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REGULARITY AND GRÖBNER BASES OF THE REES ALGEBRA OF EDGE IDEALS OF BIPARTITE GRAPHS

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Let G be a bipartite graph and $I = I(G)$ be its edge ideal. The aim of this note is to investigate different aspects of the Rees algebra $\mathcal{R}(I)$ of I. We compute its regularity and the universal Gröbner basis of its defining equations; interestingly, both of them are described in terms of the combinatorics of G.

We apply these ideas to study the regularity of the powers of I. For any $s \geq \operatorname{match}(G) + |E(G)| + 1$ we prove that $\operatorname{reg}(I^{s+1}) = \operatorname{reg}(I^s) + 2$.

1. Introduction

Let $G = (V(G), E(G))$ be a bipartite graph on the vertex set $V(G) = X \cup Y$ with bipartition $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. Let $\mathbb{K}$ be a field and let R be the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. The edge ideal $I = I(G)$, associated to G, is the ideal of R generated by the set of monomials $x_i y_j$ such that x_i is adjacent to y_j.

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One can find a vast literature on the Rees algebra of edge ideals of bipartite graphs (see [28], [22], [11], [26], [25], [27], [10]), nevertheless, in this note we study several properties that might have been overlooked. From a computational point of view we first focus on the universal Gröbner basis of its defining equations, and from a more algebraic standpoint we focus on its total and partial regularities as a bigraded algebra. Applying these ideas, we give an estimation of when $\operatorname{reg}(I^s)$ starts to be a linear function and we find upper bounds for the regularity of the powers of I.

Let $\mathcal{R}(I) = \bigoplus_{i=0}^{\infty} I^i t^i \subset R[t]$ be the Rees algebra of the edge ideal I. Let $f_1, \ldots, f_q$ be the square free monomials of degree two generating I. We can see $\mathcal{R}(I)$ as a quotient of the polynomial ring $S = R[T_1, \ldots, T_q]$ via the map

$$\begin{gathered} S = \mathbb{K}[x_1, \ldots, x_n, y_1 \ldots, y_m, T_1, \ldots, T_q] \xrightarrow{\psi} \mathcal{R}(I) \subset R[t], \\ \psi(x_i) = x_i, \quad \psi(y_i) = y_i, \quad \psi(T_i) = f_i t. \end{gathered} \tag{1}$$

Then the presentation of $\mathcal{R}(I)$ is given by $S/\mathcal{K}$ where $\mathcal{K} = \operatorname{Ker}(\psi)$. We give a bigraded structure to $S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m] \otimes_{\mathbb{K}} \mathbb{K}[T_1, \ldots, T_q]$, where $\operatorname{bideg}(x_i) = \operatorname{bideg}(y_i) = (1,0)$ and $\operatorname{bideg}(T_i) = (0,1)$. The map ψ from Eq. 1 becomes bihomogeneous when we declare $\operatorname{bideg}(t) = (-2,1)$, then we have that $S/\mathcal{K}$ and $\mathcal{K}$ have natural bigraded structures as S-modules.

The universal Gröbner basis of the ideal $\mathcal{K}$ is defined as the union of all the reduced Gröbner bases $\mathcal{G}_<$ of the ideal $\mathcal{K}$ as $<$ runs over all possible monomial orders (see [23]). In our first main result we compute the universal Gröbner basis of the defining equations $\mathcal{K}$ of the Rees algebra $\mathcal{R}(I)$.

Theorem 1.1 (Theorem 2.5). *Let G be a bipartite graph and $\mathcal{K}$ be the defining equations of the Rees algebra $\mathcal{R}(I(G))$. The universal Gröbner basis $\mathcal{U}$ of $\mathcal{K}$ is given by*

$$\begin{aligned} \mathcal{U} = &\{T_w \mid w \text{ is an even cycle}\} \\ &\cup \{v_0 T_{w^+} - v_a T_{w^-} \mid w = (v_0, \ldots, v_a) \text{ is an even path}\} \\ &\cup \{u_0 u_a T_{(w_1,w_2)^+} - v_0 v_b T_{(w_1,w_2)^-} \mid w_1 = (u_0, \ldots, u_a) \text{ and} \\ &\qquad\qquad\qquad\qquad w_2 = (v_0, \ldots, v_b) \text{ are disjoint odd paths}\}. \end{aligned}$$

From [25, Theorem 3.1, Proposition 3.1] we have a precise description of $\mathcal{K}$ given by the syzygies of I and the set even of closed walks in the graph G. The algebra $\mathcal{R}(I)$, as a bigraded S-module, has a minimal bigraded free resolution

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathcal{R}(I) \longrightarrow 0, \tag{2}$$

where $F_i = \oplus_j S(-a_{ij}, -b_{ij})$. In the same way as in [19], we can define the xy-regularity of $\mathcal{R}(I)$ by the integer

$$\operatorname{reg}_{xy}(\mathcal{R}(I)) = \max_{i,j}\{a_{ij} - i\},$$

or equivalently by

$$\operatorname{reg}_{xy}(\mathcal{R}(I)) = \max\{a \in \mathbb{Z} \mid \beta^S_{i,(a+i,b)}(\mathcal{R}(I)) \neq 0 \text{ for some } i,b \in \mathbb{Z}\},$$

where $\beta^S_{i,(a,b)}(\mathcal{R}(I)) = \dim_{\mathbb{K}}(\operatorname{Tor}^S_i(\mathcal{R}(I),\mathbb{K})_{(a,b)})$.

