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EXTENDING DIVISORS FROM COMPLETE INTERSECTION SURFACES

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The aim of this paper is to extend a theorem of Griffiths, Harris and Hulek [4], [6] on the extendability of divisors of a smooth complete intersection surface in $\mathbb{P}^n$ to the case when the ambient space is a product of projective spaces or a Grassmannian. The proofs generalize the proof of the result of Griffiths-Harris-Hulek given by Ellingsrud-Gruson-Peskine-Strømme in [3].

Introduction.

A well known theorem due to Griffiths, Harris and Hulek (see [4], [6]) asserts that if X is a complete intersection surface in $\mathbb{P}^n$, a smooth curve $Y \subset X$ is the ideal theoretic intersection of a hypersurface $H \subset \mathbb{P}^n$ if and only if the exact sequence of normal bundles

$$0 \to N_{Y/X} \to N_{Y/\mathbb{P}^n} \to N_{X/\mathbb{P}^n}|_Y \to 0$$

splits.

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A new proof of this result was given in [3]. The aim of this paper is to prove similar results when X is a complete intersection surface in a product of projective spaces or in a Grassmannian, adapting the ideas and methods of [3] to the new situations. In this sense, the paper [2] helps to clarify ideas toward some possible generalizations (cf. also [1], Chapter 8).

The main results of this paper are the following:

Theorem 01. *Let $X \subset P = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$ be a smooth surface that is complete intersection of ample hypersurfaces $D_1, \ldots, D_{k-2}$ of P, with $k = n_1 + \cdots + n_r$ and $D_i \in |\mathcal{O}_P(b_1^i, \ldots, b_r^i)|$, $i = 1, \ldots, k-2$. Let $Y \subset X$ be a smooth curve. Suppose that the canonical exact sequence of vector bundles*

$$0 \to N_{Y/X} \to N_{Y/P} \to N_{X/P}|_Y \to 0 \tag{1}$$

splits. Then there exist $a_1, \ldots, a_r \in \mathbb{Z}$ such that $\mathcal{O}_X(Y) \simeq \mathcal{O}_X(a_1, \ldots, a_r)$. In particular, Y is linearly equivalent to with an effective divisor on X which is a scheme-theoretic intersection of X with an ample hypersurface of P. Moreover, Y is itself a scheme-theoretic intersection of X with an ample hypersurface of P if one of the following conditions is satisfied:

i) *for all $j = 1, \ldots, r$, $a_j \geq b_j^1 + \ldots + b_j^{k-2} - n_j$, or*
ii) *for all $j = 1, \ldots, r$, $i = 1, \ldots, k-2$, $a_j \leq b_j^i$.*

Theorem 02. *Let X be a complete intersection surface in the grassmannian $\mathbb{G}(k, n)$ of k-planes in $\mathbb{P}^n$. Let $Y \subset X$ be a smooth Cartier divisor on X. Then Y is the scheme-theoretic complete intersection of X and a hypersurface H on $\mathbb{G}(k, n)$ if and only if the short exact sequence*

$$0 \to N_{Y/X} \to N_{Y/\mathbb{G}(k,n)} \to N_{X/\mathbb{G}(k,n)}|_Y \to 0$$

splits.

The proofs of these two theorems generalize the proof of the result of Griffiths-Harris-Hulek given by Ellingsrud-Gruson-Peskine-Strømme in [3]. This paper is organized as follows. After some preliminaries in Section 1, in Section 2 we prove Theorem 01 and in Section 3, Theorem 02.

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1. Notation and preliminaries.

For the rest of the paper, P will be an ambient variety (to be specified), X a complete intersection of ample divisors in P and Y a smooth curve in X. We will denote by $X(1)$ to the first infinitesimal neighbourhood of X in P. We will use as usual the notation Ω_Z for the cotangent sheaf of any scheme Z. The normal bundle of A in B will be written $N_{A/B}$ and $\mathrm{Pic}(A)$ and $\mathrm{Num}(A)$ will be for the Picard group and numerical class divisor group of A (respectively). We will write also H for the hyperplane section class.

