb{R}^2} G(x, y) f(y)\, dy\,, \tag{3}$$

with

$$G(x, y) = \sum_{j \in \{s,a\}} \int_0^\infty \frac{e^{i|x_2 - y_2|\sqrt{k^2 n_*^2 - \lambda}}}{2i\sqrt{k^2 n_*^2 - \lambda}} v_j(x_1, \lambda) v_j(y_1, \lambda)\, d\rho_j(\lambda)\,,$$

where, for every $\eta \in C_0^\infty([0, +\infty))$, it holds that

$$\langle d\rho_j, \eta\rangle = \sum_{m=1}^{M_j} r_j^m \eta(\lambda_j^m) + \frac{1}{2\pi} \int_{d^2}^\infty \frac{\sqrt{\lambda - d^2}}{(\lambda - d^2)\phi_j(h, \lambda)^2 + \phi_j'(h, \lambda)^2} \eta(\lambda)\, d\lambda\,.$$

3. 3-D rectilinear waveguides.

We study cylindrically symmetric optical fibers, i.e. when

$$n = n(r) = \begin{cases} n_{co}(r), & \text{if } 0 < r \le R, \\ n_{cl}, & \text{if } r > R, \end{cases}$$

where r is the distance from the fiber's axis. In this case, we separate the variables by using cylindrical coordinates (r, ϑ, x_3) and looking for solutions of the homogeneous Helmholtz equation of the form $u = e^{i\beta k x_3} e^{im\vartheta} w(r) r^{-\frac{1}{2}}$, $m \in \mathbb{Z}$. Hence, the associated eigenvalue problem is

$$w'' + \left[\lambda - q(r) - \frac{m^2 - 1/4}{r^2}\right] w = 0\,, \quad r > 0.$$

The classification of the solutions is analogous to the one obtained in the 2-D case.

In this case, we can still apply the theory of Titchmarsh in all its power (see [1]). Notice that in this case, due to the term $\frac{m^2-1/4}{r^2}$, the equation has a singularity at $r = 0$ besides the one at $r = +\infty$; this adds further technical difficulties.

Numerical results in the 3-D case are shown in Fig. 1, where we supposed n to be

$$n(r) = \begin{cases} n_{cl}, & 0 < r < a\,, \\ n_{co}, & a \le r < R\,, \\ n_{cl}, & r \ge R, \end{cases} \tag{4}$$

with n_{co} and n_{cl} constants and such that $n_{co} > n_{cl}$.

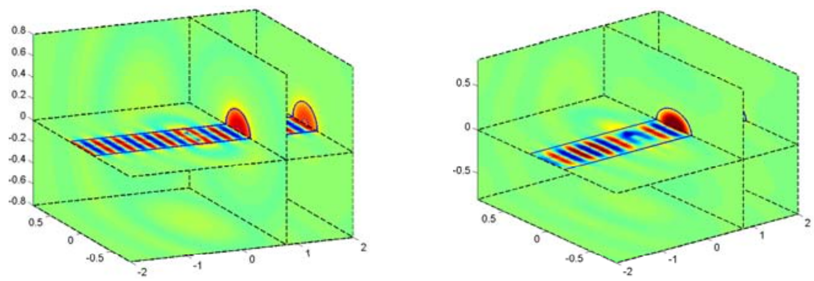


Figure 1: The figures represent the real part of the Green's function in the case of a coaxial cable, i.e. n given by (4). In the first one, the source is on the waveguide's axis, and only the symmetric modes are excited. In the second one, the source is inside the waveguide but not on the axis and all the guided modes are excited.

4. 2-D waveguides with imperfections.

Real-life waveguides are never perfect, since they might contain imperfections due to inhomogeneities or changes in the core's width and shape. When a pure guided mode is excited inside a guide with imperfections, a sort of resonation takes place and the other modes supported by the fiber are excited. This effect causes a signal distortion, since every guided mode propagates at its own characteristic velocity, and a loss in the signal power, due to the transfer of power to radiation, evanescent and the other guided modes.

These effects are not always to be avoided. It is possible to make optical devices which can "propagate the energy as desired". We will show two examples in the last part of this paper.

From the mathematical point of view, we consider the Helmholtz equation

$$L_\varepsilon u := \Delta u + k^2 n_\varepsilon(x_1, x_2)^2 u = f, \quad \text{in } \mathbb{R}^2, \tag{5}$$

where the index of refraction n_ε is supposed to be a small perturbation of n_0. We formally represent L_ε and $u := u_\varepsilon$ in terms of their Neumann series and find

$$L_0 u_0 = f, \quad L_0 u_1 = -L_1 u_0, \ldots, L_0 u_j = -\sum_{r=0}^{j-1} L_{j-r} u_r, \ \ldots$$

Each step of the above iterative method can be solved by using the resolution formula (3). It is possible to prove the existence of a solution by writing the equation $L_\varepsilon u = f$ as $L_0 u = f + (L_0 - L_\varepsilon)u$ and then as

$$u = L_0^{-1} f + \varepsilon L_0^{-1} \left(\frac{L_0 - L_\varepsilon}{\varepsilon} \right) u.$$

Consider a weight function $\mu(x) = \frac{16}{(4+|x|^2)^2}$. By using estimates on the solution (3), we are able to prove that the linear operators

$$L_0^{-1} : L^2(\mathbb{R}^2, \mu^{-1}) \to H^2(\mathbb{R}^2, \mu) \text{ and } \frac{L_0 - L_\varepsilon}{\varepsilon} : H^2(\mathbb{R}^2, \mu) \to L^2(\mathbb{R}^2, \mu^{-1})$$

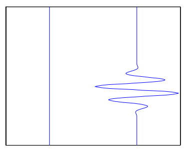
are continuous. Hence, by choosing ε small enough and using the contraction mapping theorem, we prove the existence of a solution of $L_\varepsilon u = f$.

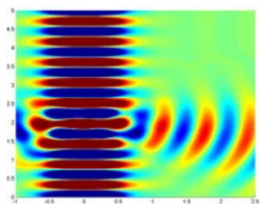
5. Numerical results.

In the following simulations, we will always suppose that the zeroth order term of the Neumann series of u is a pure guided mode and we will show the effect due to the imperfections by computing the first term u_1.

Near-field.

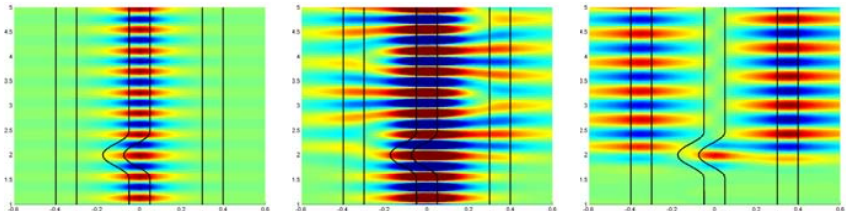
We suppose that the profile of the perturbation is as the one in the small figure on the right. In the second figure on the right, the real part of $u_0 + \varepsilon u_1$ in proximity of the waveguide is represented. We notice that most of the radiating energy is directed along a certain direction. Such energy is due to the coupling between the guided mode u_0 and the radiation and evanescent modes supported by the waveguide.





Coupling between guided modes

Grating-assisted direction couplers are optical devices where two or more waveguides are close to each other and the coupling between guided modes is made by using a perturbation of the index of refraction of the core or of the cladding. An example is shown in the figures below, where we show the real part of u_0, $u_0 + \varepsilon u_1$ and u_1, respectively.



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DISEGUAGLIANZA DI TIPO HARNACK PER LE SOLUZIONI DI EQUAZIONI ELLITTICHE NON LINEARI DI ORDINE SUPERIORE

SALVATORE D'ASERO

In questa nota abbiamo preso in esame la seguente equazione di ordine superiore al secondo:

$$(1) \qquad \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \mathrm{D}^\alpha A_\alpha(x, u, \mathrm{D}^1 u, \cdots, \mathrm{D}^m u) = 0 \qquad x \in \Omega \subset \mathbb{R}^n$$

che soddisfa una condizione di ellitticità degenere del tipo:

$$(2) \qquad \sum_{1 \leq |\alpha| \leq m} A_\alpha(x, \xi)\xi_\alpha \geq \nu_1 \Big\{ \sum_{|\alpha|=m} w_p(x)|\xi_\alpha|^p + \sum_{|\alpha|=1} w_q(x)|\xi_\alpha|^q \Big\} -$$

$$\nu_2 \sum_{1<|\alpha|<m} w_\alpha(x)|\xi_\alpha|^{p_\alpha} - g_0(x)w_q(x)|\xi_0|^q - f_0(x)w_q(x).$$

con $q > mp$ e dove $\xi = \big\{\xi_\alpha \in \mathbb{R} : |\alpha| \leq m\big\} \in \mathbb{R}^{N(m)}$, ν_1, ν_2 sono costanti positive, p_α è un numero positivo soddisfacente alcune condizioni, $w_\alpha(x)$ sono funzioni misurabili non negative e $w_q(x)$ appartiene alla classe di Muckenhoupt A_q e $g_0(x)$, $f_0(x)$ sono funzioni in opportuni spazi pesati di Lebesgue.

Osserviamo che la condizione di ellitticità considerata, anche nel caso non degenere, è più forte di quella che usualmente si considera:

$$\sum_{|\alpha|=m} A_\alpha(x,\xi)\xi_\alpha \geq \nu_1 \sum_{|\alpha|=m} |\xi_\alpha|^p - \nu_2 \sum_{|\alpha|<m} |\xi_\alpha|^{r_\alpha} - f(x), \tag{3}$$

ma tale condizione, come ben si sa, in generale non ci permette di stabilire nemmeno la limitatezza delle soluzioni se la dimensione di Ω è sufficientemente grande (vedi [9]).

La classe delle equazioni soddisfacenti la (2), fu introdotta nel 1978 da Skrypnik in [10], [9] dove ogni soluzione è limitata e Hölder continua e ciò indipendentemente da qualunque relazione tra n, m, q, p.

In questa nota viene presentato un risultato concernente una diseguaglianza di Harnack per le soluzioni non negative dell'equazione (1), i cui coefficienti soddisfano la condizione degenere (2) e la seguente condizione di crescita:

$$\begin{aligned} |A_\alpha(x,\xi)|^{\frac{p_\alpha}{p_\alpha-1}}[w_\alpha(x)]^{-\frac{1}{p_\alpha-1}} \leq \\ C_2 \sum_{1\leq|\beta|\leq m} w_\beta(x)|\xi_\beta|^{p_\beta} + [g_\alpha(x)|\xi_0|^q + f_\alpha(x)]w_q(x), \end{aligned} \tag{4}$$

con $g_\alpha(x)$ e $f_\alpha(x)$ appartenenti ad opportuni spazi pesati di Lebesgue. In particolare, utilizzando la tecnica adottata da Nicolosi e Skrypnik in [7] nel provare l'analogo risultato per equazioni non degeneri e adattandola al caso pesato, si otterrà il seguente risultato:

Teorema 1. *Supponiamo che valgano le condizioni di ellitticità* (2) *e di crescita* (4) *e sia* $u(x)$ *una soluzione non negativa dell'equazione* (1) *nella sfera* $B(x_0, 3R)$, $x_0 \in \Omega_{4R}$, $0 < R \leq 1$. *Allora vale la seguente disuguaglianza:*

$$\operatorname*{sup\,ess}_{B(x_0,R)} u(x) \leq L\left\{ \operatorname*{inf\,ess}_{B(x_0,R)} u(x) + F(x_0, R)\right\} \tag{5}$$

dove L *è una costante positiva dipendente solo da parametri noti.*

Osserviamo che, anche nel caso non pesato, è impossibile ottenere la diseguaglianza di Harnack per le soluzioni non negative dell'equazione (1) associata alla condizione di ellitticità standard (3) (proprio perché esistono esempi di soluzioni non limitate), ma è impossibile ottenerla anche per l'equazione (1) con la condizione (2) quando $q = mp$.

Infine applicando la diseguaglianza di Harnack (5) si deriva un risultato di regolarità hölderiana per le soluzioni dell'equazione (1), del tipo:

Teorema 2. *Supponiamo che le condizioni* (2) *e* (4) *siano verificate. Se* $u(x)$ *è una soluzione dell'equazione* (1) *nel dominio* Ω. *Allora per* $\rho > 0$ *vale:*

$$|u(x) - u(y)| \leq H[F(\Omega_\rho) + M_\rho]|x - y|^\gamma, \tag{6}$$

$$x, y \in \Omega_{2\rho} = \left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 2\rho\right\},$$

dove

$$F(\Omega_\rho) = 1 + \left\{ \sum_{0 \leq |\alpha| \leq m} ||f_\alpha||_{t_1, \Omega_\rho} \right\}^{\frac{1}{q}}, \quad M_\rho = \text{ess sup}\left\{|u(x)| : x \in \Omega_\rho\right\},$$

H *dipende dagli stessi parametri cui dipende* L *nel Teorema 1,* γ *dipende solo da parametri noti e dalle rispettive norme di* $g_\alpha(x)$ *nel dominio* Ω_ρ.

Lo stesso risultato, ottenuto da Nicolosi e Skrypnik in [6], è qui, pertanto, ottenuto con una tecnica differente.

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LINEAR ELLIPTIC EQUATIONS RELATED TO GAUSS MEASURE

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We study a Dirichlet problem relative to a linear second order elliptic equation with lower order terms, where ellipticity condition is given in terms of the function $\varphi(x) = (2\pi)^{-\frac{n}{2}}\exp(-|x|^2/2)$, the density in the Gaussian measure. We use the notion of rearrangement with respect to the Gauss measure to obtain apriory estimate and we study the summability of the solution when data vary in suitable Lorent-Zygmund spaces.

Let us consider the problem:

$$(1)\qquad \begin{cases} -\left(a_{ij}(x)u_{x_i}\right)_{x_j} - (d_i(x)u)_{x_i} + b_i(x)u_{x_i} + c(x)u = \\ \qquad\qquad\qquad\qquad\qquad g\varphi - (f_i\varphi)_{x_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open set of $\mathbb{R}^n$ $(n \geq 2)$, $\varphi(x) = (2\pi)^{-\frac{n}{2}}\exp\left(-|x|^2/2\right)$ is the density of the Gauss measure and $a_{ij}(x)$, $d_i(x)$, $b_i(x)$, $i, j = 1, .., n$, and $c(x)$ are measurable functions on Ω such that

(i) $a_{ij}(x)\xi_i\xi_j \geq \varphi(x)|\xi|^2$ for $a.e.$ $x \in \Omega, \forall \xi \in \mathbb{R}^n$,

(ii) $\dfrac{a_{ij}(x)}{\varphi(x)} \in L^\infty(\Omega)$,

(iii) $\left(\sum b_i^2(x)\right)^{\frac{1}{2}} \leq b(x)\varphi(x)$, $b(x) \in L^\infty(\log L)^{-\frac{1}{2}}(\varphi, \Omega)$,

(iv) $\left(\sum d_i^2(x)\right)^{\frac{1}{2}} \leq d(x)\varphi(x),\ d(x) \in L^\infty (\log L)^{-\frac{1}{2}} (\varphi, \Omega)$,

(v) $\dfrac{c(x)}{\varphi(x)} \in L^\infty (\Omega)$ and $c(x) \geq 0$,

(vi) $g(x) \in L^2 (\log L)^{-\frac{1}{2}} (\varphi, \Omega)$,

(vii) $f_i(x) \in L^2 (\varphi, \Omega)\ \ i = 1, ..n,\ \ \sum f_i^2(x) = f^2(x)$.

problem (1) is related to the generator of Ornstein-Uhlenbeck semigroup (see e.g. [3]).
First of all we observe that the natural space for searching weak solution of the problem (1) is the weighted Sobolev space $H_0^1(\varphi, \Omega)$, that is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{H_0^1(\varphi,\Omega)} = \left(\int_\Omega |\nabla u(x)|^2 \ \varphi(x)\,dx\right)^{\frac{1}{2}}.$$

We recall that $u \in H_0^1(\varphi, \Omega)$ is a weak solution of problem (1), if

$$\int_\Omega (a_{ij}(x)u_{x_i}\psi_{x_j} + d_i(x)u\psi_{x_i} + b_i(x)u_{x_i}\psi + c(x)u\psi)\,dx =$$
$$\int_\Omega \left(g + f_i\psi_{x_i}\right)\varphi(x)\,dx \quad \forall \psi \in H_0^1(\varphi, \Omega),$$

Our aim is to prove apriori estimates and regularity results for problem (1).