Similarly, we can define the T-regularity

$$\operatorname{reg}_T(\mathcal{R}(I)) = \max_{i,j}\{b_{ij} - i\}$$

and the total regularity

$$\operatorname{reg}(\mathcal{R}(I)) = \max_{i,j}\{a_{ij} + b_{ij} - i\}.$$

Our second main result is computing the total regularity and giving upper bounds for both partial regularities.

Theorem 1.2 (Theorem 4.2)**.** *Let G be a bipartite graph. Then we have:*

(i) $\operatorname{reg}(\mathcal{R}(I(G))) = \operatorname{match}(G)$*,*

(ii) $\operatorname{reg}_{xy}(\mathcal{R}(I(G))) \leq \operatorname{match}(G) - 1$*,*

(iii) $\operatorname{reg}_T(\mathcal{R}(I(G))) \leq \operatorname{match}(G)$*,*

where $\operatorname{match}(G)$ *denotes the matching number of G.*

Finally, we apply these results in order to study the regularity of the powers of the edge ideal $I = I(G)$.

It is a famous result (for a general ideal in a polynomial ring) the asymptotic linearity of $\operatorname{reg}(I^s)$ for $s \gg 0$ (see [8] and [18]). However, the exact form of this linear function and the exact point where $\operatorname{reg}(I^s)$ starts to be linear, is a problem that continues wide open even in the case of monomial ideals.

In recent years, a number of researchers have focused on computing the regularity of powers of edge ideals and on relating these values to combinatorial invariants of the graph (see e.g. [4], [1], [2], [3], [5], [17]). Most of the upper bounds given in these papers use the concept of even-connection introduced in [3]. Actually, using this idea as a central tool, in [17] it was proved the upper bound

$$\operatorname{reg}(I^s) \leq 2s + \operatorname{co-chord}(G) - 1$$

for any bipartite graph G, where $\operatorname{co-chord}(G)$ represents the co-chordal number of G (see [17, Definition 3.1]).

As a consequence of our study of the Rees algebra $\mathcal{R}(I)$, we make an estimation of when $\operatorname{reg}(I^s)$ starts to be a linear function, and we obtain the weaker upper bounds for the regularity of the powers of I (see Remark 3.9, Corollary 4.3, Corollary 3.8). Perhaps, this could give new tools and fresh ideas to pursue the stronger upper bound

$$\operatorname{reg}(I^s) \leq 2s + \operatorname{reg}(I) - 2, \tag{3}$$

that has been conjectured by Alilooee, Banerjee, Beyarslan and Hà ([4, Conjecture 7.11]).

Using the upper bound for the partial T-regularity of $\mathcal{R}(I)$, we can get the following estimation.

Corollary 1.3 (Corollary 4.4). *Let G be a bipartite graph. Then, for all $s \geq \operatorname{match}(G) + |E(G)| + 1$ we have*

$$\operatorname{reg}(I(G)^{s+1}) = \operatorname{reg}(I(G)^s) + 2.$$

The basic outline of this note is as follows.

In Section 2, we compute the universal Gröbner basis of $\mathcal{K}$ (Theorem 1.1). In Section 3, we consider a specific monomial order that allows us to get upper bounds for the xy-regularity of $\mathcal{R}(I)$. In Section 4 we exploit the canonical module of $\mathcal{R}(I)$ in order to prove Theorem 1.2 and Corollary 1.3. Finally, in Section 5 we give some general ideas about the conjectured upper bound Eq. 3.

2. The universal Gröbner basis of $\mathcal{K}$

In this section we will give an explicit description of the universal Gröbner basis $\mathcal{U}$ of $\mathcal{K}$. Our approach is the following, first we compute the set of circuits of the incidence matrix of the cone graph, and then we translate this set of circuits into a description of $\mathcal{U}$.

The following notation will be assumed in most of this note.

Notation 2.1. Let G be a bipartite graph with bipartition $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, and R be the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Let I be the edge ideal $I(G) = (f_1, \ldots, f_q)$ of G. We consider the Rees algebra $\mathcal{R}(I)$ as a quotient of $S = R[T_1, \ldots, T_q]$ by using Eq. 1. Let $\mathcal{K}$ be the defining equations of the Rees algebra $\mathcal{R}(I)$.

Let $A = (a_{i,j}) \in \mathbb{R}^{n+m,q}$ be the incidence matrix of the graph G. Then we construct the matrix M of the following form

$$M = \begin{pmatrix} a_{1,1} & \ldots & a_{1,q} & e_1 & \ldots & e_{n+m} \\ \vdots & \vdots & \vdots & & & \\ a_{n+m,1} & \ldots & a_{n+m,q} & & & \\ 1 & \ldots & 1 & & & \end{pmatrix}, \tag{4}$$

where $e_1,\dots e_{n+m}$ are the first $n+m$ unit vectors in $\mathbb{R}^{n+m+1}$ (see [11, Section 3] for more details). This matrix corresponds to the presentation of $\mathcal{R}(I)$ given in Eq. 1. For any vector $\beta \in \mathbb{Z}^{n+m+q}$ with nonnegative coordinates we shall use the notation

$$\mathbf{xyT}^{\beta} = x_1^{\beta_{q+1}}\cdots x_n^{\beta_{q+n}} y_1^{\beta_{q+n+1}}\cdots y_m^{\beta_{q+n+m}} T_1^{\beta_1}\cdots T_q^{\beta_q}.$$

A given vector $\alpha \in \mathrm{Ker}(M)\cap\mathbb{Z}^{n+m+q}$, can be written as $\alpha=\alpha^+-\alpha^-$ where α^+ and α^- are nonnegative and have disjoint support.