The terminology and notation used are standard, unless otherwise specified. When P is a product of projective spaces, we will note p_i to the i-th canonical projection, $\mathcal{O}_P(0, ..., 0, 1, 0, ..., 0)$ (with 1 on the i-th place) to the pullback by p_i of $\mathcal{O}(1)$ and H_i for the divisor class associated to this line bundle. In the case of grassmannians, we will write Q and S for the universal quotient bundle and the universal subbundle respectively.

We finish this section by recalling a result from [3] that will be very useful in the two cases we consider.

Lemma 1.1. ([3], cf. also [1], Lemma 8.1). *Let $Y \subset X \subset P$ be three smooth irreducible varieties, with Y (resp. X) closed in X (resp. in P). The canonical exact sequence of normal bundles*

$$0 \to N_{Y/X} \to N_{Y/P} \to N_{X/P}|_Y \to 0$$

splits if and only if there exist an effective Cartier divisor $\mathcal{Y} \subset X(1)$ such that $Y = \mathcal{Y} \cap X$ (scheme-theoretically).

The main idea to prove Theorems 0.1 and 0.2 is the following. First we use Lemma 1.1 to prove that (when the normal sequence splits) the composition of natural maps $\mathrm{Pic}X(1) \to \mathrm{Pic}X \to \mathrm{Num}X$ has the same image as the natural restriction $\mathrm{Num}P \to \mathrm{Num}X$ (where $\mathrm{Num}Z$, as we said, is the quotient of $\mathrm{Pic}Z$ by the numerical equvalence relationship). Then we use the simply connectedness of the surface X and Lefschetz Hyperplane Theorem to get that $\mathcal{O}_X(Y)$ is the restriction of a line bundle E on P. We then prove that the morphism $H^0(P, E) \to H^0(X, \mathcal{O}_X(Y))$ is surjective to conclude the proof.

2. Surfaces in a product of projective spaces.

This section is devoted to prove Theorem 0.1, so from now on, X will be a smooth scheme-theoretic complete intersection surface of ample divisors of $P := \mathbb{P}^{n_1} \times ... \times \mathbb{P}^{n_r}$. We start with the following technical result.

Lemma 2.1. *Let X and P be as above. Then $h^1(X, \Omega^1_P|_X) = r$.*

Proof. First of all, we consider the decomposition

$$\Omega_P|_X \simeq p_1^*(\Omega_{\mathbb{P}^{n_1}})|_X \oplus \ldots \oplus p_r^*(\Omega_{\mathbb{P}^{n_r}})|_X,$$

where $p_1, ..., p_r$ are the canonical projections. Then we have the following exact sequence as the sum of the pullbacks of the tautological exact sequence:

$$0 \to \Omega_{P|X} = p_1^*(\Omega_{\mathbb{P}^{n_1}})|_X \oplus \ldots \oplus p_r^*(\Omega_{\mathbb{P}^{n_r}})|_X \to$$

$$\to \mathcal{O}_X^{\oplus(n_1+1)}(-1, 0, ..., 0) \oplus \ldots \oplus \mathcal{O}_X^{\oplus(n_r+1)}(0, ..., 0, -1) \to \mathcal{O}_X^{\oplus r} \to 0.$$

This leads to the exact sequence

$$H^0(X, \mathcal{O}_X^{\oplus(n_1+1)}(-1, 0, ..., 0) \oplus \ldots \oplus \mathcal{O}_X^{\oplus(n_r+1)}(0, ..., 0, -1)) \to$$

$$\to H^0(X, \mathcal{O}_X^{\oplus r}) \xrightarrow{\phi}$$

$$\xrightarrow{\phi} H^1(X, \Omega_P|X) \to H^1(X, \mathcal{O}_X^{\oplus(n_1+1)}(-1, 0, ..., 0) \oplus \ldots \oplus \mathcal{O}_X^{\oplus(n_r+1)}$$

$$(0, ..., 0, -1)) \tag{2}$$

Now, since X is the complete intersection of ample divisors and all the $n_i \geq 2$, we have that all projections have general fibre of dimension zero.