It is well known that if Ω is bounded and problem (1) is uniformly elliptic it is possible to compare the solution of original problem with the solution of a simpler one which is defined in a ball and has spherical symmetric data (see for example [10], [1]). The main tools for proving this kind of results are Schwarz symmetrization and isoperimetric inequality.

In this case, using Gauss symmetrization and the isoperimetric inequality with the respect to Gauss measure we are able to compare the solution of the problem (1) with the solution of a problem which is defined in an halfspace and has data depending on one variable.

To explain our results we give some definitions.

If $\gamma_n(dx) = \varphi(x)\,dx = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{2}\right) dx,\ x \in \mathbb{R}^n$ is the n-dimensional Gauss measure on $\mathbb{R}^n$ normalized by $\gamma_n(\mathbb{R}^n) = 1$, we define the perimeter with respect to Gauss measure as

$$P(E) = (2\pi)^{-\frac{n}{2}} \int_{\partial E} \exp\left(-\frac{|x|^2}{2}\right) \mathcal{H}_{n-1}(dx),$$

where E is a $(n-1)-$rectifiable set and $\mathcal{H}_{n-1}$ denotes the $(n-1)-$dimensional Hausdorff measure. It is well known (see e.g. [9]) that for all subsets $E \subset \mathbb{R}^n$ having the same Gauss measure of the half-space takes the smallest perimeter, i.e.

$$P(E) \geq P(H(\xi, \lambda)).$$

where $H(\xi, \lambda) = \{x \in \mathbb{R}^n : (x, \xi) > \lambda\}$.

Moreover we give the notion of rearrangement. If u is a measurable function in Ω, we denote by

- $u^{\circledast}$ the decreasing rearrangement of u with respect to Gauss measure, i.e.

$$u^{\circledast}(s) = \inf\{t \geq 0 : \mu(t) \leq s\} \qquad s \in]0, 1],$$

  where $\mu(t) = \gamma_n(\{x \in \Omega : |u| > t\})$ is the distribution function of u;
- $u^{\star}$ the rearrangement with respect to Gauss measure of u, i.e.

$$u^{\star}(x) = u^{\circledast}(\Phi(x_1)) \qquad x \in \Omega^{\star},$$

  where $\Omega^{\star} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$ is the half-space such that $\gamma_n\left(\Omega^{\star}\right) = \gamma_n(\Omega)$.

In what follows we will use also the notion of pseudo-rearrangement introduced in [2]. Let u be a measurable function in Ω, $f \in L^p(\varphi, \Omega)$ with $1 \leq p \leq +\infty$, $f \geq 0$ and $\Omega^{\circledast} = (0, \gamma_n(\Omega))$. We will say that a function $\widetilde{f_u} : \Omega^{\circledast} \to \mathbb{R}$ is a Gauss pseudo-rearrangement of f with respect to u if there exists a family $\mathcal{E}(u) = \{E(s)\}_{s \in \Omega^{\circledast}}$ of measurable subsets of Ω such that

$$\begin{aligned}
&\gamma_n(E(s)) = s, \\
&s_1 \leq s_2 \quad \Rightarrow \quad E(s_1) \subseteq E(s_2) \\
&E(s) = \{x \in \Omega : |u(x)| > u^{\circledast}(s)\} \quad \text{if} \quad \exists t \in \mathbb{R},\ s = \mu(t) \quad \text{and}
\end{aligned}$$

$$\widetilde{f_u}(s) = \frac{d}{ds} \int_{E(s)} f(x)\varphi(x)\, dx \quad \text{for a.e.} \quad s \in \Omega^{\circledast}.$$

In general $\widetilde{f_u}$ is not a rearrangement of f, but it is a weak limit of a sequence of functions that have the same distribution function of f.

Finally let us recall that a measurable function u belongs to the Lorentz-Zygmund space $L^{p,q}(\log L)^{\alpha}(\varphi, \Omega)$ for $0 < q, p \leq \infty$ and $-\infty < \alpha < +\infty$, if

$$||u||_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)} = \begin{cases} \left(\displaystyle\int_0^{\gamma_n(\Omega)} \left[t^{\frac{1}{p}} (1 - \log t)^{\alpha} u^{\circledast}(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ \qquad \text{if} \quad 0 < q < \infty, \\ \displaystyle\sup_{t \in (0, \gamma_n(\Omega))} \left[t^{\frac{1}{p}} (1 - \log t)^{\alpha} u^{\circledast}(t) \right] \\ \qquad \text{if} \quad q = \infty, \end{cases}$$

is finite [1].

The following theorem is proved in [7]:

Theorem 1. *Let Ω be an open set of $\mathbb{R}^n$ with $\gamma_n(\Omega) < 1$ and let $u \in H_0^1(\varphi, \Omega)$ be solution of* (1) *under the assumptions* (i)–(vii)*; moreover, let us suppose that either $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ is small enough or $b \in L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi, \Omega)$ with $2 < a < \infty$. Let $w(x) = w^\star(x)$ be the solution of problem*

$$\text{(2)}\qquad \begin{cases} \left(w_{x_1}\varphi(x)\right)_{x_1} + (D(\Phi(x_1))w\varphi(x))_{x_1} - B(\Phi(x_1))w_{x_1}\varphi(x) = \\ \qquad g^\star(x_1)\varphi(x) - (F(\Phi(x_1))\varphi(x))_{x_1} & \text{in } \Omega^\star \\ w = 0 & \text{on } \partial\Omega^\star, \end{cases}$$

where $\Omega^\star$ is the half-space $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$, with $\lambda \in \mathbb{R}$ such that $\gamma_n(\Omega) = \gamma_n(\Omega^\star)$, $\Phi(\tau) = \gamma_n(\{x \in \mathbb{R}^n : x_1 > \tau\})$ and F, B and D are functions such that $F^2 = \widetilde{\left(f^2\right)}_u$, $B^2 = \widetilde{\left(b^2\right)}_u$ and $D^2 = \widetilde{\left(d^2\right)}_u$.
Then

$$\text{(3)}\qquad u^\star(x_1) \le w^\star(x_1) = w(x) \qquad \text{for a.e. } x \in \Omega^\star.$$

Comparison (3) provides estimates of u in terms of the solution of a problem of the same type of (1), but simpler, because it is defined in an half-space and its coefficients depend only on one variable. We observe that if $b_i(x) = 0$ or $d_i(x) = 0$ then the solution of the problem (1) can be explicitely written and then comparison (3) gives an explicite estimate; an estimate of the norm $|\nabla u|$ can be also proven (see [7] for details).

Starting by comparison result we study regularity results in the Lorentz-Zygmund spaces.

Let us observe that by Gross inequality we have that if $u \in H_0^1(\varphi, \Omega)$ is a solution of problem (1) then u belongs to Lorentz-Zygmund space $L^2(\log L)^{\frac{1}{2}}(\varphi, \Omega)$. We study how the summability of u improves by improving the summability of the data f and g in Lorentz-Zygmund spaces $L^{p,q}(\log L)^\alpha(\varphi, \Omega)$.

The following theorem states the regularity result in the case $f_i(x) \equiv d_i(x) \equiv 0$, $i = 1, \ldots, n$, all the other cases are studied in [7].

Theorem 2. *Under the assumptions of Theorem 1, when $d_i(x) \equiv f_i(x) \equiv 0$, $i = 1, \ldots, n$ and $g \in L^{p,q}(\log L)^\alpha(\varphi, \Omega)$, the following results hold:*

[1] We use the following 'arithmetic' convention: $\frac{s}{\infty} = 0$ for $s > 0$.

(a) if

$$p = 2 \ \textit{and either} \ 1 \le q \le 2 \ \textit{and} \ \alpha \ge -\frac{1}{2} \ \textit{or} \ 2 < q \le \infty \ \textit{and} \ \alpha > -\frac{1}{q}$$

or

$$2 < p < \infty, \ 1 \le q \le \infty \ \textit{and} \ -\infty < \alpha < +\infty,$$

then $u \in L^{p,q}(\log L)^{\alpha+1}(\varphi, \Omega)$ *and*

$$\|u\|_{L^{p,q}(\log L)^{\alpha+1}(\varphi,\Omega)} \le C_1 \ \|g\|_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)} ;$$

(b) if

$$p = \infty, 1 \le q \le \infty, -\infty < \alpha < +\infty \ \textit{and} \ \alpha + \frac{1}{q} < 0,$$

then $u \in L^{\infty,q}(\log L)^{\alpha}(\varphi, \Omega)$ *and*

$$\|u\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)} \le C_2 \ \|g\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)} .$$

The constants C_1, C_2 *depend on* $p, q, \alpha, \gamma_n(\Omega)$ *and* $\|b\|_{L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$.

Comparison results using rearrangement with respect to Gauss measure are proved in [4] when $d_i(x) \equiv b_i(x) \equiv c(x) \equiv f_i(x) \equiv 0 \ i = 1, \dots, n$ and in [6] when $d_i(x) \equiv f_i(x) \equiv 0$ and $c(x) \ge c_0(x)\varphi(x)$. Parabolic and nonlinear case has been studied respectively in [5] and [8].

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ANALYSIS ON METRIC SPACES AND APPLICATIONS TO PDE'S

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1. Introduction.

Harnack inequality for solutions to degenerate elliptic equations has received much attention in recent years. In several cases the underlying geometry for their study relies on quite general metric structures of homogeneous type. Moser iteration technique has been extended to non Euclidean settings, yielding Harnack inequalities for solutions to second order degenerate elliptic equations in *divergence form* with underlying Carnot-Caratheodory metric structures, see [5], [8] and references therein.

Caffarelli's technique [2], [3], to prove Harnack's inequality for uniformly elliptic fully nonlinear equations has been extended in [4] to the linearized Monge-Ampere equation, see also [7], Chapter 2. The role of the Euclidean balls in Caffarelli's original method is played, in [4], by the *sections* of a convex function. This makes clear the quasi metric character of some crucial parts of the procedure and it is the purpose of this paper to put these techniques in the frame of quasi metric spaces and apply the results to X-elliptic operators. This technique permits to handle both non divergence and divergence structure linear equations simultaneously.

We would like to point out that the present method avoids the use of the BMO John-Nirenberg type inequality, which plays a crucial role in Moser iteration technique. It is well known that this inequality is hard to prove in the settings of quasi metric doubling spaces, see [1].

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2. Critical Density, Double Ball Property and power decay.

In this section (Y, d, μ) is a doubling quasi metric Hölder space. We begin with the following two definitions. Let $\Omega \subseteq Y$ be open. We shall denote by $\mathbb{K}_\Omega$ a family of μ-measurable functions with domain contained in Ω. If $u \in \mathbb{K}_\Omega$ and its domain contains a set $A \subset \Omega$, we write $u \in \mathbb{K}_\Omega(A)$.

Definition 2.1. (Critical density) Let $0 < \epsilon < 1$. We say that $\mathbb{K}_\Omega$ satisfies the ϵ critical density property if there exists a constant $c = c(\epsilon) > 0$ such that for every ball $B_{2R}(x_0) \subset \Omega$ and for every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with $\mu(\{x \in B_R(x_0) : u(x) \geq 1\}) \geq \epsilon\, \mu(B_R(x_0))$, we have $\inf_{B_{R/2}(x_0)} u \geq c$.

Definition 2.2. (Double ball property) We say that $\mathbb{K}_\Omega$ satisfies the double ball property if there exists a positive constant γ such that for every $B_{2R}(x_0) \subset \Omega$ and every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with $\inf_{B_{R/2}(x_0)} u \geq 1$ we have $\inf_{B_R(x_0)} u \geq \gamma$.

We point out that we do not identify two functions that differ on a set of μ measure zero. We also notice that if $\mathbb{K}_\Omega$ satisfies the ϵ_0 critical density property, then $\mathbb{K}_\Omega$ satisfies the ϵ critical density for any $\epsilon > \epsilon_0$.

The ϵ critical density and the double ball properties are in general independent but if the critical density holds for ϵ sufficiently small, then the double ball property also holds.

Proposition2.3. *If c_D is the doubling constant of μ and $\mathbb{K}_\Omega$ satisfies the ϵ critical density for some $0 < \epsilon < 1/c_D^2$, then $\mathbb{K}_\Omega$ satisfies the double ball property.*

As a consequence of the critical density and the double ball properties we get the following proposition.

Proposition 2.4. (Improved Critical Density) *Assume that $\mathbb{K}_\Omega$ satisfies the double ball property and the critical density property for some $0 < \epsilon < 1$. Assume $\mathbb{K}_\Omega$ closed under multiplications by positive constants. Then there exists a structural constant $M_0 > 1$ depending on ϵ such that for any $\alpha > 0$ and any*

$u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with $\mu(\{x \in B_R(x_0) : u(x) \geq \alpha\}) \geq \epsilon\, \mu(B_R(x_0))$, we have $\inf\limits_{B_R(x_0)} u \geq \alpha/M_0$.

Definition 2.5. The family $\mathbb{K}_\Omega$ satisfies the power decay property if there exist constants $M, \eta > 1$ and $0 \leq \gamma < 1$ such that for each $u \in \mathbb{K}_\Omega(B_{\eta R}(x_0))$ with $\inf\limits_{B_r(x_0)} u \leq 1$ we have

$$\mu(\{x \in B_{r/2}(x_0) : u(x) > M^k\}) \leq \gamma^k\, \mu(B_{r/2}(x_0)), \qquad k = 1, 2, \cdots.$$

Families of functions satisfying the critical density and double ball properties satisfy the power decay property too.

Theorem 2.6. (Power decay) *Let (Y, d, μ) be a doubling quasi-metric Hölder space and let $\Omega \subset Y$ be open and such that $\mu(B_r(x)) \leq \delta\mu(B_{2r}(x))$ for a suitable $0 \leq \delta < 1$ and for every ball $B_{2r}(x) \subset \Omega$. Suppose the set $\mathbb{K}_\Omega$ is closed under multiplication by positive constants and satisfies the following conditions:*

1. *$\mathbb{K}_\Omega$ satisfies the ϵ-critical density property for some $0 < \epsilon < 1/c_D^2$.*
2. *The function $r \mapsto \mu(B_r(x))$ is continuous for each $x \in Y$.*

Then, the family $\mathbb{K}_\Omega$ satisfies the power decay property.

3. Power decay and abstract Harnack inequality.

Theorem 3.1. *Suppose that the family of functions $\mathbb{K}_\Omega$ satisfies the power decay property in Definition 2.5 and assume in addition that $\mathbb{K}_\Omega$ is closed under multiplications by positive constants and if $u \in \mathbb{K}_\Omega(B_r(x_0))$ satisfies $u \leq \lambda$ in $B_r(x_0)$, then $\lambda - u \in \mathbb{K}_\Omega(B_r(x_0))$. There exists a positive constant c independent of u, R and x_0 such that if $u \in \mathbb{K}_\Omega(B_{2\eta_0 R}(x_0))$ is nonnegative and locally bounded, then $\sup_{B_R(x_0)} u \leq c \inf_{B_R(x_0)} u$, where $\eta_0 = K(2K\eta + 1)$ and η is the constant in Definition 2.5.*

We point out that we may assume that d is a Hölder quasi distance. Indeed, by Macías and Segovia [9], Theorem 2, every quasi metric has an equivalent Hölder quasi metric. On the other hand, it is easy to see that if a family of functions satisfies the power decay property, or the Harnack inequality, with respect to a quasi distance d, then it satisfies the same properties with respect to any quasi distance d' equivalent to d.