Definition 2.2 ([23])**.** A vector $\alpha \in \mathrm{Ker}(M)\cap\mathbb{Z}^{n+m+q}$ is called a circuit if it has minimal support $\mathrm{supp}(\alpha)$ with respect to inclusion and its coordinates are relatively prime.

Notation 2.3. Given a walk $w=\{v_0,\dots,v_a\}$, each edge $\{v_{j-1},v_j\}$ corresponds to a variable T_{i_j}, and we set $T_{w^+}=\prod_{j \text{ is even}} T_{i_j}$ and $T_{w^-}=\prod_{j \text{ is odd}} T_{i_j}$ (in case $a=1$ we make $T_{w^+}=1$). We adopt the following notations:

(i) Let $w=\{v_0,\dots,v_a=v_0\}$ be an even cycle in G. Then by T_w we will denote the binomial $T_{w^+}-T_{w^-}\in\mathcal{K}$.

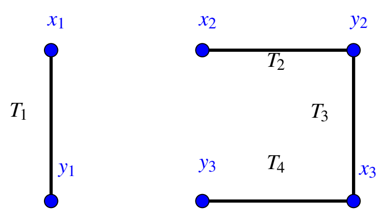
(ii) Let $w=\{v_0,\dots,v_a\}$ be an even path in G, since G is bipartite then both endpoints of w belong to the same side of the bipartition, i.e. either $v_0=x_i, v_a=x_j$ or $v_0=y_i, v_a=y_j$. Then the path w determines the binomial

$$v_0T_{w^+}-v_aT_{w^-}\in\mathcal{K}.$$

(iii) Let $w_1=\{u_0,\dots,u_a\}$, $w_2=\{v_0,\dots,v_b\}$ be two disjoint odd paths, then the endpoints of w_1 and w_2 belong to different sides of the bipartition. Let $T_{(w_1,w_2)^+}=T_{w_1^+}T_{w_2^-}$ and $T_{(w_1,w_2)^-}=T_{w_1^-}T_{w_2^+}$, then w_1 and w_2 determine the binomial

$$u_0u_aT_{(w_1,w_2)^+}-v_0v_bT_{(w_1,w_2)^-}\in\mathcal{K}.$$

Example 2.4. In the bipartite graph shown below



we have that the odd paths $w_1=(x_1,y_1)$ and $w_2=(x_2,y_2,x_3,y_3)$ determine the binomial $x_1y_1T_2T_4-x_2y_3T_1T_3$.

Let $\mathcal{U}$ be the universal Göbner basis of $\mathcal{K}$. In general we have that the set of circuits is contained in $\mathcal{U}$ ([23, Proposition 4.11]). But from the fact that M is totally unimodular ([11, Theorem 3.1]), we can use [23, Proposition 8.11] and obtain the equality

$$\mathcal{U} = \{\mathbf{xyT}^{\alpha^+} - \mathbf{xyT}^{\alpha^-} \mid \alpha \text{ is a circuit of } M\}.$$

Therefore we shall focus on determining the circuits of M, and for this we will need to introduce the concept of the cone graph $C(G)$. The vertex set of the graph $C(G)$ is obtained by adding a new vertex z to G, and its edge set consists of the edges in $E(G)$ together with the edges $\{x_1,z\},\ldots,\{x_n,z\},\{y_1,z\},\ldots,\{y_m,z\}$.

Theorem 2.5. *Let G be a bipartite graph and $I = I(G)$ be its edge ideal. The universal Gröbner basis $\mathcal{U}$ of $\mathcal{K}$ is given by*

$$\begin{aligned}\mathcal{U} = \{&T_w \mid w \text{ is an even cycle}\}\\ &\cup \{v_0T_{w^+} - v_aT_{w^-} \mid w = (v_0,\ldots,v_a) \text{ is an even path}\}\\ &\cup \{u_0u_aT_{(w_1,w_2)^+} - v_0v_bT_{(w_1,w_2)^-} \mid w_1 = (u_0,\ldots,u_a) \text{ and}\\ &\qquad\qquad w_2 = (v_0,\ldots,v_b) \text{ are disjoint odd paths}\}.\end{aligned}$$

Proof. Let $\mathbb{K}[C(G)]$ be the monomial subring of the graph $C(G)$, which is generated by the monomials

$$\mathbb{K}[C(G)] = \mathbb{K}\big[\{x_iy_j \mid \{x_i,y_j\} \in E(G)\} \cup \{x_iz \mid i=1,\ldots,n\} \cup \{y_iz \mid i=1,\ldots,m\}\big].$$

As we did for the Rees algebra $\mathcal{R}(I)$, we can define a similar surjective homomorphism

$$\pi : S \longrightarrow \mathbb{K}[C(G)] \subset R[z],$$
$$\pi(x_i) = x_iz, \quad \pi(y_i) = y_iz, \quad \pi(T_i) = f_i.$$

We have a natural isomorphism between $\mathcal{R}(I)$ and $\mathbb{K}[C(G)]$ [24, Excercise 7.3.3]. For instance, we can define the homomorphism $\varphi : R[t] \to R[z,z^{-1}]$ given by $\varphi(x_i) = x_iz$, $\varphi(y_i) = y_iz$ and $\varphi(t) = 1/z^2$, then the restriction $\varphi\mid_{\mathcal{R}(I)}$ of φ to $\mathcal{R}(I)$ will give us the required isomorphism because both algebras are integral domains of the same dimension (see Proposition 4.1 (i)).