This means that all bundles $\mathcal{O}_X(0, ..., 0, 1, 0, ..., 0)$ are at least 1-ample and then (by a vanishing theorem of Sommese)

$$H^0(X, \mathcal{O}_X^{\oplus(n_1+1)}(-1, 0, ..., 0) \oplus \ldots \oplus \mathcal{O}_X^{\oplus(n_r+1)}(0, ..., 0, -1)) = 0,$$

so ϕ is injective. Now we decompose ϕ as the product of all $\phi_i \colon H^0(X, \mathcal{O}_X) \to H^1(X, p_i^*(\Omega_{\mathbb{P}^{n_i}})_{|X})$.

To show that ϕ is onto, we have to show that all $H^1(\mathcal{O}_X(0, ..., 0, -1, 0, ..., 0))$ vanish. So let $k = n_1 + ... + n_r$ and consider $P = X_k \supset X_{k-1} \supset ... \supset X_0 = X$ such that X_{i-1} is an ample divisor of X_i. Set $E = \mathcal{O}_P(0, ..., 0, 1, 0, ..., 0)$. Since $E_{|X_i} := \mathcal{O}_{X_i}(0, ..., 0, 1, 0, ..., 0)$ is nef, we get that $E_{|X_i} \otimes \mathcal{O}_{X_i}(X_{i-1})$ is ample in X_i. Now we consider the short exact sequence

$$0 \to E^\vee_{|X_i} \otimes \mathcal{O}_{X_i}(-X_{i-1}) \to E^\vee_{|X_i} \to E^\vee_{|X_{i-1}} \to 0$$

which leads to

$$H^1(E^\vee_{|X_i} \otimes \mathcal{O}_{X_i}(-X_{i-1})) \to H^1(E^\vee_{|X_i}) \to H^1(E^\vee_{|X_{i-1}}) \to$$

$$\to H^2(E^\vee_{|X_i} \otimes \mathcal{O}_{X_i}(-X_{i-1}))$$

The left and the right vector spaces are zero because of the following claim:

- $H^j(X_i, E^\vee(a_1, ..., a_r)) = 0$ for all $a_1, ..., a_r < 0$, $j = 1, ..., i-1$, $i = 1..., k$.

We prove this claim by induction. It is clear by Kodaira vanishing that

$$H^j(P, E^\vee(a_1, ..., a_r)) = 0 \quad \text{for all} \quad a_1, ..., a_r < 0 \quad \text{and} \quad j = 1, ..., k-1.$$

So now let us suppose we have this statement for X_i. Let $a_1, ..., a_r$ be strictly negative integers. Since X_{i-1} is an ample divisor, by Lefschetz hyperplane theorem we have that there exist strictly positive integers $b_1, ..., b_r$ such that $\mathcal{O}_{X_i}(X_{i-1}) \simeq \mathcal{O}_{X_i}(b_1, ..., b_r)$. Then the result for X_{i-1} follows by induction using the cohomology exact sequence associated to the following exact sequence:

$$0 \to E^\vee_{X_i}(a_1 - b_1, ..., a_r - b_r) \to E^\vee_{X_i}(a_1, ..., a_r) \to E^\vee_{X_{i-1}}(a_1, ..., a_r) \to 0.$$

Therefore

$$H^1(\mathcal{O}_X(0, ..., 0, -1, 0, ..., 0)) \simeq H^1(\mathcal{O}_P(0, ..., 0, -1, 0, ..., 0)).$$

On the other side, $\mathcal{O}_P(0, ..., 0, -1, 0, ..., 0)$ can be seen as $K_P \otimes \mathcal{O}(n_1 + 1, ..., n_{i-1} + 1, n_i, n_{i+1} + 1, ..., n_r + 1)$, where K_P is the canonical bundle of P, so by Kodaira vanishing theorem again we get

$$H^1(\mathcal{O}_X(0, ..., 0, -1, 0, ..., 0)) = H^1(\mathcal{O}_P(0, ..., 0, -1, 0, ..., 0)) = 0. \quad \square$$