From Harnack inequality in the previous theorem a Hölder regularity result readily follows.

Theorem 3.2. *Suppose $\mathbb{K}$ satisfies, together with the hypotheses of Theorem 3.1 the following one: for any ball $B_R(x_0)$,if $u \in \mathbb{K}(B_R(x_0))$ satisfies $u \geq \lambda$ in $B_R(x_0)$, then $u - \lambda \in \mathbb{K}(B_R(x_0))$. There exist positive constants C and α independent of u, R and x_0 such that if $u \in \mathbb{K}(B_{\frac{2R}{\eta}}(x_0))$, then*

$$|u(x) - u(y)| \leq C \left(\frac{d(x, y)}{R}\right)^{\alpha} \sup_{B_R(x_0)} |u| , \quad \forall x, y \in B_R(x_0).$$

4. X-elliptic operators.

Let $X = (X_1, \ldots, X_m)$ be a family of vector fields, with locally Lipschitz continuous coefficients, defined in an open set $Y \subset \mathbb{R}^N$.

We will consider "degenerate" operators which are elliptic with respect to the given system X of vector fields.

Definition 4.1. Let $(a_{i,j})_{i,j=1,\ldots,N}$ be a symmetric matrix with measurable entries. We say that the operator

$$L \equiv \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) \tag{4, 1}$$

is uniformly X-elliptic in an open set $\Omega \subset Y$ if there exist two positive constants λ, Λ such that

$$\lambda \sum_{i=1}^{m} |\langle X_i(x), \xi\rangle|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq \Lambda \sum_{i=1}^{m} |\langle X_i(x), \xi\rangle|^2 , \tag{4.2}$$

$$\forall \xi \in \mathbb{R}^N , \quad \text{a.e. } x \in \Omega .$$

Regarding the given system of vector fields, we shall assume that the Carnot-Caratheodory distance d related to X is well defined and *continuous* with respect to the Euclidean topology. Moreover, we assume (Y, d, μ) is a doubling metric space, where $d\mu = dx$ denotes the Lebesgue measure. We also assume the following

(P) (Poincaré inequality): There exists a positive constant C such that

$$\left(\fint_{B_R} (u - u_R)^2 dx\right)^{1/2} \leq C\, R \left(\fint_{B_R} |Xu|^2\, dx\right)^{1/2} \quad \forall u \in C^1(B_R),$$

and for every d-ball with $B_R = B_R(x_0)$, $x_0 \in Y$, $R > 0$.

Given a set E we denote $u_E \equiv \fint_E u$. If E is a metric ball B_r we put $u_r = u_{B_r}$. Moreover, Xu denotes the X- gradient of u, i.e. $Xu = (X_1 u, \ldots, X_m u)$ where, if $X = (c_j^1, \ldots, c_j^N)$,

$$X_j u = \sum_{k=1}^{N} c_j^k \partial_{x_k} u \,, \quad j = 1, \ldots, m \,.$$

We define the (generalized) Sobolev space $W^1(\Omega, X)$ related to the family X and the open set Ω, as follows

$$W^1(\Omega, X) = \{u \in L^2(\Omega) : |Xu| \in L^2(\Omega)\} \,.$$

We recall that $W^1(\Omega, X)$ is the closure of $\{u \in C^1(\Omega) : u \in L^2(\Omega) : |Xu| \in L^2(\Omega)\}$, with respect to the norm $u \to \|u\|_{L^2(\Omega)} + \|Xu\|_{L^2(\Omega)}$, see [6]. We say that $u \in W^1_{loc}(\Omega, X)$ if $u\varphi \in W^1(\Omega, X)$ for any $\varphi \in C^1_0(\Omega)$.

Let us now consider the bilinear form

$$\mathcal{L}(u, \varphi) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x)\, u_{x_i} \varphi_{x_j}\, dx$$

defined in $C^1_0(\Omega) \times C^1_0(\Omega)$. Since

$$|\mathcal{L}(u, \varphi)| \le \Lambda \|Xu\|\, \|X\varphi\|, \ \ \forall u, \varphi \in C^1_0(\Omega) \,,$$

$\mathcal{L}$ can be extended to $W^1_{loc}(\Omega, X) \times C^1_0(\Omega)$.

Definition 4.2. (weak solutions) We say that a function $u \in L^2(\Omega)$ is a weak subsolution (supersolution) to $Lu = 0$ in Ω, if $Xu \in L^2(\Omega)$ and

$$\mathcal{L}(u, \varphi) \le (\ge) 0 \,, \tag{4.3}$$

for all $\varphi \in C^1_0(\Omega)$, $\varphi \ge 0$. We say u is a solution if it is both a sub and a super–solution.

The set of nonnegative solutions to X-elliptic equations satisfies the critical density property in Definition 2.1 for all $0 < \epsilon < 1$. First of all we point out that weak subsolutions to X-elliptic equations are (essentially) locally bounded. This has been shown in [8], inequality (4.12).

Theorem 4.3. *(Local boundedness) Let $u \geq 0$ be a weak subsolution of $Lu = 0$ in Ω and let $B_R(x_0)$ be a d-ball such that $B_{2R}(x_0) \subset \Omega$. Then there exists a constant c such that*

$$\operatorname*{ess\,sup}_{B_R(x_0)} u \leq c \left(\fint_{B_{2R}(x_0)} u^2 \, dx \right)^{1/2} . \tag{4, 4}$$

It is a standard fact that if u is a weak solution to $Lu = 0$ in Ω then $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ are (non negative) sub solutions (see [8]). Then, $u = u^+ - u^-$ is essentially locally bounded. As a consequence, there exists a locally bounded function $\hat{u} \in W^1_{loc}(\Omega, X)$ such that $L\hat{u} = 0$ and $\hat{u} = u$ a.e. in Ω. From now on we identify u with $\hat{u}$. The critical density is a consequence of the previous results.

Theorem 4.4. (Critical density for all $0 < \epsilon < 1$) *Let $B_{2R}(x_0) \subset \Omega$ be a metric ball. Given $0 < \epsilon < 1$, there exists $c = c(\epsilon) > 0$ such that for any nonnegative supersolution to $Lu = 0$ in the ball $B_{2R}(x_0)$ satisfying $|\{x \in B_R(x_0) : u(x) \geq 1\}| \geq \epsilon\, |B_R(x_0)|$, we have $\inf_{B_{R/2}(x_0)} u \geq c$.*

Let us assume now that the open set Ω is such that

$$\mu(B_r(x)) < \delta\mu(B_{2r}(x)) , \tag{4.5}$$

for every d-ball $B_{2r}(x) \subset \Omega$. Since we are interested in local properties of the solutions we may assume that (4.5) holds. Actually, it holds true if the d-diameter of Ω is sufficiently small. Let us now define

$$\mathbb{K}_\Omega := \left\{ u \in W^1_{loc}(A) \ : \ A \subset \Omega \, , A \text{ open} : Lu = 0 \text{ in } A, u \in L^\infty_{loc} \, , \ u \geq 0 \right\} .$$

Obviously if $u \in \mathbb{K}_\Omega(A)$ and $\lambda_1 \leq u \leq \lambda_2$ then $\lambda_2 - u$ and $u - \lambda_1$ belong to $\mathbb{K}_\Omega(A)$.

Thus, keeping in mind all our previous results applied to the family $\mathbb{K}_\Omega(A)$, we get the following result.

Theorem 4.5. (Harnack inequality for X-elliptic operators) *Let $u \in W^1_{loc}(\Omega, X)$ be a non negative solution to $Lu = 0$ in Ω. There exist structural constants $c, \theta > 1$ such that $\sup_{B_R(x_0)} u \leq c \inf_{B_R(x_0)} u$, for every d-ball such that $B_{\theta R}(x_0) \subset \Omega$.*

Remark 4.6. Since the metric balls are relatively compact and connected, a standard argument can be used to prove that in the previous theorem one can replace the constant θ by any constant bigger than one.

Theorem 4.7. (Hölder continuity for X-harmonic functions) *Let* $u \in W^1_{loc}(\Omega, X)$ *be a solution to* $Lu = 0$ *in* Ω. *There exists structural positive constants* c *and* $0 < \alpha \leq 1$, *such that*

$$|u(x) - u(y)| \leq c \left(\frac{d(x, y)}{R} \right)^{\alpha} \sup_{B_R(x_0)} |u| ,$$

for every d*-ball* $B_R(x_0) \subseteq \Omega$.

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$C^{1,\alpha}_{loc}$ REGULARITY FOR SUBELLIPTIC P-HARMONIC FUNCTIONS IN GRUŠIN PLANE

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The main result of our work ([3]) is the $C^{1,\alpha}_{loc}$ regularity for subelliptic p-harmonic functions in the case of the Grušin vector fields. To this goal we prove a Calderón-Zygmund inequality and an estimate for strong solutions of a linear subelliptic equation in nondivergence form with L^∞ coefficients.

Let Ω be a bounded domain in $\mathbb{R}^n$, $q \le n$ and $X = (X_1, X_2, \cdots, X_q)$ be a system of C^∞ vector fields defined in Ω. We suppose that the system X satisfies the following Hörmander or bracket generating condition of step m in Ω (see [6]):

the vector fields X_i together with their commutators of length at most m span $\mathbb{R}^n$ at every point of Ω.

In particular we will discuss about the following Grušin vector fields

$$X_1 = \frac{\partial}{\partial x}\ , \quad X_2 = x\,\frac{\partial}{\partial t}\ .$$

They satisfy the Hörmander condition of step 2 at every point of $\mathbb{R}^2$.

In Ω we consider the Carnot–Carathéodory distance defined with respect to the system X of vector fields. For more details see [7].

For $k \in \mathbb{N}$ and $p > 1$ we define the Sobolev spaces associated to the vector fields X_i as

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \ : \ X_{i_1} ... X_{i_j} u \in L^p(\Omega) \ , \ 1 \leq j \leq k \right\} ,$$

and we denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the following norm in $W^{k,p}(\Omega)$:

$$\|u\|_{W^{k,p}(\Omega)} \equiv \|u\|_{L^p(\Omega)} + \sum_{j=1}^{k} \sum_{i_j=1}^{q} \|X_{i_1} \ldots X_{i_j} u\|_{L^p(\Omega)} .$$

We denote by Δ_X the sub-elliptic Laplace operator $\sum_{j=1}^q X_j^2$. Our first result is the following Calderón-Zygmund inequality.

Theorem 1. *Let $p \in (1, \infty)$. Then there exists a constant $\mathcal{C}_p$ such that for all $u \in W_0^{2,p}(\Omega)$ we have*

$$\|X^2 u\|_{L^p(\Omega)} \leq \mathcal{C}_p \|\Delta_X u\|_{L^p(\Omega)} .$$

Remark 1. In the case of Grušin plane we have $\mathcal{C}_2 = \sqrt{3}$ as in the Heisenberg group $\mathbb{H}^1$ (see [4]) and this constant is sharp.

Now, let us consider the non variational linear operator

$$\mathcal{A} u = \sum_{i,j}^{q} a_{ij}(x) X_i X_j u ,$$

where $a_{ij} \in L^\infty(\Omega)$.

We assume that the operator $\mathcal{A}$ satisfies the following Cordes condition $K_{\varepsilon,\sigma}$ (see [2]):

there exist $\varepsilon \in (0, 1]$ and $\sigma > 0$ such that for a.e. $x \in \Omega$

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{q} a_{ij}^2(x) \leq \frac{1}{q - 1 + \varepsilon} \left(\sum_{i=1}^{q} a_{ii}(x) \right)^2 .$$

We denote by I the $q \times q$ identity matrix and by $A(x) = \{a_{ij}(x)\}_{i,j=1,\ldots,q}$ the matrix of the coefficients. The next result follows from Cordes condition and Theorem 1.

Theorem 2. *Let* $0 < \varepsilon \leq 1$, $\sigma > 0$ *such that* $\gamma = \sqrt{1-\varepsilon}\,\mathcal{C} < 1$ *(* $\mathcal{C} = \mathcal{C}_2$ *is the constant in Theorem 1) and the operator* $\mathcal{A}$ *satisfies the condition* $K_{\varepsilon,\sigma}$. *Then for all* $u \in W_0^{2,2}(\Omega)$ *we have*

$$\|X^2 u\|_{L^2(\Omega)} \leq \frac{\mathcal{C}}{1-\gamma}\|\alpha\|_{L^\infty(\Omega)}\|\mathcal{A}u\|_{L^2(\Omega)},$$

where $\alpha(x) = \frac{\langle A(x), I\rangle}{\|A(x)\|^2}$.

In the case of the Grušin vector fields, if $p > 1$, consider the p-Laplace equation

$$(1) \qquad -\Delta_X^p u = -X_1\Big(|Xu|^{p-2} X_1 u\Big) - X_2\Big(|Xu|^{p-2} X_2 u\Big) = 0 \ , \ \text{in } \Omega$$

and the "regularized" nondegenerate p-Laplace equation

$$(2) \qquad -\sum_{i=1}^{2} X_i\Big(\big(\lambda + |Xu|^2\big)^{\frac{p-2}{2}} X_i u\Big) = 0 \ , \quad \lambda > 0.$$

A subelliptic p-harmonic function is a weak solutions of (1), that is a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying

$$\sum_{i=1}^{2}\int_\Omega |Xu|^{p-2} X_i u\, X_i\varphi \, dxdt = 0 \ , \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

A weak solution of equation (2) is a function $u_\lambda \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying

$$\sum_{i=1}^{2}\int_\Omega \big(\lambda + |Xu_\lambda|^2\big)^{\frac{p-2}{2}} X_i u_\lambda\, X_i\varphi \, dxdt = 0 \ , \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

The next theorem is a simply restatement of a result proved in the case of Heisenberg group $\mathbb{H}^1$ (see e.g. [4]).

Theorem 3. *For* $\frac{\sqrt{17}-1}{2} \leq p < 4$ *any weak solution* u_λ *of the nondegenerate subelliptic* p*-Laplace equation (2) belongs to* $W^{2,2}_{\mathrm{loc}}(\Omega)$.

Theorem 3 allows to differentiate the equation (2) and obtain the linear operator

$$L_\lambda = \sum_{i=1}^{2} a_{ij}^{\lambda} \; X_i X_j \, ,$$

where

$$a_{ij}^{\lambda} = \delta_{ij} + (p-2)\frac{X_i u_\lambda \; X_j u_\lambda}{\lambda + |X u_\lambda|^2} \, .$$

If p belongs to a neighborhood of 2, the operator L_λ satisfies the Cordes condition uniformly with respect to λ. Then, using Theorem 2, we get

Theorem 4. *For* $\frac{\sqrt{17}-1}{2} \leq p < \frac{5+\sqrt{5}}{2}$ *any weak solution of the subelliptic p-Laplace equation (1) belongs to* $W^{2,2}_{\mathrm{loc}}(\Omega)$.

To obtain our main theorem we use the $W^{2,2}_{loc}$ regularity and some properties of the linear operator L_λ that, under suitable hypothesis on p, is "near" to the subelliptic Laplace operator (for the definition of near operators see [1]). Finally, we have

Theorem 5. *There exist* $p_0 < 2 < p_1$ *such that if* $u \in W^{1,p}(\Omega)$ *is a subelliptic p-harmonic function, then* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ *with* $0 < \alpha < 1$.