Hence we will identify the ideal $\mathcal{K}$ with the kernel of π. Let N be the incidence matrix of the cone graph $C(G)$. From [25, Proposition 4.2], we have that a vector $\alpha \in \operatorname{Ker}(N) \cap \mathbb{Z}^{m+n+q}$ is a circuit of N if and only if the monomial walk defined by α corresponds to an even cycle or to two edge disjoint odd cycles joined by a path.

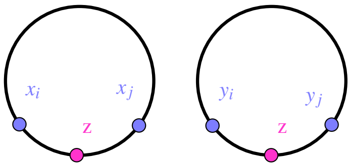
Since the graph G is bipartite, then an odd cycle in $C(G)$ will necessarily contain the vertex z. Therefore the monomial walks defined by the circuits of N are of the following types:

(i) An even cycle in $C(G)$ that does not contain the vertex z.

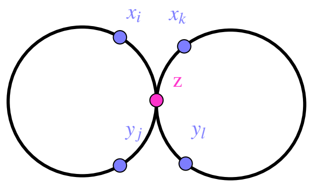
(ii) An even cycle in $C(G)$ that contains the vertex z.

(iii) Two odd cycles in $C(G)$ whose intersection is exactly the vertex z.

The figure below shows how the cases (ii) and (iii) may look.



(a) The two possible cycles of (ii).



(b) The graph of (iii).

Since the circuits of the matrices M and N coincide, now we translate these monomial walks in $C(G)$ into binomials of $\mathcal{K}$. An even cycle in $C(G)$ not containing z, is also an even cycle in G, and it determines a binomial in $\mathcal{K}$ using Notation 2.3. In the cases (ii) and (iii), we delete vertex z in order to get a subgraph H of G. Thus we have that H is either an even path or two disjoint odd paths, and we translate these into binomials in $\mathcal{K}$ using Notation 2.3. □

Remark 2.6. Alternatively in Theorem 2.5, we can see that the matrices M and N have the same kernel because they are equivalent. We multiply the last row of M by -2 and then we successively add the rows $1,\dots,n+m$ to the last row; with these elementary row operations we transform M into N.

Example 2.7. Using Theorem 2.5, the universal Gröbner basis of the defining equations of the Rees algebra of the graph in Example 2.4 is given by

$$\{x_2y_2T_1 - x_1y_1T_2,\ x_2y_3T_1T_3 - x_1y_1T_2T_4,\ x_3T_2 - x_2T_3,\ x_3y_2T_1 - x_1y_1T_3,$$
$$x_3y_3T_1 - x_1y_1T_4,\ y_3T_3 - y_2T_4,\ x_3y_3T_2 - x_2y_2T_4\}.$$

It can also be checked in [12] using the command `universalGroebnerBasis`.

Corollary 2.8. *Let G be a bipartite graph and $I = I(G)$ be its edge ideal. The universal Gröbner basis $\mathcal{U}$ of $\mathcal{K}$ consists of square free binomials with degree at most linear in the variables x_i's and at most linear in the variables y_i's.*

3. Upper bound for the xy-regularity

In this section we get an upper bound for the xy-regularity of $\mathcal{R}(I)$, and the important point is that we will choose a special monomial order. Using the xy-regularity we can find an upper bound for the regularity of all the powers of the edge ideal I.

Since most of the upper bounds for the regularity of the powers of edge ideals are based on the technique of even-connection [3], then a strong motivation for this section is trying to give new tools for the challenging conjecture:

Conjecture 3.1 (Alilooee, Banerjee, Beyarslan and Hà)**.** Let G be an arbitrary graph then

$$\operatorname{reg}(I(G)^s) \leq 2s + \operatorname{reg}(I(G)) - 2$$

for all $s \geq 1$.

The following theorem will be crucial in our treatment.

Theorem 3.2. ([19, Theorem 5.3], [14, Proposition 10.1.6]) *The regularity of each power I^s is bounded by*

$$\operatorname{reg}(I^s) \leq 2s + \operatorname{reg}_{xy}(\mathcal{R}(I)).$$

By fixing a particular monomial order $<$ in S, then we can see the initial ideal $\operatorname{in}_<(\mathcal{K})$ as the special fibre of a flat family whose general fibre is $\mathcal{K}$ (see e.g. [14, Chapter 3] or [9, Chapter 15]), and we can get a bigraded version of [14, Theorem 3.3.4, (c)].

Theorem 3.3. *Let $<$ be a monomial order in S, then we have*

$$\operatorname{reg}_{xy}(\mathcal{R}(I)) \leq \operatorname{reg}_{xy}(S/\operatorname{in}_<(\mathcal{K})).$$

Let $\mathcal{M}$ be an arbitrary maximal matching in G with $|\mathcal{M}| = r$. We assume that the vertices of G are numbered in such a way that $\mathcal{M}$ consists of the edges

$$\mathcal{M} = \big\{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_r, y_r\}\big\},$$

and also we assume that $n = |X| \leq |Y| = m$.