The following result is crucial in the proof of Theorem 0.1 (which in fact is the main technical novelty in [3]):

Lemma 2.2. *Let X and P be as above. Let $X(1)$ be the first infinitesimal neighbourhood of X in P. Then*

$$\Im(\mathrm{Pic}(X(1)) \to \mathrm{Pic}(X)) = \Im(\mathrm{Pic}(P) \to \mathrm{Pic}(X)) = \mathbb{Z}^r.$$

Proof. The inclusion

$$\Im((\mathrm{Pic}(P) \to \mathrm{Pic}(X)) \subseteq \Im(\mathrm{Pic}(X(1)) \to \mathrm{Pic}(X))$$

is trivial. For the reverse inclusion we shall adapt the proof of Lemma 2 of [3] (cf. also the proof of [1, Lemma 8.3]) in our situation. Since X is a complete intersection surface in P, by Lefschetz theorem the restriction map $\mathrm{Pic}(P) = \mathbb{Z}^r \to \mathrm{Pic}(X)$ is injective with torsion free cokernel. Moreover, X is a simply connected surface. In particular, the linear equivalence and the numerical equivalence of divisors on X coincide. Let $\alpha = \mathrm{dlog}_X : \mathrm{Pic}(X) \to H^1(X, \Omega_X)$ be the logarithmic derivative map defined in the following way. Every $L \in \mathrm{Pic}(X) \simeq H^1(\mathcal{O}_X^*)$, can be considered, via Cech cohomology, as $\{\psi_{ij}\}_{i,j}$ where $\psi_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$. Then $L \mapsto \left\{\frac{d\psi_{ij}}{\psi_{ij}}\right\}_{i,j}$. X is simply connected, the map α is injective. Taking into account that X is a smooth projective surface, from [7, Exercise IV.1.8] it follows that the non-degenerate intersection pairing on $\mathrm{Pic}(X) = \mathrm{Num}(X)$ (where $\mathrm{Num}(X)$, is the quotient of $\mathrm{Pic}(X)$ modulo the numerical equivalence) is compatible via the injective map α with a non-degenerate pairing on $H^1(X, \Omega_X)$ coming from Serre duality ($H^1(X, \Omega_X)$ is self-dual). This easily implies that the map $\alpha_{\mathbb{C}} : \mathrm{Pic}(X) \otimes \mathbb{C} \to H^1(X, \Omega_X)$ induced by α is injective.

Now it is a simple fact (see [3]) that the canonical map $\Omega_P|_X \to \Omega_{X(1)}|_X$ is an isomorphism. This follows from the exact sequence

$$\mathcal{I}_X^2/\mathcal{I}_X^4 \to \Omega_P|_{X(1)} \to \Omega_{X(1)} \to 0,$$

in which the first map becomes zero when restricted to X.

Then we get the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Pic}(X(1)) & \xrightarrow{\mathrm{dlog}_{X(1)}} H^1(X(1), \Omega_{X(1)}) \to H^1(X, \Omega_{X(1)}|_X) \simeq & & & H^1(X, \Omega_P|_X) \\ \downarrow & & & & \downarrow \\ \mathrm{Pic}(X) & \xrightarrow{\beta} & \mathrm{Pic}(X) \otimes \mathbb{C} & \xrightarrow{\alpha_{\mathbb{C}}} & H^1(X, \Omega_X) \end{array}$$