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INFINITE DIMENSIONAL LAGRANGEAN THEORY AND APPLICATIONS TO GENERALIZED COMPLEMENTARITY PROBLEMS

SOFIA GIUFFRÈ - GIOVANNA IDONE

The authors express Generalized Complementarity Problems in terms of suitable optimization problems and provide some necessary optimality conditions by means of the infinite dimensional Lagrangean and Duality Theories.

1. Introduction.

In this paper we are interested in the so-called Generalized Complementarity Problem

$$\begin{cases} \mathcal{B}(u)\mathcal{L}(u) = 0 \\ u \in \mathbb{K}, \end{cases} \tag{1}$$

where S is a nonempty subset of a real linear space X, Y is a partially ordered real normed space with the ordering cone C, Z is the set of nonnegative measurable functions. Moreover $\mathcal{L} : S \to Z$, $\mathcal{B} : S \to Z$ are two operators such that $\mathcal{L}(v) \geq 0$, $\mathcal{B}(v) \geq 0$ $\forall\, v \in S$, $g : S \to Y$ is a given constraint mapping and $\mathbb{K} = \{v \in S : g(v) \in -C\}$.

Let us observe that the Generalized Complementarity Problem (1) expresses many economic and physical equilibrium problems. In fact, starting from the classical Signorini problem, it has been observed that the Obstacle problem, the Elastic–Plastic Torsion problem, the Traffic Equilibrium problem both in the discrete and continuous cases, the Spatial Price Equilibrium problem, the Financial Equilibrium problem and many others (see [5], [7], [8], [12]) satisfy the Generalized Complementarity Problem (1).

Our aim is to express Problem (1) in terms of a suitable optimization problem and to associate some necessary optimality conditions by means of infinite dimensional Lagrangean and Duality Theories.

The paper improve the results in [9], where a restrictive condition on the set $\operatorname{qri}(g(S)+C)$ has been assumed, while here that condition has been removed. Different approaches for establishing solvability of nonlinear complementarity problems has been presented in [13].

In the sequel, for the sake of simplicity, we confine ourselves to a less general case. Let us suppose that X and Y are real Hilbert spaces with the usual inclusion $X \subseteq Y \subseteq X^*$; let it be C the ordering convex cone of Y and let $\mathcal{L}$, $\mathcal{B}$, g be three functions defined on X with values in Y. Let us suppose that the set $\mathbb{K} = \{v \in X : g(v) \in -C\}$ is nonempty.

Let us observe that (1) can be written as the Optimization Problem:

$$\begin{cases} \min \mathcal{B}(v)\ \mathcal{L}(v) = 0 \\ v \in \mathbb{K} \end{cases} \tag{2}$$

Assuming that Problem (2) holds in the sense of the scalar product on Y and that $\langle \mathcal{L}v, \mathcal{B}v\rangle \geq 0\ \forall v \in X$, it becomes

$$\min_{v\in\mathbb{K}} \langle \mathcal{L}v,\ \mathcal{B}v\rangle = 0\,. \tag{3}$$

The main result of this paper is the following:

Theorem 1. *Let the function* $(\langle \mathcal{L}(v), \mathcal{B}(v)\rangle, g(v))$ *be convex-like. Let us assume that* $\operatorname{qri}[g(X)+C] \neq \emptyset$, $\operatorname{qri} C \neq \emptyset$. *In addition suppose that* C *is closed,* $\overline{C-C} = Y$ *and there exists* $\bar{v} \in X$ *such that* $g(\bar{v}) \in -\operatorname{qri} C$. *Then if the functions* $\mathcal{L}$, $\mathcal{B}$, g *are Fréchet differentiable and Problem (3) admits a solution* $u \in \mathbb{K}$, *then there exists an element* $\bar{l} \in C^*$ *such that*

$$\langle \mathcal{L}_u(u)v,\ \mathcal{B}(u)\rangle + \langle \mathcal{L}(u),\ \mathcal{B}_u(u)v\rangle + \langle \bar{l},\ g_u(u)v\rangle = 0 \quad \forall v \in X, \tag{4}$$

$$\langle l, g(u)\rangle \leq 0,\ \forall l \in C^*\,, \qquad \langle \bar{l}, g(u)\rangle = 0\,. \tag{5}$$

We can generalize the result obtained in Theorem 1 assuming that the set of the constraints is given by $\mathbb{K} = \{v \in S : g(v) \in -C\}$, where S is a nonempty convex subset of X and that $\mathcal{L}(v)$, $\mathcal{B}(v)$ and $g(v)$ are defined on S. In this case the following result holds:

Theorem 2. *Under the same assumptions of Theorem 1, if there exists $\bar{v} \in S$ such that $g(\bar{v}) \in -\operatorname{qri} C$ and Problem (3) admits a solution $u \in \mathbb{K}$, then there exists an element $\bar{l} \in C^*$ such that*

$$\langle \mathcal{L}(u),\ \mathcal{B}_u(u)(v-u)\rangle + \langle \mathcal{L}_u(u)(v-u),\ \mathcal{B}(u)\rangle + \langle \bar{l},\ g_u(u)(v-u)\rangle \geq 0\ \forall v \in S\,,$$

$$\langle l, g(u)\rangle \leq 0,\ \forall l \in C^*, \quad \langle \bar{l}, g(u)\rangle = 0\,.$$

2. The Lagrangean and Duality Theory.

Let us introduce the dual cone C^*, that, in virtue of the usual identification $Y = Y^*$, can be written $C^* = \{l \in Y : \langle l, v\rangle \geq 0, \forall v \in C\}$. Then, using the same technique used by J. Jhan in [11], we may show the following result:

Theorem 3. *Let the ordering cone C be closed. Then u is a minimal solution of (3) if and only if u is a solution of the problem*

$$\min_{v\in X}\ \sup_{l\in C^*} \{\langle \mathcal{L}v, \mathcal{B}v\rangle + \langle l, g(v)\rangle\} \tag{6}$$

and the extremal values of the two problems are equal.

Now let us introduce the Dual Problem

$$\max_{l\in C^*}\ \inf_{v\in X} \{\langle \mathcal{L}v, \mathcal{B}v\rangle + \langle l, g(v)\rangle\}\,. \tag{7}$$

It is known (see Theorem 6.7 of [11]) that if $\operatorname{int} C$ is nonempty, if Problem (3) (or (6)) is solvable and the generalized Slater condition is satisfied, namely there exists $\bar{v} \in X$ with $g(\bar{v}) \in -\operatorname{int} C$, then problem (7) is also solvable and the extremal values of the two problems are equal. However in many concrete situations the request that $\operatorname{int} C$ is non-empty is not verified: for example if X, Y are Lebesgue spaces. For this reason in [1] the authors develop the notation of quasi-relative interior of a convex set that is an extension of the relative interior in finite dimension and that may constitute a first contribute to the search of effective regularity assumptions. Let us recall the definition of quasi-relative interior (see [6] for the properties of $\operatorname{qri} C$).

Definition 1. Let C be a convex subset of a real Hilbert space Y. The quasi-relative interior of C, denoted by $\operatorname{qri} C$, is the set of those $x \in C$ for which $\overline{\operatorname{Cone}(C - x)} = \overline{\{\lambda y : \lambda \geq 0, \ y \in C - x\}}$ is a subspace.

Using this concept of quasi-relative interior, more general separation theorems can be proved (see [6]) and by means of the new separation theorems, it is possible to show that problem (7) is solvable and that the extremal values of the two problems are equal.
At first we recall the definition of a convex–like function.

Definition 2. Let X be a real linear space and let Y be a real linear space partially ordered by a convex cone C. A function $f : X \to Y$ is called convex–like if the set $f(X) + C$ is convex.

Theorem 4. *Let the function $\varphi(v) = (\langle \mathcal{L}(v), \mathcal{B}(v)\rangle, g(v))$ be convex-like with respect to the product cone $\mathbb{R}^+ \times C$ in $\mathbb{R} \times Y$. Let $\operatorname{qri}[g(X) + C] \neq \emptyset$, $\operatorname{qri} C \neq \emptyset$ and $\overline{C - C} = Y$. If Problem (3) is solvable and there exists $\bar{v} \in X$ with $g(\bar{v}) \in -\operatorname{qri} C$, then also Problem (7) is solvable and the extremal values of the two problems are equal. Moreover, if u is a solution to Problem (3) and $\bar{l} \in C^*$ of (7), it turns out to be $\langle \bar{l}, g(u)\rangle = 0$.*

3. Proof of the Theorems.

Let us consider the Lagrangean functional $L : X \times C^* \to \mathbb{R}$, $L(v, l) = \langle \mathcal{L}(v), \mathcal{B}(v)\rangle + \langle l, g(v)\rangle$. Using the preceding theorems we are able to state the following theorem (see for the proof [5]).

Theorem 5. *Let the assumptions of Theorem 4 be fulfilled, with C closed. Then a point $(u, \bar{l}) \in X \times C^*$ is a saddle point of L, namely*

$$L(u, l) \leq L(u, \bar{l}) \leq L(v, \bar{l}), \quad \forall v \in X, \ \forall l \in C^* \tag{8}$$

if and only if u is a solution of problem (3) (or (6)), $\bar{l}$ is a solution of Problem (7) and the extremal values of the two problems are equal, namely

$$\begin{aligned}\min_{v\in X} \sup_{l\in C^*} \{\langle \mathcal{L}v, \mathcal{B}v\rangle + \langle l, g(v)\rangle\} &= \max_{l\in C^*} \inf_{v\in X} \{\langle \mathcal{L}v, \mathcal{B}v\rangle + \langle l, g(v)\rangle\} \\ &= \langle \mathcal{L}u, \mathcal{B}u\rangle + \langle \bar{l}, g(u)\rangle = 0 .\end{aligned}$$

From (8), bearing in mind that $\mathcal{L} : X \to Y$, $\mathcal{B} : X \to Y$, $g : X \to Y$ are Fréchet differentiable functions, we may derive (4), (5).

In a similar way, using a suitable version of Theorem 4 for Problem (7) with X replaced by S, we obtain Theorem 2.

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ESISTENZA E CLASSIFICAZIONE DI PUNTI CRITICI PER FUNZIONI NON DIFFERENZIABILI

ROBERTO LIVREA

The aim of this lecture is to extend a general min-max principle established by Ghoussoub to the case of nondifferentiable functions. A non smooth version of the Brézis-Nirenberg's critical point theorem in presence of splitting is also presented and accompanied with an application to a class of elliptic variational-hemivariational eigenvalue problem. Finally a study of the critical set at a suitable level is pointed out.

1. Due teoremi di punto critico.

Uno dei punti di partenza per lo studio dei punti critici di funzionali di classe C^1 definiti su uno spazio di Banach infinito dimensionale X è il ben noto teorema di passo di montagna (in acronimo TPM) di Ambrosetti e Rabinowitz [1], Theorem 2.1. Successivamente, autori come Chang [3], Szulkin [9] e, più recentemente, Motreanu e Panagiotopoulos [8] hanno fornito versioni del TPM per funzionali sempre meno regolari.

Nella presente nota ci proponiamo, inizialmente, di presentare un risultato che, facendo uso della seguente ipotesi di struttura,

(H'_f) $f = \Phi(x) + \psi(x)$ *per ogni* $x \in X$*, dove* $\Phi : X \to \mathbb{R}$ *è localmente lipschitziana e* $\psi : X \to \mathbb{R} \cup \{+\infty\}$ *è convessa, propria e semicontinua inferiormente. Inoltre,* ψ *è continua su ogni insieme compatto* $A \subset X$ *tale che* $\sup_{x\in A} \psi(x) < +\infty$

estende un principio generale di min-max di Ghoussoub [4], Theorem 1.bis, ottenuto in ambito C^1.

Nel contesto dell'ipotesi (H'_f) ricordiamo che

$$\Phi^0(x;z) := \limsup_{w\to z, t\to 0^+} \frac{\Phi(w+tz)-\Phi(w)}{t}$$

è la derivata direzionale generalizzata di Φ in x lungo la direzione z, mentre $x \in X$ è un punto critico di f se soddisfa la seguente disequazione variazionale-emivariazionale

$$\Phi^0(x; z-x) + \psi(z) - \psi(x) \geq 0 \ \ \forall z \in X.$$

La classica condizione di Palais-Smale diventa:

$(\mathrm{PS})_f$ *Ogni successione* $\{x_n\} \subset X$ *tale che* $\{f(x_n)\}$ *è limitata e*

$$\Phi^0(x_n; z-x_n) + \psi(z) - \psi(x_n) \geq -\varepsilon_n \|z - x_n\|$$

per ogni $n \in \mathbb{N}$, $z \in X$, *dove* $\varepsilon_n \to 0^+$, *possiede un'estratta convergente.*

Nel seguito, per un dato $\lambda \in \mathbb{R}$, poniamo

$$K_\lambda(f) := \{x \in X : f(x) = \lambda, \ x \text{ è punto critico di } f\}.$$

Faremo infine uso del seguente insieme $\Gamma := \{\gamma \in C^0(Q, X) : \gamma_{|Q_0} = \gamma_0\}$, dove $Q \subset X$ è un insieme compatto, Q_0 è un sottoinsieme non vuoto e chiuso di Q e γ_0 appartiene a $C^0(Q_0, X)$. Ad esempio, se $x_0, x_1 \in X$ e consideriamo $Q = [x_0, x_1]$, $Q_0 = \{x_0, x_1\}$ e $\gamma_0 = \mathrm{id}_{|Q_0}$, si ha

$$\Gamma := \{\gamma \in C^0([x_0, x_1], X) : \ \gamma(x_i) = x_i, \ i = 0, 1\}. \tag{1}$$

Vale quindi il seguente risultato.

Theorem 1.1. ([5], Theorem 3.3) *Assumiamo che la funzione* f *soddisfi le seguenti ipotesi oltre a* (H'_f) *e* $(\mathrm{PS})_f$.

(a_1) $\sup_{x\in Q} f(\gamma(x)) < +\infty$ *per qualche* $\gamma \in \Gamma$.

(a_2) *Esiste un sottoinsieme chiuso* S *di* X *tale che*

$$(\gamma(Q) \cap S) \setminus \gamma_0(Q_0) \neq \emptyset \quad \textit{per ogni} \ \ \gamma \in \Gamma \tag{2}$$

e

$$\sup_{x\in Q_0} f(\gamma_0(x)) \leq \inf_{x\in S} f(x). \tag{3}$$

Poniamo $c := \inf_{\gamma\in\Gamma} \sup_{x\in Q} f(\gamma(x))$. *Allora l'insieme* $K_c(f)$ *è non vuoto. Se, inoltre,* $\inf_{x\in S} f(x) = c$ *allora* $K_c(f) \cap S \neq \emptyset$.

Un opportuno e articolato utilizzo del Teorema 1.1, nel caso in cui $X := X_1 \oplus X_2$, dove $\dim(X_1) > 0$ e $0 < \dim(X_2) < \infty$, consente di ottenere la seguente versione non-smooth di un noto risultato di Brézis-Nirenberg [2], Theorem 4.

Theorem 1.2. ([6], Theorem 3.1) *Assumiamo che f sia limitata inferiormente e soddisfi (H'_f) e $(\mathrm{PS})_f$. Sia x_0 un punto di minimo globale di f. Se $\inf_{x\in X} f(x) < f(0)$, $f(0) = 0$ e inoltre*

(f_1) *l'insieme $\{x \in X : f(x) < a\}$ è aperto per qualche $a > 0$,*
(f_2) *esiste $r \in \left]0, \frac{\|x_0\|}{2}\right[$ tale che $f_{|\overline{B}_r\cap X_1} \geq 0$, $f_{|\overline{B}_r\cap X_2} \leq 0$ e $f_{|\partial B_r\cap X_2} < 0$,*

allora la funzione f ammette almeno due punti critici non banali.