In $R = \mathbb{K}[x_1 \ldots, x_n, y_1, \ldots, y_m]$ we consider the lexicographic monomial order induced by

$$x_n > \ldots > x_2 > x_1 > y_m > \ldots > y_2 > y_1.$$

We choose an arbitrary monomial order $<^{\#}$ on $\mathbb{K}[T_1, \ldots, T_q]$, then we define the following monomial order $<^{\mathcal{M}}$ on $S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m, T_1, \ldots, T_q]$: for two monomials $\mathbf{x}^{\alpha_1}\mathbf{y}^{\beta_1}\mathbf{T}^{\gamma_1}$ and $\mathbf{x}^{\alpha_2}\mathbf{y}^{\beta_2}\mathbf{T}^{\gamma_2}$ we have

$$\mathbf{x}^{\alpha_1}\mathbf{y}^{\beta_1}\mathbf{T}^{\gamma_1} <^{\mathcal{M}} \mathbf{x}^{\alpha_2}\mathbf{y}^{\beta_2}\mathbf{T}^{\gamma_2}$$

if either

(i) $\mathbf{x}^{\alpha_1}\mathbf{y}^{\beta_1} < \mathbf{x}^{\alpha_2}\mathbf{y}^{\beta_2}$ or

(ii) $\mathbf{x}^{\alpha_1}\mathbf{y}^{\beta_1} = \mathbf{x}^{\alpha_2}\mathbf{y}^{\beta_2}$ and $\mathbf{T}^{\gamma_1} <^{\#} \mathbf{T}^{\gamma_2}$.

Let $\mathcal{G}_{<^{\mathcal{M}}}(\mathcal{K})$ be the reduced Gröbner basis of $\mathcal{K}$ with respect to $<^{\mathcal{M}}$. The possible type of binomials inside $\mathcal{G}_{<^{\mathcal{M}}}(\mathcal{K})$ were described in Theorem 2.5, now we focus on obtaining a more refined information about the type (iii) in Notation 2.3.

Notation 3.4. In this section, for notational purposes (and without loss of generality) we shall assume that w_1 and w_2 are disjoint odd paths of the form

$$w_1 = (x_e, u_1, \ldots, u_{2a}, y_f),$$
$$w_2 = (x_g, v_1, \ldots, v_{2b}, y_h).$$

Then we analyze the binomial $x_e y_f T_{(w_1,w_2)^+} - x_g y_h T_{(w_1,w_2)^-}$.

Lemma 3.5. *Let $x_e y_f T_{(w_1,w_2)^+} - x_g y_h T_{(w_1,w_2)^-} \in \mathcal{G}_{<^{\mathcal{M}}}(\mathcal{K})$, then we have*

(i) at least one of the vertices x_e, y_f is in the matching $\mathcal{M}$, i.e. $e \le r$ or $f \le r$;

(ii) at least one of the vertices x_g, y_h is in the matching $\mathcal{M}$, i.e. $g \le r$ or $h \le r$.

Proof. (i) First, assume that $a = 0$, i.e. w_1 has length one. Since $\mathcal{M}$ is a maximal matching then we necessarily get that $e \le r$ or $f \le r$.

Now let $a > 0$, and by contradiction assume that $e > r$ and $f > r$. From the maximality of $\mathcal{M}$, we get that $u_1 = y_j$ where $j \le r$. We consider the even path

$$w_3 = (y_j, \ldots, u_{2a}, y_f),$$

then using Notation 2.3 we get the binomial

$$F = y_j T_{w_3^+} - y_f T_{w_3^-} \in \mathcal{K}.$$

We have $\mathrm{in}_{<^{\mathcal{M}}}(F) = y_f T_{w_3^-}$ because $f > j$. So we obtain that $\mathrm{in}_{<^{\mathcal{M}}}(F)$ divides $x_e y_f T_{(w_1,w_2)^+}$, and this contradicts that $\mathcal{G}_{<^{\mathcal{M}}}(\mathcal{K})$ is reduced.

(ii) Follows identically. $\square$

In the rest of this note we assume the following.

Notation 3.6. $b(G)$ represents the minimum cardinality of the maximal matchings of G and $\mathrm{match}(G)$ denotes the maximum cardinality of the matchings of G

Theorem 3.7. *Let G be a bipartite graph and $I = I(G)$ be its edge ideal. The xy-regularity of $\mathcal{R}(I)$ is bounded by*

$$\mathrm{reg}_{xy}(\mathcal{R}(I)) \le \min\left\{|X|-1,\ |Y|-1,\ 2b(G)-1\right\}.$$

Proof. From Theorem 3.3, it is enough to prove that

$$\operatorname{reg}_{xy}(S/\operatorname{in}_{<\mathcal{M}}(\mathcal{K})) \le \min\big\{|X|-1,\ |Y|-1,\ 2r-1\big\}.$$

Let $\{m_1,\ldots,m_c\}$ be the monomials obtained as the initial terms of the elements of $\mathcal{G}_{<\mathcal{M}}(\mathcal{K})$. We consider the Taylor resolution (see e.g. [14, Section 7.1])

$$0 \longrightarrow T_c \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow S/\operatorname{in}_{<\mathcal{M}}(\mathcal{K}) \longrightarrow 0,$$

where each T_i as a bigraded S-module has the structure

$$T_i = \bigoplus_{1\le j_1<\ldots<j_i\le c} S\big(-\deg_{xy}(\operatorname{lcm}(m_{j_1},\ldots,m_{j_i})), -\deg_T(\operatorname{lcm}(m_{j_1},\ldots,m_{j_i}))\big).$$