in which β is the canonical injective map. Now, from Lemma 2.1 we have that $h^1(X, \Omega_P|_X) = r$. Since the maps $\alpha_{\mathbb{C}}$ and β are injective, it follows that the image $\Im(\mathrm{Pic}(X(1)) \to \mathrm{Num}(X))$ is isomorphic to the image of the composition $\mathrm{Pic}(X(1)) \to H^1(X, \Omega_X)$, which is contained in the image of the right vertical map. Since by theorem of Néron-Severi, $\mathrm{Num}(X) = \mathrm{Pic}(X)$ is a finitely generated group, it follows that $\Im(\mathrm{Pic}(X(1)) \to H^1(X, \Omega_X))$ is a finitely generated subgroup of rank $\leq r$. On the other hand, by Lefschetz theorem the classes of $\mathcal{O}_X(1, 0, ..., 0)$,..., $\mathcal{O}_X(0, ..., 0, 1)$ are linearly independent in $\mathrm{Num}(X)$, the rank of $\Im(\mathrm{Pic}(X(1)) \to \mathrm{Num}(X))$ is $\geq r$, whence this rank is precisely r. Moreover, since $\mathrm{Num}(X)$ is a free abelian group of finite rank, it follows that $\Im(\mathrm{Pic}(X(1)) \to \mathrm{Num}(X))$ is a free abelian group of rank r, which, together with the fact that coker($\mathrm{Pic}(P) \to \mathrm{Pic}(X)$) has no torsion (by Lefschetz's theorem), concludes the proof. □

Lemma 2.3. *Let X and P as above and let k be the sum $n_1 + ... + n_r$. Let $D_1, ..., D_{k-2}$ be the hypersurfaces whose ideal-theoretic intersection is X and $(b_1^i, ..., b_r^i)$ the multidegree of D_i. Let $a_1, ..., a_r \in \mathbb{Z}$ be integers satisfying one of the following conditions:*

i) *for all* $j = 1, ..., r,\ a_j \geq b_j^1 + ... + b_j^{k-2} - n_j$,

ii) *for all* $j = 1, ..., r,\ i = 1, ..., k-2,\ a_j \leq b_j^i$.

Then the restriction map $H^0(P, \mathcal{O}_P(a_1, ..., a_r)) \to H^0(X, \mathcal{O}_X(a_1, ..., a_r))$ is surjective.

Proof. Let $X_i := D_1 \cap ... \cap D_{k-i}$ (as in the proof of Lemma 2.1) with $X_k = P$. Then we have to prove that all the restrictions

$$H^0(X_i, \mathcal{O}_{X_i}(a_1, ..., a_r)) \to H^0(X_{i-1}, \mathcal{O}_{X_{i-1}}(a_1, ..., a_r))$$

are surjective.

To prove this consider the exact sequence

$$0 \to \mathcal{O}_{X_i}(a_1 - b_1^{k-i+1}, ..., a_r - b_r^{k-i+1}) \to \mathcal{O}_{X_i}(a_1, ..., a_r) \to$$

$$\to \mathcal{O}_{X_{i-1}}(a_1, ..., a_r) \to 0$$

I is easy to check that the first sheaf is either $\mathcal{O}(-K_{X_i})$ twisted with an ample line bundle or the dual of an ample line bundle. Therefore, by Kodaira vanishing theorem we get that $H^1(X_i, \mathcal{O}_{X_i}(a_1 - b_1^{k-i+1}, ..., a_r - b_r^{k-i+1})) = 0$, which finishes the proof. □

After all this preparation we can now prove Theorem 0.1:

Proof of Theorem 9.1. For the nontrivial implication assume that the exact sequence (1) splits. Since X is the smooth complete intersection of ample divisors of dimension 2 in P and since P is simply connected, X is also simply connected by Lefschetz theorem. Therefore $\mathrm{Num}(X) = \mathrm{Pic}(X)$. By Lemma 1.1, there exists an effective Cartier divisor $\mathcal{Y}$ on $X(1)$ such that, scheme theoretically, $Y = \mathcal{Y} \cap X$; in particular, the isomorphism class of $[\mathcal{O}_X(Y)]$ lies in the image of the restriction map $\mathrm{Pic}(X(1)) \to \mathrm{Pic}(X)$. Therefore by Lemma 2.2, there are integers $a'_1, ..., a'_r$ and a positive integer m such that $\mathcal{O}_X(mY) \cong O_X(a'_1, ..., a'_r)$. On the other hand, by Lefschetz theorem again the cokernel of the restriction map $\mathrm{Pic}(P) \to \mathrm{Pic}(X)$ is torsion-free. This implies that m divides a'_i, $i = 1, ..., r$, and if we set $a'_i = ma_i$, $i = 1, ..., r$, then it follows that $\mathcal{O}_X(Y) \cong \mathcal{O}_X(a_1, ..., a_r)$, with $a_1, ..., a_n$ non-negative integers, which gives the first part of the theorem.