2. Un'applicazione ad una classe di disequazioni variazionali - emivariazionali.

Sia Ω un sottoinsieme aperto e limitato di $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$, con frontiera regolare $\partial\Omega$. Poniamo

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^2\, dx\right)^{1/2}$$

per ogni $u \in H_0^1(\Omega)$. Assegnata una funzione $a \in L^\infty(\Omega)$, si consideri il problema

$$\begin{cases} -\Delta u + a(x)u = \lambda u & \text{in } \Omega \\ u = 0 & \text{su } \partial\Omega. \end{cases} \tag{4}$$

È noto che (4) possiede una successione $\{\lambda_n\}$ di autovalori tale che $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$. Sia $\{\varphi_n\}$ la corrispondente successione di autofunzioni normalizzate in modo tale che, per ogni $n \in \mathbb{N}$

$$\int_\Omega \left[|\nabla\varphi_n(x)|^2 + a(x)\varphi^2\_n(x)\right] dx = \lambda_n \int_\Omega \varphi_n^2(x)\, dx = \lambda_n$$

$$\int_\Omega \left[\nabla\varphi_m(x)\cdot\nabla\varphi_n(x) + a(x)\varphi_m(x)\varphi_n(x)\right] dx = \int_\Omega \varphi_m(x)\varphi_n(x)\, dx = 0$$

a condizione che $m, n \in \mathbb{N}$ e $m \neq n$.

Assumiamo che $\lambda_s < 0 < \lambda_{s+1}$ per qualche $s \in \mathbb{N}$.

Sia $g : \mathbb{R} \to \mathbb{R}$ una funzione tale che:

(g_1) *g è misurabile,*
(g_2) *esistono $a_1 > 0$, $p \in]2, 2^*[$ tali che*

$$|g(t)| \leq a_1(1 + |t|^{p-1}) \quad \textit{per ogni} \;\; t \in \mathbb{R},$$

e si considerino le funzioni $G : \mathbb{R} \to \mathbb{R}$ e $\mathscr{G} : H_0^1(\Omega) \to \mathbb{R}$ date da

$$G(\xi) := \int_0^{\xi} -g(t)\, dt \quad \forall \xi \in \mathbb{R}, \quad \mathscr{G}(u) := \int_{\Omega} G(u(x))\, dx \quad \forall u \in H_0^1(\Omega).$$

Per la nostra applicazione, supponiamo inoltre che

(g_3) $\lim_{t \to 0} \frac{g(t)}{t} = 0$,
(g_4) $\limsup_{|t| \to +\infty} \frac{g(t)}{t} < 0$,
(g_5) *esiste $\xi_0 \in \mathbb{R}$ tale che $G(\xi_0) < 0$.*

(g_2) e (g_4) assicurano l'esistenza di due costanti positive β, γ tali che

$$g(t) \geq -\beta t - \gamma \quad \forall\, t \leq 0\,, \quad g(t) \leq -\beta t + \gamma \quad \forall t \geq 0\,.$$

Per $\lambda, \mu > 0$, se c_1 è la costante dell'immersione $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$, poniamo

$$r_{\lambda,\mu} := \lambda \gamma c_1 + \sqrt{(\lambda \gamma c_1)^2 + 2\mu}\,.$$

Un insieme $K_\lambda \subseteq H_0^1(\Omega)$ è detto di tipo (K_λ^g) se

(K_λ^g) *K_λ è convesso e chiuso in $H_0^1(\Omega)$. Inoltre, esiste $\mu > 0$ tale che $\overline{B}_{r_{\lambda,\mu}} \subseteq K_\lambda$.*

Se $X := H_0^1(\Omega)$, $X_2 := \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_s\}$, $X_1 := X_2^{\perp}$, $f := \Phi + \psi$, con

$$\Phi(u) := \frac{1}{2} \int_{\Omega} \left(|\nabla u(x)|^2 + a(x) u^2(x) \right) dx + \lambda \mathscr{G}(u),$$

$$\psi := \begin{cases} 0 & \text{se} \;\; u \in K_\lambda, \\ +\infty & \text{altrimenti,} \end{cases}$$

dal Teorema 1.2 discende il seguente risultato.

Theorem 2.1. ([6], Theorem 4.1) *Assumiamo che siano verificate* (g_1)–(g_5). *Allora, per ogni* λ *abbastanza grande e* K_λ *di tipo* (K_λ^g)*, il problema* (P_λ) *:*

Trovare $u \in K_\lambda$ *tale che*

$$- \int_\Omega \nabla u(x) \cdot \nabla(v-u)(x)\, dx - \int_\Omega a(x)u(x)(v-u)(x)\, dx \\ \leq \lambda \mathscr{G}^0(u; v-u) \quad \forall v \in K_\lambda$$

ammette almeno due soluzioni non banali.

Osserviamo che se u è soluzione di (P_λ) allora si ha anche che per ogni $v \in K_\lambda$

$$- \int_\Omega \nabla u(x) \cdot \nabla(v-u)(x)\, dx - \int_\Omega a(x)u(x)(v-u)(x)\, dx \\ \leq \lambda \int_\Omega G^0(u(x); (v-u)(x))\, dx.$$

Quindi, se g è continua e $K_\lambda := H_0^1(\Omega)$, $u \in H_0^1(\Omega)$ risulta una soluzione debole del problema di Dirichlet

$$-\Delta u + a(x)u = \lambda g(u) \text{ in } \Omega, \quad u = 0 \text{ su } \partial\Omega.$$

3. Sulla struttura dell'insieme critico al livello *c*.

Se $x_0, x_1 \in D_\psi := \{x \in X : \psi(x) < +\infty\}$, poniamo $c := \inf_{\gamma\in\Gamma} \sup_{x\in[x_0,x_1]} f(\gamma(x))$, dove, in questo caso, Γ è come in (1).

Theorem 3.1. *Supponiamo che la funzione* f *soddisfi* (H_f') *e* $(PS)_f$*. Inoltre, assumiamo che* $\{x \in X : f(x) \geq c\}$ *sia chiuso e che*

(a_3) x_0 *sia un punto di minimo locale di* f,
(a_4) $f(x_1) \leq f(x_0)$.

Se $c > f(x_0)$*, allora* $K_c(f)$ *contiene un punto che non è di minimo locale. Altrimenti, per ogni* $r > 0$ *sufficientemente piccolo, esiste un punto di minimo locale* $x_r \in \partial B(x_0, r)$ *con* $f(x_r) = f(x_0)$.
In particolare, ciò avviene ogni volta che x_0 *e* x_1 *sono entrambi punti di minimo locale.*

Ricordiamo che un punto critico $x \in X$ si dice di sella per f se per ogni $\delta > 0$ esistono $x', x'' \in B(x, \delta)$ tali che $f(x') < f(x) < f(x'')$.

Theorem 3.2. *Supponiamo che* $\dim(X) = +\infty$ *e che la funzione* f *soddisfi* (H'_f) *e* $(\mathrm{PS})_f$. *Se*

(a_5) $\{x \in X : f(x) \geq c\}$ *è chiuso,*
(a_6) $\max\{f(x_0), f(x_1)\} < c$,

allora $K_c(f)$ *contiene un punto di sella.*

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A DIRECT METHOD FOR THE CALCULATION OF THE EQUILIBRIUM IN THE TRAFFIC NETWORK PROBLEM

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We propose an algorithm for the determination of the equilibrium in the traffic network problem. Our algorithm belongs to the class of direct (i.e. non iterative) methods for variational inequalities and extends the range of applicability of previous methods of the same class to equilibrium problems whose size is not necessarily small.

1. Introduction.

In the last decades many equilibrium problems arising from various fields of applied science have been formulated and studied in the framework of variational inequalities (v.i.) [7], [3], [2]. The v.i. formulation provides a unifying tool for their theoretical analysis and numerical solution. Among the numerical methods for (finite-dimensional) v.i., iterative methods are probably the most popular.

On the contrary, direct methods have often been discarded because they are, in general, difficult to apply to large problems and also, in our opinion, because their implementation is more involved with respect to iterative methods. However, when applicable, direct methods do not introduce iteration errors and this can be very useful, e.g., when finite dimensional v.i. are used to approximate infinite dimensional v.i., to isolate the error due to discretization (see e.g. [4]). Thus, starting from the consideration that direct methods deserve more attention, we focus on a particular problem, the celebrated traffic equilibrium problem, and present a direct algorithm that can be applied to networks whose size is not necessarily small.

Our algorithm is based on a general procedure proposed by O. Mancino and G. Stampacchia long ago [5], which we adapt to the particular problem under consideration, taking advantage of the particular structure of the convex polytope described by the constraints. The application of direct methods to the traffic equilibrium problem dates to A. Maugeri [6].

We shall assume that the problem has a solution and that the cost operator is strictly monotone. Then, we shall associate to the initial v.i. a hierarchy of simpler subproblems of decreasing dimension.

The maximum number of subproblems to be solved equals the total number of polytope faces. We point out that, as can be easily computed, the number of faces of the polytope associated to our problem increases exponentially with its dimension. Therefore a selection rule is needed if one wants to deal with problems with a large number of variables.

Following our algorithm, after solving *each subproblem*, either we find the solution, or we are able to reduce the maximum number of subproblems of lower dimensionality to be solved. Moreover, the formulation of each subproblem is achieved by using directly the equilibrium principle equivalent to the v.i., making the implementation of the algorithm straightforward.

We shall give the proofs of the theorems related to our algorithm and perform extensive numerical computations in a subsequent paper.

2. The traffic equilibrium problem.

The traffic assignment problem has a relatively recent history. For a variational inequality formulation of this problem we refer to the influential papers by Smith [9] and Dafermos [1]. For a comprehensive treatment of models and methods we refer to [8].

Let us first introduce the notation commonly used to state the standard traffic equilibrium problem from the user point of view. A traffic network consists of a triple (N, A, W) where $N = \{N_1, \dots, N_p\}$ is the set of nodes, $A = (A_1, \dots, A_n)$ represents the set of the directed arcs connecting couples of nodes and $W = \{W_1, \dots, W_m\} \subset N \times N$ is the set of the origin-destination (O-D) pairs. The flow on the arc A_i is denoted by f_i, $f = (f_1, \dots, f_n)$. We call a set of consecutive arcs a path, and assume that each O-D pair W_j is connected by $r_j \geq 1$ paths whose set is denoted by P_j $(j = 1, \dots, m)$. All the paths in the network are grouped in a vector $(R_1, \dots, R_k)$. We can describe the link structure of the paths by using the arc - path incidence matrix $\Delta = \{\delta_{ir}\}_{i=1,\dots,n;r=1,\dots,k}$, whose entries take the value 1 if $A_i \in R_r$, 0 if $A_i \notin R_r$. To each path R_r there corresponds a flow F_r. The path flows are grouped in a vector $(F_1, \ldots, F_k)$ which is called the path (network) flow. The flow f_i on the arc A_i is equal to the sum of the path flows which contain A_i, so that $f = \Delta F$. Let us now introduce the cost of going through A_i as a function $c_i(f) \geq 0$ of the flows on the network, so that $c(f) = (c_1(f), \ldots, c_n(f))$ denotes the arc cost vector on the network . Analogously, one can define a cost on the paths as $C(F) = (C_1(F), \ldots, C_k(F))$. In most applications the cost $C_r(F)$ associated to path r is just the sum of the costs on the arcs which build that path;

$$C_r(F) = \sum_{i=1}^{n} \delta_{ir} c_i(f) \tag{1}$$

or in compact form, $C(F) = \Delta^T C(\Delta F)$. For each O-D pair W_j there is a given traffic demand $D_j \geq 0$, so that $(D_1, \ldots, D_m)$ is the demand vector on the network. Feasible flows are nonnegative flows which satisfy the demands, i.e., which belong to the set

$$\mathcal{K} := \{F \in \mathbb{R}^k \mid F \geq 0, \quad \Phi F = D \},$$

where Φ is the well known O-D pair-path incidence matrix whose elements $\phi_{j,r}$ $(j = 1, \ldots, m; r = 1, \ldots, k)$ are set equal 1 if the path R_r connects the pair W_j, 0 else.

A path flow H is called an equilibrium flow (or *Wardrop Equilibrium* flow), if $H \in \mathcal{K}$ and $\forall W_j \in W, \forall R_q, R_s \in P_j$, there holds:

$$C_q(H) < C_s(H) \Longrightarrow H_s = 0 . \tag{2}$$

This statement is equivalent to:

$$H \in \mathcal{K} \text{ and } [C(H)]^T [F - H] \geq 0 \quad \forall F \in \mathcal{K} . \tag{3}$$

Roughly speaking, the meaning of Wardrop Equilibrium is that the road users choose minimum cost paths, and the meaning of the cost is usually that of traversal time.

3. The algorithm.

As stated in Section 2, we consider a network with m O-D couples W_j, each joined by r_j paths. Moreover we assume that (3) has a solution and that the cost function is strictly monotone.

In our algorithm we shall consider a family of subproblems, obtained by assuming that the flows on some set of paths are equal to zero, and search for the subproblem that yields to the desired solution. In view of the implementation it is then necessary to introduce a new object, the collection of "candidate subproblems", which, given the nature of the algorithm, is not known *a priori* but will be populated during the execution of the algorithm.

Let $I = \{1, \dots, k\}$ and denote by $I_j = \{r : R_r \in P_j\}$ the set of indexes of the paths connecting W_j. Let $Q \subset I$ be the set of indexes corresponding to a generic set of flows which we shall set to zero, and q its cardinality. We require that each couple W_j is still connected by at least one path, that is $I_j \setminus Q \neq \{\emptyset\}$. We can then generate a system which contains only the quantities connected to the paths $R_r, r \notin Q$.

$$
(4) \qquad \begin{cases}
\displaystyle\sum_{r \in I_1 \setminus Q} F_r = D_1 \\
C_r(F) = C_s(F) \quad r, s \in I_1 \setminus Q,\ s \neq r \\
\qquad \cdots\cdots\cdot \\
\displaystyle\sum_{r \in I_j \setminus Q} F_r = D_j \\
C_r(F) = C_s(F) \quad r, s \in I_j \setminus Q,\ s \neq r \\
\qquad \cdots\cdots\cdot \\
\displaystyle\sum_{r \in I_m \setminus Q} F_r = D_m \\
C_r(F) = C_s(F) \quad r, s \in I_m \setminus Q,\ s \neq r
\end{cases}
$$

The first equation in group j is the conservation law for W_j; the subsequent equations express the condition that the costs on the paths of W_j, associated to the flows not set to zero, are equal. Let us notice that the number of the possible equations $C_r = C_s$ in group j is $p_j(p_j - 1)/2$ (where p_j is the cardinality of the set $I_j \setminus Q$ of the nonzero flows in the group j), but only $p_j - 1$ of them are independent. For this reason, we understand that system (4) is made up of only independent equations, hence it consists of $k - q$ equations in $k - q$ unknowns.

After solving system (4), we build a vector $H \in \mathbb{R}^k$, where $H_r = 0$ for $r \in Q$, and the remaining components are obtained from the solution of (4). H

is the solution of (3) if the following feasibility and compatibility conditions are met:

$$(5) \quad H_s \geq 0 \quad \text{and} \quad C_r(H) \geq C_s(H), \qquad \forall r \in I_j \cap Q,\ \forall s \in I_j \setminus Q,\ \forall j.$$

We note that for $p_j = 1$, in the system it will not appear any equality between costs in group j, and if $p_j = r_j$ no compatibility conditions are needed for group j.