From it, we get the upper bound

$$\operatorname{reg}_{xy}(S/\operatorname{in}_{<\mathcal{M}}(\mathcal{K})) \le \max\big\{\deg_{xy}(\operatorname{lcm}(m_{j_1},\ldots,m_{j_i}))-i \mid \{j_1,\ldots,j_i\}\subset\{1,\ldots,c\}\big\}.$$

When $\deg_{xy}(m_{j_i}) \le 1$, then we have

$$\deg_{xy}(\operatorname{lcm}(m_{j_1},\ldots,m_{j_i}))-i \le \deg_{xy}(\operatorname{lcm}(m_{j_1},\ldots,m_{j_{i-1}}))-(i-1). \quad (5)$$

So, according with Theorem 2.5, we only need to consider subsets $\{j_1,\ldots,j_i\}$ such that for each $1\le k\le i$ we have $m_{j_k} = \operatorname{in}_{<\mathcal{M}}(F_k)$ and F_k is a binomial as in Notation 3.4. We use the notation $\operatorname{in}_{<\mathcal{M}}(F_k) = x_{e_k}y_{f_k}B_k$, where B_k is a monomial in the T_i's. Also, we can assume that $x_{e_1}y_{f_1}, x_{e_2}y_{f_2},\ldots,x_{e_k}y_{f_k}$ are pairwise relatively prime, because we can make a reduction like in Eq. 5 if this condition is not satisfied.

Thus, in order to finish the proof, we only need to show that we necessarily have $i\le\min\{|X|-1,\ |Y|-1,\ 2r-1\}$ under the two previous conditions. Since the two paths that define each F_k are disjoint, then by the monomial order chosen we have that $e_k>1$ for each k, and by a "pigeonhole" argument follows that $i\le|X|-1\le|Y|-1$. Also, from Lemma 3.5 there are at most $2r-1$ available positions to satisfy the condition of being co-primes. Thus we have $i\le 2r-1$, and the result of the theorem follows because $\mathcal{M}$ is an arbitrary maximal matching. □

Corollary 3.8. *Let G be a bipartite graph and $I=I(G)$ be its edge ideal. For all $s\ge 1$ we have*

$$\operatorname{reg}(I^s) \le 2s+\min\big\{|X|-1,\ |Y|-1,\ 2b(G)-1\big\}.$$

Proof. It follows from Theorem 3.7 and Theorem 3.2. □

Remark 3.9. From the fact that co-chord$(G) \leq$ match$(G) \leq \min\{|X|,|Y|\}$ (see [17]) and match$(G) \leq 2b(G)$ (see [15, Proposition 2.1]), then we have the following relations

$$\text{co-chord}(G)-1 \leq \text{match}(G)-1 \leq \min\left\{|X|-1,\ |Y|-1,\ 2b(G)-1\right\}.$$

Although the last upper bound is weaker, it is interesting that an approach based on Gröbner bases can give a sharp answer in several cases.

In the last part of this section we deal with the case of a complete bipartite graph. The Rees algebra of these graphs was studied in [26].

Notation 3.10. By G we will denote a complete bipartite graph with bipartition $X=\{x_1,\ldots,x_n\}$ and $Y=\{y_1,\ldots,y_m\}$. Let $I=\{x_iy_j \mid 1\leq i\leq n, 1\leq j\leq m\}$ be the edge ideal of G and let T_{ij} be the variable that corresponds to the edge x_iy_j. Thus we have a canonical map

$$\begin{gathered} S=\mathbb{K}[x_i\text{'s},\ y_j\text{'s},\ T_{ij}\text{'s}] \xrightarrow{\psi} \mathcal{R}(I)\subset R[t], \\ \psi(x_i)=x_i, \quad \psi(y_i)=y_i, \quad \psi(T_{ij})=x_iy_jt. \end{gathered} \tag{6}$$

Let $\mathcal{K}$ be the kernel of this map. For simplicity of notation we keep the same monomial order $<^{\mathcal{M}}$.

Exploiting our characterization of the universal Gröbner basis of $\mathcal{K}$, we shall prove that all the powers of the edge ideal of G have a linear free resolution.

Lemma 3.11. *Let G be a complete bipartite graph. The reduced Gröbner basis $\mathcal{G}_{<\mathcal{M}}(\mathcal{K})$ consists of binomials with linear xy-degree.*

Proof. From Theorem 2.5 we only need to show that any binomial determined by two disjoint odd paths is not contained in $\mathcal{G}_{<\mathcal{M}}(\mathcal{K})$. Let $x_ey_fT_{(w_1,w_2)^+} - x_gy_hT_{(w_1,w_2)^-}$ be a binomial like in Notation 3.4. By contradiction assume that $x_ey_fT_{(w_1,w_2)^+} - x_gy_hT_{(w_1,w_2)^-} \in \mathcal{G}_{<\mathcal{M}}(\mathcal{K})$.

Without loss of generality we assume that $e>g$. Since G is complete bipartite, we choose the edge x_ey_h and we append it to w_2, that is

$$w_3=(x_g,v_1,\ldots,v_{2b},y_h,x_e).$$

Using Notation 2.3 we get the binomial

$$F=x_gT_{w_3^+}-x_eT_{w_3^-}\in\mathcal{K},$$

with initial term $\text{in}_{<\mathcal{M}}(F)=x_eT_{w_3^-}$ because $e>g$. Thus we get that $\text{in}_{<\mathcal{M}}(F)$ divides $x_ey_fT_{(w_1,w_2)^+}$, a contradiction. $\square$

Corollary 3.12. *Let G be a complete bipartite graph and $I = I(G)$ be its edge ideal. For all $s \geq 1$ we have* $\operatorname{reg}(I^s) = 2s$.