Then the proof can be concluded as follows. Let $s \in H^0(X, \mathcal{O}_X(Y)) = H^0(X, \mathcal{O}_X(a_1, ..., a_n))$. By Lemma 2.3, the section s can be lifted to a section s' of $H^0(P, \mathcal{O}_P(a_1, ..., a_r))$. Setting $H = div_P(s')$ we get a hypersurface H of P such that $X \cap H = Y$ (scheme-theoretically). □

3. Surfaces in Grassmannians.

In this section we study the extendability of divisors of complete intersection surfaces in grassmannians, just applying the method in [3] or in former section.

So let this time $P = \mathbb{G}(k, n)$, where $0 \leq k \leq \frac{n}{2}$, be the grassmannian of k-planes in $\mathbb{P}^n$ and let X be a scheme-theoretic complete intersection surface in P. We begin as before with a well known result in cohomology:

Lemma 3.1. *Let $\mathbb{G}(k, n)$ be the grassmannian of k-planes in $\mathbb{P}^n$. Then:*

i) $h^i(\mathbb{G}(k, n), \Omega^1_{\mathbb{G}(k,n)}(t)) = 0$ *for all* $i = 2, ..., (k+1)(n-k)-1$, $t \in \mathbb{Z}$.

ii) $h^1(\mathbb{G}(k,n), \Omega^1_{\mathbb{G}(k,n)}(t)) = 0$ *for all* $t < 0$.

iii) $h^1(\mathbb{G}(k,n), \Omega^1_{\mathbb{G}(k,n)}) = 1$.

Proof. The cohomology of these bundles can be easily computed using the classic Littlewood-Richardson rule, which can be found in [8]. □

Lemma 3.2. *Let X and P be as above. Then* $h^1(X, \Omega^1_P|X) = 1$.

Proof. Consider a chain $X = X_2 \subset \ldots \subset X_{(k+1)(n-k)} = P$ such that X_i is a divisor in X_{i+1} and $\mathcal{O}_{X_{i+1}}(X_i) \simeq \mathcal{O}_{X_{i+1}}(b_i)$ for some possitive integer b_i. Now we claim the following:

- $h^j(X_i, \Omega^1_P|_{X_i}(t)) = 0$ for all $i = 3, \ldots, (k+1)(n-k)$, $j = 2, \ldots, i-1$, $t < 0$.
- $h^1(X_i, \Omega^1_P|_{X_i}(t)) = 0$ for all $i = 3, \ldots, (k+1)(n-k)$, $t < 0$.
- $h^1(X_i, \Omega^1_P|_{X_i}) = 1$ for all $i = 3, \ldots, (k+1)(n-k)$.

We prove all three properties together by induction on i. While for $i = (k+1)(n-k)$ it is Lemma 3.1, for the case "i implies $i-1$" we consider the cohomology sequence associated to the short exact sequence

$$0 \to \Omega_P|_{X_i}(t - b_{i-1}) \to \Omega_P|_{X_i}(t) \to \Omega_P|_{X_{i-1}}(t) \to 0.$$

Let us take $j \in \{1, \ldots, i-2\}$, $t \leq 0$ and consider

$$\ldots \to H^j(X_i, \Omega_P|_{X_i}(t - b_{i-1})) \to H^j(X_i, \Omega_P|_{X_i}(t)) \to$$

$$\to H^j(X_{i-1}, \Omega_P|_{X_{i-1}}(t)) \to H^{j+1}(X_i, \Omega_P|_{X_i}(t - b_{i-1})) \to \ldots \tag{3}$$

Here the first and last groups are zero because $t - b_{i-1} < t \leq 0$ and by induction hypothesis. Then, since the second group in 3 satisfies the claim by induction hypothesis (it is $\{0\}$ or $\mathbb{C}$ depending on t and j), the third one satisfies the claim.