Since the solution of (3) exists, it will be found in correspondence to at least one set Q. Thus a possible approach could be to explore the subproblems corresponding to *all* the possible choices of Q, which however are $\Pi_j(2^{r_j} - 1)$. The exponential proliferation of subproblems can be connected, from a geometrical point of view, with the fact that the polytope described by the constraints has a number of faces which increases exponentially with its dimension. In order to decrease the computational effort in the case of large problems, a selection strategy is obviously needed.

Initial step. Consider the set $Q^0 = \{\emptyset\}$, and let S^0 be the corresponding system (4) and H^0 its solution. If $H_r^0 \geq 0, \forall r$, then H^0 is the solution of (3) and the algorithm stops. Otherwise we shall start to collect and solve a family of subproblems, identified by a corresponding family $\mathcal{Q}$ of subsets of I. Note that $\mathcal{Q}$ will be updated in the subsequent steps, since after the solution of each subproblem a certain number of sets will be added to it. Initially $\mathcal{Q}$ consists of the sets $Q^1 = \{t_1^0\}, \ldots, Q^l = \{t_l^0\}$, where $t_1^0, \ldots, t_l^0$ are the indexes t such that $H_t^0 < 0$. In this way we select only l among the possible (in general k) subproblems where one flow is set to zero.

All the subsequent steps of the algorithm can be described by the following general procedure.

General step. We choose a set $Q^i \in \mathcal{Q}$ (among those not previously considered), solve the corresponding system S^i, and build the vector $H^i \in \mathbb{R}^k$ where $H_r = 0$ for $r \in Q^i$, and the other components are given by the solution of S^i. If H^i satisfies (5), then it is the solution of (3), otherwise there are two possibilities. If H^i is feasible but the compatibility conditions are not satisfied, then the subproblem does not provide further information on the solution of (3), and we iterate the general step. If H^i is not feasible, then $H_r^i < 0$ for some r and we shall update $\mathcal{Q}$ according to the following rule. If $t_1^i, \ldots, t_{l_i}^i$ are the indexes t such that $H_t^i < 0$, then we add to $\mathcal{Q}$ the sets $Q^i \cup \{t_1^i\}, \ldots, Q^i \cup \{t_{l_i}^i\}$.

The algorithm described above yields the solution independently of the criterion used to choose the subproblem to solve at each step. A possible criterion is to solve always all the problems of a given dimension d, before solving those of immediately lower dimension $d-1$. Thus, after S^0, we solve all the subproblems of dimension $k-1$, generating in this way all the subproblems of dimension $k-2$, which in turn generate all the subproblems of dimension $k-3$, and proceed in this way until we get the solution. It is then evident that, analogously to the initial step, where we selected only a fraction of the possible $k-1$-dimensional subproblems, we continue to operate a selection also for the subproblems of lower dimension.

4. Conclusion and further remarks.

We have presented an algorithm for the solution of the traffic equilibrium problem which belongs to the class of direct methods and which provides a strategy for reducing the computational effort with respect to methods of the same type. We point out that the family of subproblems that we have considered can be explored in different ways and that the search for an optimal strategy is an open problem.

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TRENDS AND TOPICS ON SOME PARTIAL DIFFERENTIAL EQUATIONS AND SYSTEMS

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We present some results concerning partial differential equations and systems, we point out our attention to operators having discontinuous coefficients of the higher order derivatives.

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$ with $\partial\Omega$ sufficiently smooth, $0 < \lambda < n$, $1 < p < \infty$.

We say that a function $f \in L^1_{loc}(\Omega)$ belongs to the Morrey Space $L^{p,\lambda}(\Omega)$ if it is finite

$$\|f\|^p_{L^{p,\lambda}(\Omega)} \equiv \sup_{x\in\Omega,\ \rho>0} \frac{1}{\rho^\lambda} \int_{B_\rho(x)\cap\Omega} |f(y)|^p \, dy$$

where $B_\rho(x)$ is a ball of radius ρ centered at the point x.

We also say that a function f belongs to the John-Nirenberg space BMO, (see [13]), or that f has "bounded mean oscillation", if

$$\|f\|_* \equiv \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty$$

where f_B is the integral average $\frac{1}{|B|}\int_B f(x)\,dx$ of the function $f(x)$ over the set B and B belongs to the class of balls of $\mathbb{R}^n$.

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Let, for a function $f \in BMO$

$$\eta(r) = \sup_{x \in \mathbb{R}^n,\ \rho \le r} \frac{1}{|B_\rho|} \int_{B_\rho} |f(x) - f_{B_\rho}|\, dx.$$

A function $f \in BMO$ belongs to the class VMO, (see [17]) or f has "vanishing mean oscillation" if

$$\lim_{r \to 0^+} \eta(r) = 0.$$

We recall the following classical Sobolev spaces

$$W^{k,p}(\Omega) = \left\{ f(x) : D^\alpha f \in L^p(\Omega), \quad |\alpha| \le k \right\}$$

where k is an integer, equipped with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|D^k f\|_{L^p(\Omega)}.$$

The closure of the space $C_0^\infty(\Omega)$ with respect to the norm in $W^{k,p}(\Omega)$ will be denoted, as usual, by $W_0^{k,p}(\Omega)$.

We set $W^{k,p,\lambda}(\Omega)$ the Banach space of functions belonging to $W^{k,p}(\Omega)$ and having k-th order derivatives lying in the Morrey space $L^{p,\lambda}(\Omega)$. A natural norm in this space is

$$\|f\|_{W^{k,p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + \|D^k f\|_{L^{p,\lambda}(\Omega)}.$$

At first let us consider the Dirichlet problem associated to an uniformly elliptic operator $\mathcal{L}$ of second order in nondivergence form having coefficients of higher order derivatives bounded and belong to the vanishing mean oscillation class and the known term f in $L^{p,\lambda}(\Omega)$. Specifically let us study regularity results in Morrey spaces of solutions to the above problem.

If the coefficients a_{ij} are Hölder continuous regularizing properties of $\mathcal{L}$ in Hölder spaces and unique classical solvability of the Dirichlet problem for $\mathcal{L}u = f$ have been studied in the book by Gilbarg and Trudinger [11].

If the coefficients a_{ij} are uniformly continuous Agmon, Douglis and Nirenberg in the note [1] (see also the book by Gilbarg and Trudinger) prove that if $\mathcal{L}u \in L^p(\Omega)$ then the strong solution u of $\mathcal{L}u = f$ belongs to $W^{2,p}(\Omega)$, $\forall\, p \in]1, +\infty[$.

The case of coefficients a_{ij} not uniformly continuous is less studied because the L^p-theory of $\mathcal{L}$ and the strong solvability of the Dirichlet problem for $\mathcal{L}u = f$ do not hold any more.

If $n = 2$ and $p = 2$ Talenti in [18] suppose a_{ij} measurable and bounded functions and prove the isomorphism of the map $\mathcal{L}$ from $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ into $L^2(\Omega)$.

Let $n \geq 3$, if $a_{ij} \in W^{1,n}(\Omega)$ (see [14] by Miranda) or a_{ij} satisfy other assumptions, as the "Cordes condition" (see the note by Campanato [5]), the local $W^{2,p}$ regularity has been proved for $p \in (2-\epsilon, 2+\epsilon)$ for an opportune $\epsilon > 0$.

An important result in local and global Sobolev regularity of strong solutions to $\mathcal{L}u = f$ is the study made by Chiarenza, Frasca and Longo, in [6] and [7], of the Dirichlet problem for $\mathcal{L}u = f$ in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, where $a_{ij} \in VMO \cap L^\infty(\Omega)$. In these notes the authors suppose $\mathcal{L}u \in L^p(\Omega)$ and prove that for all $p \in (1, \infty)$ the function u belongs to $W^{2,p}(\Omega)$. It is also studied the well posedness of the related Dirichlet problem.

Regularity properties of the operator $\mathcal{L}$ in Morrey spaces if $a_{ij} \in VMO$ are studied in [3]. The author prove that, if

$$f \in L^{n,n\alpha}_{loc}(\Omega), \quad \alpha \in (0,1),$$

a $W^{2,p}-$viscosity solution of

$$\mathcal{L}u = f$$

belongs to $C^{1+\alpha}_{loc}(\Omega)$. Using the same technique, the hypothesis for the function f could not be weakened for $f \in L^{p,\lambda}_{loc}(\Omega)$, $p < n$, $\lambda > 0$.

Interior estimates in Morrey spaces for the second derivatives of the $W^{2,p}$ solutions to

$$\mathcal{L}u = f,$$

where $a_{ij} \in VMO \cap L^\infty$ are proved in the paper [10]. The authors showed that if the known term $f \in L^{p,\lambda}_{loc}(\Omega)$, the second derivatives of a strong solution u of the above nondivergence form equation belong to the same space. Then it is later extended the regularity property from local to global in the paper [9]. In this note the authors suppose Ω be an open bounded set in $\mathbb{R}^n$ with $C^{1,1}$ smooth boundary and the coefficients a_{ij} of $\mathcal{L}$ such that

$$a_{ij}(x) \in L^\infty(\Omega) \cap VMO, \quad a_{ij}(x) = a_{ji}(x),$$

$$\exists\, \kappa > 0 : \ \kappa^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \kappa|\xi|^2, \forall \xi \in \mathbb{R}^n, \ q.o.\, x \in \Omega.$$

The boundary regularity in Morrey spaces for second order derivatives of solutions to the Dirichlet problem for the operator $\mathcal{L}$ is proved. Combining these results with the $W^{2,p}$−strong solvability of the Dirichlet problem proved in [7], it is proved in [9] the well-posedness of

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. in } \Omega, \\ u \in W^{2,p,\lambda}(\Omega) \cap W_0^{1,p}(\Omega). \end{cases}$$

Moreover, there exists a constant c independent on u and f such that

$$\|D_{ij}u\|_{L^{p,\lambda}(\Omega)} \leq c\|f\|_{L^{p,\lambda}(\Omega)}.$$

We observe that the result obtained for the homogeneous Dirichlet problem can be applied in the nonhomogeneous case. More precisely, if $f \in L^{p,\lambda}(\Omega)$, $\varphi \in W^{2,p,\lambda}(\Omega)$ and

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. in } \Omega \\ u = \varphi & \text{on } \partial\Omega, u - \varphi \in W_0^{1,p}(\Omega) \end{cases}$$

the authors have $\mathcal{L}\varphi \in L^{p,\lambda}(\Omega)$, then $u(x) - \varphi(x)$ satisfy the homogeneous Dirichlet problem and the strong solutions of the nonhomegeneous problem will belong to $W^{2,p,\lambda}(\Omega)$.

As a consequence in [9] is proved that, if $n - p < \lambda < n$, $u \in W^{2,p}(\Omega)$ is a strong solution to the nonhomogeneous problem with $f \in L^{p,\lambda}(\Omega)$ and $\varphi \in W^{2,p,\lambda}(\Omega)$, the gradient Du is a Hölder continuous function on Ω with exponent $\alpha = 1 - (n - \lambda)/p$.

Moreover, known properties of Morrey spaces (see the note by Campanato [4]) for suitable values of $p \in (1, \infty)$ and $\lambda \in (0, n)$ allow to obtain global Hölder regularity for the gradient Du of the strong solution of the nonhomogeneous problem.

This kind of Morrey estimates has been studied also for elliptic equations in divergence form.

In the paper [15] is considered

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u) = \operatorname{div} f$$

$f = (f_1, \ldots, f_n)$; $f_i \in L^{p,\lambda}(\Omega)$, $i = 1, \ldots, n$, $1 < p < \infty$, $0 < \lambda < n$.

In this case, with respect to the nondivergence one, the representation formula used is that one in [8], we point out that it is not written for the second derivatives of the solution, $D_{ij}u$, but for the first derivatives $D_i u$.

We obtain that if $f_i \in L^{p,\lambda}(\Omega)$ $\forall i$, it follows that $\nabla u \in L^{p,\lambda}(\Omega)$ and, as a consequence, $u \in C^{0,\alpha}(\Omega)$, $\alpha = 1 - \frac{n-\lambda}{p}$, $0 < \lambda < n$, $p > n - \lambda$.

In the sequel we are interested in the study of elliptic systems in nondivergence form. In the note [16] local L^p regularity for highest order derivatives of an elliptic system of arbitrary order in nondivergence form has been proved. We point out that in [16] cannot be applied the deep technique used by Caffarelli in [3] because of it relies on the Aleksandrov-Pucci maximum principle.

Let $n \geq 3$, α a multi-index and set

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Let us consider the system

$$\begin{aligned} (1.1) \quad \mathcal{L}u \equiv \sum_{j=1}^{N} \sum_{|\alpha|=2s} a_{ij}^{(\alpha)}(x) D^\alpha u_j(x) + \\ + \sum_{j=1}^{N} \sum_{|\alpha|\leq 2s-1} b_{ij}^{(\alpha)}(x) D^\alpha u_j(x) = f_i(x), \quad i = 1, \ldots, N \end{aligned}$$

where $f_i \in L^{p,\lambda}(\Omega)$, $i = 1, \ldots, N$, $1 < p < \infty$, $0 \leq \lambda < n$.

We define a local solution of the above equation a vector function $u = (u_1, \ldots, u_N)$ with $u_i \in W_{loc}^{2s,p}(\Omega)$, $\forall i = 1, \ldots, N$, $s \in \mathbb{N}$ satisfying the above equation almost everywhere in Ω.

Throughout the paper we consider

1) $\quad a_{ij}^{(\alpha)} \in VMO(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $i, j = 1, \ldots, N$, $|\alpha| = 2s$.

Let us set

$$\mathcal{S} = \max_{\substack{i,j=1,\ldots,N \\ |\alpha|=2s}} \sup_{\Omega} | a_{ij}^{(\alpha)} |.$$

Let us define $\sum_{|\alpha|=2s} a_{ij}^{(\alpha)} \phi^\alpha$, $\forall i, j = 1, \ldots, N$, for almost every $x \in \Omega$, the homogeneous polynomial of degree $2s$ in ϕ obtained substituting D^α with $\phi^\alpha = \phi_1^{\alpha_1} \ldots \phi_n^{\alpha_n}$.

2) (ellipticity condition)

$$\exists\, \Lambda > 0 : \det\Big(\sum_{|\alpha|=2s} a_{ij}^{(\alpha)}(x) \phi^\alpha \Big) \geq \Lambda |\phi|^{2sN}, \ \forall \phi \in \mathbb{R}^n, \ \text{for a. e. } x \in \Omega.$$

The hypothesis on the lower order terms is to belong to a suitable Lebesgue space.

Let us now define the Calderón-Zygmund kernel, useful in the sequel.

Definition 1.1. A function $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is a Calderón-Zygmund kernel (C-Z kernel) if satisfies the following assumptions

$\mathcal{C}Z$ 1) $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$;

$\mathcal{C}Z$ 2) $k(x)$ is homogeneous of degree $-n$;

$\mathcal{C}Z$ 3) $\int_\Sigma k(x)\,dx = 0$, where Σ is the surface of the unit sphere in $\mathbb{R}^n$

$$\Sigma = \{ x \in \mathbb{R}^n : |x| = 1 \}.$$

The following two Lemmas are contained in [10].

Lemma 1.2. *Let us consider Ω be an open subset of $\mathbb{R}^n$, $f \in L^{p,\lambda}(\Omega)$, $1 < p < \infty, 0 \le \lambda < n$, $a \in VMO \cap L^\infty(\mathbb{R}^n)$. Let us also set $k(x, z)$ is a Calderón–Zygmund kernel in the z variable for almost all $x \in \Omega$ such that*

$$\max_{|j| \le 2n} \left\| \frac{\partial^j}{\partial z^j} k(x, z) \right\|_{L^\infty(\Omega \times \Sigma)} = \mathcal{M} < +\infty,$$

where Σ is defined as above.