Proof. Using Lemma 3.11 and repeating the same argument of Theorem 3.7 we can get $\operatorname{reg}_{xy}(\mathcal{R}(I)) = 0$. Again, the result follows by Theorem 3.2. $\square$

We remark that this previous result also follows from [17] since it is easy to check that co-chord$(G) = 1$ (i.e. it is a co-chordal graph) in the case of complete bipartite graphs.

4. The total regularity of $\mathcal{R}(I)$

In the previous sections we heavily exploited the fact that the matrix M (corresponding to $\mathcal{R}(I)$) is totally unimodular in the case of a bipartite graph G. From [11, Theorem 2.1] we have that $\mathcal{R}(I)$ is a normal domain, then a famous theorem by Hochster [16] (see e.g. [6, Theorem 6.10] or [7, Theorem 6.3.5]) implies that $\mathcal{R}(I)$ is Cohen-Macaulay. So, the Rees algebra $\mathcal{R}(I)$ of a bipartite graph G is also special from a more algebraic point of view (see [22]).

For notational purposes we let N be $N = n + m$. It is well known that the canonical module of S (with respect to our bigrading) is given by $S(-N, -q)$ (see e.g. [6, Proposition 6.26], or [7, Example 3.6.10] in the $\mathbb{Z}$-graded case). The Rees cone is the polyhedral cone of $\mathbb{R}^{N+1}$ generated by the set of vectors

$$\mathcal{A} = \big\{ v \mid v \text{ is a column of } M \text{ in Eq. } 4 \big\},$$

and we will denote it by $\mathbb{R}_+\mathcal{A}$. The irreducible representation of the Rees cone for a bipartite graph was given in [11, Section 4].

Proposition 4.1. *Adopt Notation 2.1. The following statements hold:*

(i) The Krull dimension of $\mathcal{R}(I)$ is $\dim(\mathcal{R}(I)) = N + 1$.

(ii) The projective dimension of $\mathcal{R}(I)$ as an S-module is equal to the number of edges minus one, that is, $p = \operatorname{pd}_S(\mathcal{R}(I)) = q - 1$.

(iii) The canonical module of $\mathcal{R}(I)$ is given by

$$\omega_{\mathcal{R}(I)} = {}^*\operatorname{Ext}_S^p\big(\mathcal{R}(I),\, S(-N, -q)\big).$$

(iv) The bigraded Betti numbers of $\mathcal{R}(I)$ and $\omega_{\mathcal{R}(I)}$ are related by

$$\beta^S_{i,(a,b)}(\mathcal{R}(I)) = \beta^S_{p-i,(N-a,q-b)}(\omega_{\mathcal{R}(I)}).$$

Proof. (i) The Rees cone $\mathbb{R}_+\mathcal{A}$ has dimension $N+1$ and the Krull dimension of $\mathcal{R}(I)$ is equal to it (see e.g. [23, Lemma 4.2]). More generally, it also follows from [21, Proposition 2.2].

Since clearly $\mathcal{R}(I)$ is a finitely generated S-module, then the statements (ii) and (iii) follow from [6, Theorem 6.28] (see [7, Proposition 3.6.12] for the $\mathbb{Z}$-graded case).

The statement (iv) follows from [6, Theorem 6.18]; also, see [6, page 224, equation 6.6]. □

Due to a formula of Danilov and Stanley (see e.g. [6, Theorem 6.31] or [7, Theorem 6.3.5]), the canonical module of $\mathcal{R}(I)$ is the ideal given by

$$\omega_{\mathcal{R}(I)} = \left(\{x_1^{a_1}\cdots x_n^{a_n}y_1^{a_{n+1}}\cdots y_N^{a_N}t^{a_{N+1}} \mid a=(a_i)\in(\mathbb{R}_+\mathcal{A})^\circ\cap\mathbb{Z}^{N+1}\}\right),$$

where $(\mathbb{R}_+\mathcal{A})^\circ$ denotes the topological interior of $\mathbb{R}_+\mathcal{A}$.

Now we can compute the total regularity of $\mathcal{R}(I)$.

Theorem 4.2. *Let G be a bipartite graph and $I=I(G)$ be its edge ideal. The total regularity of $\mathcal{R}(I)$ is given by*

$$\operatorname{reg}(\mathcal{R}(I)) = \operatorname{match}(G).$$

Proof. In the case of the total regularity, we can see $\mathcal{R}(I)$ as a standard graded S-module (i.e. $\deg(x_i)=\deg(y_i)=\deg(T_i)=1$), and since $\mathcal{R}(I)$ is a Cohen-Macaulay S-module then the regularity can be computed with the last Betti numbers (see e.g. [20, page 283] or [9, Exercise 20.19]). Thus, from Proposition 4.1 we get

$$\begin{aligned}
\operatorname{reg}(\mathcal{R}(I)) &= \max\left\{a+b-p \mid \beta^S_{p,(a,b)}(\mathcal{R}(I))\neq 0\right\}\\
&= \max\left\{a+b-p \mid \beta^S_{0,(N-a,q-b)}(\omega_{\mathcal{R}(I)})\neq 0\right\}\\
&= N+1-\min\left\{a+b \mid \beta^S_{0,(a,b)}(\omega_{\mathcal{R}(I)})\neq 0\right\},
\end{aligned}$$

and by the bigrading that we are using ($\operatorname{bideg}(x_i)=\operatorname{bideg}(y_i)=(1,0)$ and $\operatorname{bideg}(t)=(-2,1)$) then we obtain

$$\operatorname{reg}(\mathcal{R}(I)) = N+1-\min\left\{a_1+\cdots+a_N-a_{N+1} \mid a=(a_i)\in(\mathbb{R}_+\mathcal{A})^\circ\cap\mathbb{Z}^{N+1}\right\}.$$