To finish the proof, put $t = 0$, and $j = 3$ in the sequence (3) and applying the claim we get immediately $H^1(X, \Omega_P|_X) \simeq \mathbb{C}$. □

Now we have the following analogue of [1, Lemma 8.3].

Lemma 3.3. *Let X and P be as above, so that* $\dim P = (k+1)(n-k) \geq 2$. *Let $X(1)$ be the first infinitesimal neighbourhood of X in P. Then the image of the natural maps*

$$\mathrm{Pic}(X(1)) \to \mathrm{Pic}(X) \to \mathrm{Num}(X)$$

is isomorphic to $\mathbb{Z}$

Proof. We will not repeat the part of the proof in Lemma 2.2 (since it is independent of the ambient space P) that leads to a commutative diagram similar to the one occurring in that proof. To finish, the only thing remaining is to prove that $H^1(X, \Omega_P|_X) \simeq \mathbb{Z}$, but this is exactly Lemma 3.2. □

Before proving Theorem 02, we need a new result on cohomology:

Lemma 3.4. *Let X and P be as above. Then for all $a \in \mathbb{Z}$, the restriction $H^0(O_P(a)) \to H^0(O_X(a))$ is surjective.*

Proof. Since $\mathrm{Pic}P \simeq \mathbb{Z}$, X is a complete intersection of ample divisors, we can find a chain $X = X_2 \subset X_3 \subset .... \subset X_{(k+1)(n-k)} = P$ such that $O_{X_i}(X_{i-1}) \simeq \mathcal{O}_{X_i}(b_i)$ for some $b_i > 0$. On the other side, it is well known that all line bundles on a grassmannian have vanishing intermediate cohomology (see for example [9]). Therefore we can prove by induction that every line bundle of type $\mathcal{O}_{X_i}(a)$, $i = 3, ..., (k+1)(n-k)$ has vanishing intermediate cohomology applying the arguments in the proof of Lemma 3.2 but substituting $\Omega_P|_{X_i}(t)$ by $O_{X_i}(a)$.

Now, since all the intermediate cohomology of $\mathcal{O}_{X_i}(a)$ vanishes, we get that, in particular, $H^1(X_i, \mathcal{O}_{X_i}(a - b_{i-1})) \simeq \{0\}$ for all $i = 2, ..., (k+1)(n-k)$. But this group contains the cokernel of the restriction map $H^0(X_i, \mathcal{O}_{X_i}(a)) \to H^0(X_{i-1}, \mathcal{O}_{X_{i-1}}(a))$ hence such map is surjective. So the map $H^0(P, \mathcal{O}_P(a)) \to H^0(X, \mathcal{O}_X(a))$ is a composition ofsurjective maps and then it is surjective. □

And now we finish with the proof of the second main result of this paper.

Proof of Theorem 0.2. For the nontrivial implication, let us suppose the normal short exact sequence splits. Then by Lemma 1.1 we have that there exists $\mathcal{Y} \subset X(1)$ Cartier divisor such that $Y = \mathcal{Y} \cap X$.

On the other side, since a grassmannian is simply connected, by Lefschetz hyperplane theorem, X is simply connected and then $\mathrm{Pic}X = \mathrm{Num}X$. Therefore by Lemma 3.3 there exist positive integers a', m such that $\mathcal{O}_X(mY) \cong O_X(a')$. On the other hand, by Lefschetz theorem again the cokernel of the restriction map $\mathrm{Pic}(\mathbb{G}(k,n)) \to \mathrm{Pic}(X)$ is torsion-free. This implies that m divides a', and if we set $a' = ma$, then it follows that $\mathcal{O}_X(Y) \cong \mathcal{O}_X(a)$, with a non-negative integers. At this point the proof can be concluded as follows. Let $s \in H^0(X, \mathcal{O}_X(Y)) = H^0(X, \mathcal{O}_X(a))$. By Lemma 3.4, the section s can be lifted to a section s' of $H^0(P, \mathcal{O}_{\mathbb{G}(k,n)}(a))$. Setting $H = \mathrm{div}_P(s')$ we get a hypersurface H of P such that $X \cap H = Y$ (scheme-theoretically).

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