For any $\varepsilon > 0$, we set

$$K_\varepsilon f(x) = \int_{\substack{|x-y|>\varepsilon \\ y \in \Omega}} k(x, x-y) f(y)\,dy,$$

$$C_\varepsilon(a, f)(x) = \int_{\substack{|x-y|>\varepsilon \\ y \in \Omega}} k(x, x-y)(a(x) - a(y)) f(y)\,dy.$$

Then there exist Kf, $C(a, f) \in L^{p,\lambda}(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \| K_\varepsilon f - Kf \|_{L^{p,\lambda}(\Omega)} = 0$$

and

$$\lim_{\varepsilon \to 0} \| C_\varepsilon(a, f) - C(a, f) \|_{L^{p,\lambda}(\Omega)} = 0$$

Also there exists a positive constant c independent on f and such that

$$\| Kf \|_{L^{p,\lambda}(\Omega)} \le c \| f \|_{L^{p,\lambda}(\Omega)},$$

and

$$\| C(a, f) \|_{L^{p,\lambda}(\Omega)} \le c \| a \|_* \| f \|_{L^{p,\lambda}(\Omega)}.$$

Lemma 1.3. *If a function a belongs to $VMO(\mathbb{R}^n)$ then, for every $\epsilon > 0$, there exists $\rho_0 > 0$ such that, if B_r is a ball with radius r such that $0 < r < \rho_0$, $k(x, z)$ verifies the hypothesis of the previous theorem in B_r and $f \in L^{p,\lambda}(B_r)$ for every $1 < p < \infty$ and $0 \leq \lambda < n$, we have*

$$\|C(a, f)\|_{L^{p,\lambda}(B_r)} \leq c\,\epsilon \|f\|_{L^{p,\lambda}(B_r)}$$

for some constant c independent on f.

Theorem 1.4. *Let $s \in N$, $2 < p < n$, $0 \leq \lambda < n$, $0 \leq \omega < n$. Assume that $a_{ij}^{(\alpha)}$ verify 1) and 2), $u \in W^{2s,p}(B_\sigma)$ is a solution of (1.1) in a ball $B_\sigma \subset\subset \Omega$ and $f_i \in L^{p,\lambda}(B_\sigma)$, $\forall i = 1, \dots, N$.*

In addition let us set

$$b_{ij}^{(\alpha)} D^\alpha u_j \in L^{p,\omega}(B_\sigma), \quad |\alpha| \leq 2s - 1, \quad j = 1, \dots, N. \tag{1.2}$$

Then there exists $\overline{\sigma} \in \,]0, \sigma[$, such that for every B_ρ concentric to B_σ, $\rho \leq \overline{\sigma}$, we have that

$$D^\alpha u_j \in L^{p,\delta}(B_{\frac{\rho}{2}}),\ |\alpha| = 2s,\ \forall\, j = 1, \dots, N,\ \delta = min(\omega, \lambda). \tag{1.3}$$

Moreover there exists a constant k independent of u_j, $b_{ij}^{(\alpha)}$, f_i such that

$$\sum_{j=1}^{N} \sum_{|\alpha|=2s} \|D^\alpha u_j\|_{L^{p,\delta}(B_{\frac{\rho}{2}})} \leq \tag{1.4}$$

$$\leq k \left(\sum_{j=1}^{N} \sum_{|\alpha|\leq 2s-1} \|b_{ij}^{(\alpha)} D^\alpha u_j\|_{L^{p,\omega}(B_\rho)} + \|f_i\|_{L^{p,\lambda}(B_\rho)} \right), \ \forall i = 1, \dots, N.$$

Proof. We will consider a fixed ball B_ρ concentric to B_σ, where $\rho \leq \sigma$, and a cut-off function $\theta \in C_0^\infty(B_\rho)$ such that $\theta \cdot u \in W_0^{2s,p}(B_\rho)$.

Let $v = \theta \cdot u \in W_0^{2s,p}(B_\rho)$, it is possible to write $v(x)$ in terms of the John fundamental solution (see [12]). Integrating by parts and adapting an idea of Bureau (see [2]) the derivatives $D^\alpha u_j(x)$, $|\alpha| = 2s$, $j = 1, \dots, N$, can be written almost everywhere in B_ρ as a combination of the terms

$$\sum_{j=1}^{N} \left(\sum_{|\alpha|=2s} a_{ij}^{(\alpha)}(x) D^\alpha u_j(x) \right),$$

$$\text{P.V.} \int_B D^{\gamma+\alpha}\Gamma(x, x-y) \sum_{j=1}^{N} \left(\sum_{|\alpha|=2s} a_{ij}^{(\alpha)}(y) D^\alpha u_j(y) \right) dy, \tag{1.5}$$

$$\text{P.V.} \int_B D^{\gamma+\alpha}\Gamma(x, x-y) \sum_{k,j=1}^{N} \sum_{|\alpha|=2s} \left(a_{kj}^{(\alpha)}(x) - a_{kj}^{(\alpha)}(y) \right) D^\alpha u_j(y)\, dy.$$

where $|\gamma| = 2s(N-1)$. Following the arguments used in [7] based on the uniqueness of the fixed point of a contraction it is possible to prove the existence of $\tilde{\sigma} \in \,]0, \sigma[$ such that for every B_ρ concentric to B_σ, with $\rho \leq \tilde{\sigma}$, we obtain $D^\alpha u_j \in L^{p,\delta}(B_{\frac{\rho}{2}})$, $|\alpha| = 2s$, $\forall\, j = 1, \ldots, N$ and using the integral representation formula, Lemma 1.2 and Lemma 1.3 we immediately have

$$\sum_{j=1}^{N} \sum_{|\alpha|=2s} \|D^\alpha u_j\|_{L^{p,\delta}(B_{\frac{\rho}{2}})} \leq$$

$$\leq k \| f_i - (\sum_{j=1}^{N} \sum_{|\alpha| \leq 2s-1} b_{ij}^{(\alpha)} D^\alpha u_j) \|_{L^{p,\delta}(B_\rho)} \leq \qquad \forall i = 1, \ldots, N$$

$$\leq k \left(\|f_i\|_{L^{p,\lambda}(B_\rho)} + \| \sum_{j=1}^{N} \sum_{|\alpha| \leq 2s-1} b_{ij}^{\alpha} D^\alpha u_j \|_{L^{p,\delta}(B_\rho)} \right) \leq \qquad \forall i = 1, \ldots, N$$

$$\leq k \left(\|f_i\|_{L^{p,\lambda}(B_\rho)} + \sum_{j=1}^{N} \sum_{|\alpha| \leq 2s-1} \|b_{ij}^{\alpha} D^\alpha u_j\|_{L^{p,\omega}(B_\rho)} \right) \qquad \forall i = 1, \ldots, N.$$

□

In this direction we wish to continue the study of regularity results for highest order derivatives of u and of the Hölder regularity of $D^{2s-1}u$, dependent on the order s of the derivatives. We wish to remove the additional assumption (1.2) and consider the coefficients of the lower order terms in suitable Morrey spaces.

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SOME DECAY RESULTS FOR SOLUTIONS OF PARABOLIC PROBLEMS

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Si studia un problema parabolico semilineare con condizioni alla frontiera di Robin e, sotto opportune condizioni sui dati e sulla geometria del dominio, si ottiene una stima decrescente nel tempo per la soluzione. Sono inoltre indicate possibili estensioni dei risultati ottenuti.

1. Introduction.

Let $u(x, t)$ a classical solution of the following problem

$$\begin{cases} \Delta u + f(u, |\nabla u|^2) = u_t, & (\mathbf{x}, t) \in \Omega \times (0, T), \\ \alpha u + \beta \dfrac{\partial u}{\partial n} = 0, & (\mathbf{x}, t) \in \partial\Omega \times (0, T), \\ u(\mathbf{x}, 0) = g(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $C^{2+\epsilon}$ boundary $\partial\Omega$, T any time prior to eventual blow-up time, f, g are assumed to be differentiable and nonnegative functions, α and β nonnegative constants and

$$\alpha g(\mathbf{x}) + \beta \frac{\partial g(\mathbf{x})}{\partial n} = 0, \quad \mathbf{x} \in \partial\Omega. \tag{1.2}$$

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Here $\dfrac{\partial}{\partial n}$ indicates the normal derivative directed outward from $\partial\Omega$.
As a consequence of assumptions u will be non negative.
If $f = 0$, $\alpha = 0$ and $\beta = 1$ (Neumann condition) and Ω a strictly convex domain in $\mathbb{R}^N$, in [4] Payne and Philippin prove for the solution u and its gradient the following estimate

$$|\nabla u|^2 + \delta u^2 \leq \Gamma^2 e^{-2\delta t}, \quad \text{in}\, \Omega \times (t > 0), \tag{1.3}$$

with

$$\Gamma^2 = \max_{\Omega}\{|\nabla g|^2 + \delta g^2\}, \tag{1.4}$$

$\int_\Omega g = 0$ and in (1.3), (1.4) $0 \leq \delta < \left(\dfrac{\pi^2}{4D^2}\right)$, D the diameter of Ω.
If $f \neq 0$, $f = f(u)$, $\alpha = 1$ and $\beta = 0$ (Dirichlet condition) and Ω a convex domain in $\mathbb{R}^N$, in the same paper Payne and Philippin, by assuming $sf'(s) \leq f(s) \leq 0$, $s > 0$ and under restriction on initial data in order to prevent a possible blow up and to have $T = \infty$, prove that

$$|\nabla u|^2 + \delta u^2 + 2F(u) \leq G^2 e^{-2\delta t}, \quad \text{in}\, \Omega \times (t > 0), \tag{1.5}$$

with $F(u) = \int_0^u f(s)\, ds$, $G^2 = \max_\Omega\{|\nabla g|^2 + \delta g^2 + 2F(g)\}$.

Moreover in (1.5) $0 \leq \delta < \left(\dfrac{\pi^2}{4d^2}\right)$, d the inradius of Ω.
If $f = f(x, u, |\nabla u|^2)$, $\alpha = 1$, $\beta = 0$ in a joint paper with Payne and Philippin ([5]), we remove the convexity condition on Ω and prove that

$$|\nabla u|^2 + \lambda_1 u^2 \leq \tilde{\Gamma}^2 e^{-2\delta t}, \quad \text{in}\ \Omega \times (t > 0), \tag{1.6}$$

with λ_1 the first eigenvalue of the fixed membrane problem and δ and $\tilde{\Gamma}$ in (1.6) suitable constants.
If $f = 0$, $\beta = 1$ (Robin condition) and Ω a convex domain in $\mathbb{R}^2$, Payne and Schaefer in [6] extend to this case the estimate in (1.3), i.e.

$$|\nabla u|^2 + \delta u^2 \leq \Gamma^2 e^{-2\delta t}, \quad \text{in}\ \Omega \times (t > 0), \tag{1.7}$$

with Γ defined similar to (1.4) and where the positive constant δ in (1.7) is restricted by the conditions

$$\sqrt{\delta}\, tg(\sqrt{\delta} d < \alpha, \quad \sqrt{\delta}\, d < \frac{\pi}{2} \tag{1.8}$$

and

$$\Delta g + (\alpha^2 + 2\delta)\, g \geq 0, \tag{1.9}$$

provided g has bounded second derivatives. In (1.8) d is the inradius of Ω.
If $f \neq 0$, in [6] similar results are obtained only in one dimension.

Aim of this paper is to obtain decay results for the solution of (1.1) and its gradient, with $f = f(u, |\nabla u|^2)$, with $\beta = 1$, Ω a convex domain in $\mathbb{R}^2$, T any time before the eventual blow up time (Sec.2). Moreover some open problems as the possibility to avoid the blow up, to find explicit exponential decay bounds for the gradient of the solution, extensions in $\mathbb{R}^N$ are presented in Sec.3.

2. Decay results for $u(x, t)$ and its gradient in problem (1.1) with Robin boundary condition.

As annunced in the introduction, in this section we consider (1.1) with $\beta = 1$ (Robin condition) and Ω a bounded convex domain in $\mathbb{R}^2$ with $\partial\Omega \in C^{2+\epsilon}$. Since it is well known that the solution u may blow up ([1],[4]) in finite time T^*, we consider $u(\mathbf{x}, t)$ for $\mathbf{x} \in \Omega$ and $t \in (0, T)$, with $T < T^*$.
Concerning the function f, we suppose that there exists a function $\Psi(u, |\nabla u|^2)$ which is nondecreasing w.r.t. u and $|\nabla u|^2$ such that

$$\max\left\{\frac{f}{u},\ f' - 2\gamma u \dot{f}\right\} \leq \Psi(u, |\nabla u|^2), \qquad f' = \frac{\partial f}{\partial u},\ \dot{f} = \frac{\partial f}{\partial(|\nabla u|^2)}, \tag{2.1}$$

where γ is a positive constant to be specified later on. For examples see [5]. We prove that u decays in time in $\Omega \times (0, T)$ in the following

Theorem 1. *Let $u(x, t)$ be a classical solution of problem* (1.1) *and let*

$$\Psi_M := \Psi(u_M, |\nabla u|^2_M),$$

with $\Psi(\mathbf{x}, t)$ in (2.1) *and*

$$u_M := \max_{\Omega\times(0,T)} \{u(\mathbf{x}, t)\} \quad \textit{and} \quad |\nabla u|_M := \max_{\Omega\times(0,T)} |\nabla u(\mathbf{x}, t)|.$$

Then

$$0 \leq u(\mathbf{x}, t) \leq \Gamma_\delta\, e^{-(\delta - \Psi_M)t}, \qquad (\mathbf{x}, t) \in \Omega \times (0, T), \tag{2.2}$$

with $\Gamma_\delta = \dfrac{\Gamma}{\sqrt{\delta}} := \max_{\Omega}\left[g^2 + \dfrac{|\nabla g|^2}{\delta}\right]^{1/2}$*, and δ, α and g satisfying* (1.8), (1.9).

Proof. By using (1.1) and (2.1) we have

$$0 = \Delta u - u_t + \left\{ \frac{f(u, |\nabla u|^2)}{u} \right\} u \leq \Delta u - u_t + \Psi_M\, u.$$

We deduce that $u(\mathbf{x}, t)$ satisfies the following initial boundary value problem

$$\begin{cases} \Delta u - u_t + \Psi_M u \geq 0, & (\mathbf{x}, t) \in \Omega \times (0, T), \\ \dfrac{\partial u}{\partial n} = -\alpha\, u, & (\mathbf{x}, t) \in \partial\Omega \times (0, T), \\ u(\mathbf{x}, 0) = g(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{2.3}$$

In order to obtain a decay estimate for $u(\mathbf{x}, t)$, we introduce the auxiliary function $\phi(\mathbf{x}, t) = u(\mathbf{x}, t)\, e^{-\Psi_M t}$, which satisfies

$$\begin{cases} \Delta \phi - \phi_t \geq 0, & (\mathbf{x}, t) \in \Omega \times (0, T), \\ \dfrac{\partial \phi}{\partial n} = -\alpha\, \phi, & (\mathbf{x}, t) \in \partial\Omega \times (0, T), \\ \phi(\mathbf{x}, 0) = g(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases} \tag{2.4}$$

Now if $\tilde{\phi}$ is a solution of the following problem

$$\begin{cases} \Delta \tilde{\phi} - \tilde{\phi}_t = 0, & (\mathbf{x}, t) \in \Omega \times (0, T), \\ \dfrac{\partial \tilde{\phi}}{\partial n} = -\alpha\, \tilde{\phi}, & (\mathbf{x}, t) \in \partial\Omega \times (0, T), \\ \tilde{\phi}(\mathbf{x}, 0) = g(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{2.5}$$

we know that, from a classical comparison theorem ([7]),

$$\phi(\mathbf{x}, t) \leq \tilde{\phi}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times (0, T). \tag{2.6}$$

From Theorem 1 in [6] the solution of (2.5) satisfies

$$\tilde{\phi}(\mathbf{x}, t) \leq \Gamma_\delta\, e^{-\delta t}, \tag{2.7}$$

since we have assumed (1.8),(1.9).
The conclusion (2.2) of the theorem follows from (2.6), (2.7) and the definition of ϕ. We note that if in (2.2) $\delta \geq \Psi_M$, the bound for u is decay in time. In this case we deduce $u \leq \Gamma_\delta$ in $\Omega \times (0, T)$.