One can check that the number

$$-\min\left\{a_1+\cdots+a_N-a_{N+1} \mid a=(a_i)\in(\mathbb{R}_+\mathcal{A})^\circ\cap\mathbb{Z}^{N+1}\right\}$$

coincides with the a-invariant of $\mathcal{R}(I)$ with respect to the $\mathbb{Z}$-grading induced by $\deg(x_i)=\deg(y_i)=1$ and $\deg(t)=-1$. This last formula can be evaluated with the irreducible representation of the Rees cone [11, Corollary 4.3], it was done in [11, Proposition 4.5], and from it we get

$$\operatorname{reg}(\mathcal{R}(I)) = N - \beta_0,$$

where β_0 denotes the maximal size of an independent set of G. The minimal size of a vertex cover is equal to $N - \beta_0$, and we finally get

$$\operatorname{reg}(\mathcal{R}(I)) = \operatorname{match}(G)$$

from König's theorem. $\square$

The following bound was obtained for the first power of the edge ideal in [13, Theorem 6.7].

Corollary 4.3. *Let G be a bipartite graph and $I = I(G)$ be its edge ideal. For all $s \geq 1$ we have*

$$\operatorname{reg}(I^s) \leq 2s + \operatorname{match}(G) - 1.$$

Proof. It is enough to prove that $\operatorname{reg}_{xy}(\mathcal{R}(I)) \leq \operatorname{reg}(\mathcal{R}(I)) - 1$. In the minimal bigraded free resolution Eq. 2 of $\mathcal{R}(I)$, suppose that $\operatorname{reg}_{xy}(\mathcal{R}) = a_{ij} - i$ for some $i, j \in \mathbb{N}$. Since necessarily $b_{ij} \geq 1$ and

$$a_{ij} + b_{ij} - i \leq \operatorname{reg}(\mathcal{R}(I)),$$

then we get the expected inequality. $\square$

This previous upper bound is sharp in some cases (see [5, Lemma 4.4]). In the following corollary we get information about the eventual linearity.

Corollary 4.4. *Let G be a bipartite graph and $I = I(G)$ be its edge ideal. For all $s \geq \operatorname{match}(G) + q + 1$ we have*

$$\operatorname{reg}(I^{s+1}) = \operatorname{reg}(I^s) + 2.$$

Proof. With the same argument of Corollary 4.3 we can prove that $\operatorname{reg}_T(\mathcal{R}(I)) \leq \operatorname{reg}(\mathcal{R}(I))$, here the difference is that in the minimal bigraded free resolution Eq. 2 we can have free modules of the type $S(0, -b_{ij})$ (for instance, in the syzygies of $\mathcal{R}(I)$ the ones that come from even cycles). Then the statement of the corollary follows from [8, Proposition 3.7]. $\square$

5. Some final thoughts

In the last part of this note we give some ideas and digressions about Conjecture 3.1. Using a "refined Rees approach" with respect to the one of this note, one might get an answer to this conjecture for general graphs or perhaps for special families of graphs:

- Restricting the minimal bigraded free resolution Eq. 2 of $\mathcal{R}(I)$ to a graded T-part gives an exact sequence

$$0 \longrightarrow (F_p)_{(*,k)} \longrightarrow \cdots \longrightarrow (F_1)_{(*,k)} \longrightarrow (F_0)_{(*,k)} \longrightarrow (\mathcal{R}(I))_{(*,k)} \longrightarrow 0$$

for all k. This gives a (possibly non-minimal) graded free R-resolution of

$$(\mathcal{R}(I))_{(*,k)} \cong I^k(2k).$$

But in the case $k = 1$ one can check that

$$0 \longrightarrow (F_p)_{(*,1)} \longrightarrow \cdots \longrightarrow (F_1)_{(*,1)} \longrightarrow (F_0)_{(*,1)} \longrightarrow I(2) \longrightarrow 0$$

is indeed the minimal free resolution of $I(2)$. Thus, one can read the regularity I from Eq. 2, and a solution to Conjecture 3.1 can be given by proving that

$$\max_{i,j} \{a_{ij} - i\} = \max_{i,j} \{a_{ij} - i \mid b_{ij} = 1\}.$$

- For bipartite graphs, Gröbner bases techniques can give very good results (for instance, in the case of complete bipartite graphs). Perhaps, for special families of bipartite graphs one can give "good" monomial orders.

- The existence of a canonical module in the case of bipartite graphs could give more information about the minimal bigraded free resolution of $\mathcal{R}(I)$. From [6, Theorem 7.26] we have that the maximal xy-degree and the maximal T-degree on each F_i of Eq. 2 form weakly increasing sequences of integers, that is

$$\max_j \{a_{ij}\} \leq \max_j \{a_{i+1,j}\} \quad \text{and} \quad \max_j \{b_{ij}\} \leq \max_j \{b_{i+1,j}\}$$

(see e.g. [9, Exercise 20.19] for the $\mathbb{Z}$-graded case). Thus a more detailed analysis of the polyhedral geometry of the Rees cone $\mathbb{R}_+\mathcal{A}$ could give better results.

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