Now to investigate the behaviour of $|\nabla u|^2$ we introduce in $\Omega \times (0, T)$, the auxiliary function

$$\Phi(\mathbf{x}, t) = \left\{ |\nabla u|^2 + \gamma u^2 \right\} e^{2\delta t}, \tag{2.8}$$

with γ and δ two positive constant to be determined in order to have a parabolic inequality for Φ. We prove the following

Theorem 2. *Assume Theorem 1 holds. If*

$$\gamma - \delta \geq \Psi_M, \tag{2.9}$$

then

$$\Delta\Phi + |\nabla u|^{-2} w_k \Phi_k - \Phi_t \geq 0\,, \tag{2.10}$$

where w_k is the k-th component of a vector field, regular in $\Omega \times (0, T)$.

Proof. We obtain, after some computation and by using Schwarz inequality

$$\Delta\Phi - \Phi_t \geq \Big[2|\nabla u|^2\Big\{2\dot{f}\gamma u - f' + (\gamma - \delta)\Big\} + \\ 2\gamma u^2\Big\{(\gamma - \delta) - f/u\Big\}\Big]e^{2\delta t} - |\nabla u|^{-2} w_k \Phi_k.$$

Since we have assumed (2.9), the coefficients of $|\nabla u|^2$ and u^2 are non negative and (2.10) is proved.
As a consequence of the maximum principle ([2],[3]), Φ can assume its maximum value either (i) at a point $(\hat{\mathbf{x}}, \hat{t})$ with $\hat{\mathbf{x}} \in \partial\Omega$, or (ii) at a critical point $(\overline{\mathbf{x}}, \overline{t})$, such that $\nabla u(\overline{\mathbf{x}}, \overline{t}) = 0$, or (iii) at a point $(\tilde{\mathbf{x}}, 0)$, $\tilde{\mathbf{x}} \in \Omega$. Since we have assumed (1.8), following [6], the second possibility cannot hold.
We derive

$$\Big\{|\nabla u|^2 + \gamma u^2\Big\} \leq K e^{-2\delta t}, \quad (\mathbf{x}, t) \in \Omega \times (0, T),$$

where $K = \max\{K_0, K_1\}$, with

$$K_0 = \max_{\Omega}\Big\{|\nabla g|^2 + \gamma g^2\Big\} \quad \text{and} \quad K_1 = \max_{\partial\Omega\times(0,T)} \Phi(\mathbf{x}, t).$$

3. Open problems.

The goal of this section is to indicate further results that can be obtained for u and its gradient under Sec.2 assumptions.
One can try to obtain an explicit estimate for $|\nabla u|^2$ as in [4], [5] and [6]. To this end usually we introduce an auxiliary function , where both the solution and its gradient are involved. In the case $f = f(u, |\nabla u|^2)$ a candidate can be the function defined in (2.8); in the case in which $f = f(u)$ a candidate can be

$$\Phi(\mathbf{x}, t) = \Big\{|\nabla u|^2 + \gamma u^2 + 2F(u)\Big\}e^{2\sigma t},$$

$F(u) = \int_0^u f(s)\,ds$, γ and σ suitable constants. Then sufficient conditions must be introduced in order to avoid the possibility that the maximum is reached at a boundary point or at a singular point, so that Φ assumes its maximum value at a point $(\mathbf{x}, 0)$, $\mathbf{x} \in \Omega$ and an explicit exponential decay estimate is obtained in terms of initial data $g(\mathbf{x})$. A second result can be to extend the estimate obtained in the interval (0,T) to the whole interval, i.e. $T = \infty$. Sufficient conditions must be introduced in order to avoid blow up: to this end initial data must be restricted in some way. A further result can be the extension to $\mathbb{R}^N$ of the results obtained in $\mathbb{R}^2$, as suggested by Payne.

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STRONG A_∞ WEIGHTS AND QUASILINEAR ELLIPTIC EQUATIONS

GIUSEPPE DI FAZIO - PIETRO ZAMBONI

1. Introduction.

In this paper we investigate the regularity of the weak solutions for degenerate elliptic equations of the following kind

$$\operatorname{div} A(x, u(x), \nabla u(x)) + B(x, u(x), \nabla u(x)) = 0\,, \tag{1.1}$$

under the following structure conditions

$$\begin{cases} |A(x, u, \xi)| \le a\omega(x)|\xi|^{p-1} + b|u|^{p-1} + e \\ |B(x, u, \xi)| \le c|\xi|^{p-1} + d|u|^{p-1} + f \\ \xi \cdot A(x, u, \xi) \ge \omega(x)|\xi|^p - d|u|^p - g \end{cases} \tag{1.2}$$

where ν is a strong A_∞ weight, $\omega = \nu^{1-\frac{p}{n}}$ and $1 < p < n$. Equations like (1.1) have been studied by many Authors - when $\omega(x) \equiv 1$ (see e.g. [1] and the references therein) or when ω is an A_2 Muckenhoupt weight ([5] and [7]).

The novelty here is the degeneracy condition given by choice of the weight ω to be a power of a strong A_∞ weight. Moreover, we assume very mild integrability conditions on the lower order terms and known term in equation (1.1). These conditions are sharp and - at least in some instances - are necessary and sufficient (see e.g. [3] or [6]).

The main result is a Harnack inequality for nonnegative weak solutions of equation (1.1) (see Theorem 4.2) and, as a direct consequence, smoothness for weak solutions. In particular, we have a continuity result under Stummel - Kato type assumptions and Hölder continuity result under Morrey type assumptions (see Theorems 4.4 and 4.5). This note is a condensed version of our paper [4].

2. Strong A_∞ weights.

Let ν be an A_∞ weight in $\mathbb{R}^n$. This means that, for any $\varepsilon > 0$ there exists $\delta > 0$, such that if Q is a cube in $\mathbb{R}^n$ and E is a measurable subset of Q for which $|E| \leq \delta|Q|$ holds, then $\nu(E) \leq \varepsilon\ \nu(Q)$. If $\nu \in A_\infty$ we can define a quasi distance in the following way: for $x, y \in \mathbb{R}^n$ let $B_{x,y}$ the euclidean ball containing both points having diameter $|x - y|$. We set

$$\delta(x, y) = \left(\int_{B_{x,y}} \nu(t)\, dt \right)^{1/n} .$$

The function δ is a quasi distance. We can define the length of a curve as the limsup of the δ-length of the approximating polygonals.

On the other side, we can actually define a distance related to the weight ν. We take, as the distance between two points x and y, the infimum of the δ-length of the curves connecting x and y. Namely we set,

$$d_\nu(x, y) = \inf\{\delta\text{-length of the curves connecting } x \text{ and } y\} .$$

In general, the function δ is not comparable to a distance.

Definition 2.1. If ν is an A_∞ weight there exists a positive constant c such that $\delta(x, y) \leq c\, d_\nu(x, y)$, for any $x, y \in \mathbb{R}^n$ (see [2]). If, in addition,

$$\delta(x, y) \sim d_\nu(x, y)\,,\ \forall x, y \in \mathbb{R}^n\,, \tag{2.1}$$

then we say that ν is a strong A_∞ weight.

The measure $\nu\, dx$ is Ahlfors regular and, as a consequence, doubling.

Remark 2.2. Any strong A_∞ weight is a A_∞ weight. For any $1 < p < \infty$ there exists an A_p weight which is not a strong A_∞ weight.

3. Function spaces.

Using strong A_∞ weights we define Lebesgue and Sobolev classes.

Definition 3.1. Let ν be a strong A_∞ weight and $\omega = \nu^{1-p/n}$, $\Omega \subset \mathbb{R}^n$. For any $u \in C_0^\infty(\Omega)$ we set,

$$\|u\|_{p,\nu} = \left(\int_\Omega |u(x)|^p\,\omega(x)\,dx\right)^{1/p}, \quad 1 \le p < \infty . \tag{3.2}$$

We define $L^p_\nu(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the above norm. In a similar way we define Sobolev classes. For any $u \in C^\infty(\Omega)$ we set,

$$\|u\|_{1,p,\nu} = \left(\int_\Omega |u(x)|^p\,\omega(x)\,dx\right)^{1/p} + \left(\int_\Omega |\nabla u(x)|^p\,\omega(x)\,dx\right)^{1/p}, \quad 1 \le p < n . \tag{3.3}$$

We define $H^{1,p}_{0,\nu}(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the above norm and $H^{1,p}_\nu(\Omega)$ to be the completion of $C^\infty(\Omega)$ with respect to the same norm.

In the above definitions we put ν in the symbol of the norm and ω into the integrals. This is because we want to stress the dependence on the strong A_∞ weight ν.

Now we define more function spaces which we need later.

Definition 3.2. Let f be a locally integrable function in $\Omega \subset \mathbb{R}^n$ and let ν be a strong A_∞ weight. We set,

$$\phi(f;R) = \sup_{x\in\Omega}\left(\int_{B(x,R)} \frac{1}{\nu(B(x,d_\nu(x,y)))^{1-\frac{1}{n}}} \left(\int_{B(x,R)} \frac{|f(z)|}{\nu(B(z,d_\nu(z,y)))^{1-\frac{1}{n}}}\,\nu(z)^{1-\frac{p}{n}}\,dz\right)^{\frac{1}{p-1}} \nu(y)\,dy\right)^{p-1} \tag{3.4}$$

We shall say that f belongs to the class $\tilde{S}_\nu(\Omega)$ if $\phi(f;R)$ is just a bounded function in a neighborhood of the origin. If, moreover, $\lim_{R\to 0}\phi(f;R) = 0$ then we say that f belongs to the Stummel-Kato class $S_\nu(\Omega)$. If there exists $\rho > 0$ such that

$$\int_0^\rho \frac{\phi(f;t)^{1/p}}{t}\,dt < +\infty , \tag{3.5}$$

then we say that the function f belongs to the class $S'_\nu(\Omega)$.

Definition 3.3. (Morrey spaces) Let $p \in [1, +\infty[$ and ν be a strong A_∞ weight. We say that f belongs to $L_\nu^{p,\lambda}(\Omega)$, for some $\lambda > 0$, if

$$\|f\|_{L_\nu^{p,\lambda}(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<r<d_0}} \left(\frac{r^\lambda}{|B(x,r)\cap\Omega|} \int_{B(x,r)\cap\Omega} |f(y)|^p \nu(y)^{1-\frac{p}{n}}\, dy \right)^{\frac{1}{p}} < +\infty,$$

where $d_0 = \operatorname{diam}(\Omega)$.

It is easy to compare the function classes previously defined.

Proposition 3.4. *Let* $1 < p < n$, $0 < \varepsilon < p$. *We have*

$$\phi\,(V;r) \le C\|V\|_{L_\nu^{1,p-\varepsilon}} r^{\frac{\varepsilon}{p-1}}$$

for any $0 < r < d_0$ *and then* $L_\nu^{1,p-\varepsilon}(\Omega) \subseteq S'_\nu(\Omega)$.

4. Regularity.

Definition 4.1. A function $u \in H_\nu^{1,p}(\Omega)$ is a local weak solution of (1.1) in Ω if

$$\int_\Omega A(x,u(x),\nabla u(x))\,\nabla\varphi(x)\,dx + \int_\Omega B(x,u(x),\nabla u(x))\,\varphi(x)\,dx = 0\,, \tag{4.6}$$

for every $\varphi \in H_{0,\nu}^{1,p}(\Omega)$.

Theorem 4.2. (Harnack inequality). *Let* ν *be a strong* A_∞ *weight and* $1 < p < n$. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$, u *be a non negative weak solution of* (1.1). *Let us assume that the structure conditions* (1.2) *hold true with*

$$a \in \mathbb{R}\,,\ \left(\frac{b}{\omega}\right)^{p/p-1},\ \left(\frac{c}{\omega}\right)^{p},\ \frac{d}{\omega},\ \left(\frac{e}{\omega}\right)^{p/p-1},\ \frac{f}{\omega},\ \frac{g}{\omega},\ \in S'_\nu(\Omega)\,, \tag{4.7}$$

where $\omega(x) = \nu^{1-\frac{p}{n}}(x)$. *Then, there exists a positive constant c, independent of u, such that, for any* $B_r = B(x_0,r)$ *for which* $B(x_0,4r) \subset \Omega$, *we have*

$$\max_{B_r} u \le c\left\{\min_{B_r} u + h(r)\right\}$$

where $h(r) = \left[\phi\left(\left(\frac{e}{\omega}\right)^{\frac{p}{p-1}};2r\right) + \phi\left(\frac{g}{\omega};2r\right)\right]^{\frac{1}{p}} + \left[\phi\left(\frac{f}{\omega};2r\right)\right]^{\frac{1}{p-1}}$.

Remark 4.3. The proof of Theorem 4.2 works also with weak subsolutions of (1.1) and the proof of Theorem 4.2 provides a weak Harnack inequality for non negative weak supersolutions.

It is an easy task now to show smoothness for weak solutions. We have

Theorem 4.4. (Continuity of weak solutions). *Let ν be a strong A_∞ weight and $1 < p < n$. Let Ω be a bounded domain in $\mathbb{R}^n$, u be a weak solution of* (1.1). *Let us assume that the structure conditions* (1.2) *hold true with*

$$a \in \mathbb{R}\,,\ \left(\frac{b}{\omega}\right)^{p/p-1},\ \left(\frac{c}{\omega}\right)^{p},\ \frac{d}{\omega},\ \left(\frac{e}{\omega}\right)^{p/p-1},\ \frac{f}{\omega},\ \frac{g}{\omega},\ \in S'_\nu(\Omega)\,, \tag{4.8}$$

where $\omega(x) = \nu^{1-\frac{p}{n}}(x)$. Then, u is continuous in Ω.

If we want to obtain better regularity we have to restrict our assumptions on the lower order terms. Indeed, even in the linear uniformly elliptic case with non negative known term, Stummel – Kato classes are necessary and sufficient for the weak solutions to be continuous ([3]).

Theorem 4.5. (Hölder Continuity of weak solutions). *Let ν be a strong A_∞ weight and $1 < p < n$. Let Ω be a bounded domain in $\mathbb{R}^n$, u be a weak solution of* (1.1). *Let us assume that the structure conditions* (1.2) *hold true with*

$$\begin{aligned} a \in \mathbb{R}\,,\ \left(\frac{b}{\omega}\right)^{p/p-1},\ \left(\frac{c}{\omega}\right)^{p},\ \frac{d}{\omega},\ \left(\frac{e}{\omega}\right)^{p/p-1}, \\ \frac{f}{\omega},\ \frac{g}{\omega},\ \in L^{1,p-\varepsilon}_\nu(\Omega)\,, \quad \varepsilon > 0\,, \end{aligned} \tag{4.9}$$

where $\omega(x) = \nu^{1-\frac{p}{n}}(x)$. Then, u is locally Hölder continuous in Ω.

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