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A SEMIPRIME FILTER-BASED IDENTITY-SUMMAND GRAPH OF A LATTICE

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Let F be a proper filter of a lattice L with the least element 0 and the greatest element 1. The filter-based identity-summand graph of L with respect to F, denoted by $\Gamma_F(L)$, is the graph with vertices $I_F^*(L) = \{x \in L \setminus F : x \vee y \in F$ for some $y \in L \setminus F\}$, and distinct vertices x and y are adjacent if and only if $x \vee y \in F$. We will make an intensive study of the notions of diameter, girth, chromatic number, clique number, independence number, domination number and planar property of this graph. Moreover, Beck's conjecture is proved for $\Gamma_F(L)$.

1. Introduction

Over the last few years, there has been an explosion of interest in associating a graph to an algebraic structure (see for instance, [1, 2, 4, 8, 10, 12–14, 19–22, 24–26, 28]). Most of the attention has focused on the zero-divisor graph of a commutative ring. By studying these graphs, we can gain a broader insight into the concepts and properties that involve both graphs and rings. The concept of zero-divisor graph for a commutative ring R was introduced by I.

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Beck [4], where he was mainly interested in colorings. In his work, all elements of R were vertices of the graph, and distinct vertices x and y were adjacent if and only if $xy = 0$. This investigation of colorings of a commutative ring was then continued by D. D. Anderson and M. Naseer in [1]. Let $Z(R)$ be the set of zero-divisors of R. In [2], D. F. Anderson and P. S. Livingston associated a (simple) graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. The zero-divisor graph $\Gamma(R)$ of R has been studied extensively; see the survey paper [8]. Let I be a proper ideal of R and $Z_I(R) = \{x \in R \setminus I : xy \in I \text{ for some } y \in R \setminus I\}$. In [28], S. P. Redmond introduced the ideal-based zero-divisor graph of R with respect to I, denoted by $\Gamma_I(R)$, with vertices $Z_I(R)$, and distinct vertices x and y are adjacent if and only if $xy \in I$. In [28], he explored the relation between $\Gamma_I(R)$ and $\Gamma(R/I)$ and showed, among other things, $\Gamma_I(R)$ is connected with $\mathrm{diam}(\Gamma_I(R)) \in \{0,1,2,3\}$ and $\mathrm{gr}(\Gamma_I(R)) \in \{3,4,\infty\}$.

Many of the deepest and most interesting applications of lattice theory concern (partially) ordered mathematical structures having also a binary addition or multiplication: ordered sets, lattice-ordered groups, monoids, vector spaces, and rings [7, 18–20]. There are many papers which interlink graph theory and lattice theory (see for example [5, 9, 10, 16, 17]). These papers discuss the properties of graphs derived from partially ordered sets and lattices. Halaš and Länger [22] have introduced the zero-divisor graph of a quasi-ordered set $(Q, \leq)$ (i.e. $\leq$ is a reflexive and transitive relation on Q) and they have shown that Beck's conjecture holds true for these graphs. Also, Halaš and Jukl [21] introduced the zero-divisor graphs of posets and answered affirmatively to the Beck's conjecture. The study of the zero-divisor graphs of lattices was then continued by Estaji and Khashyarmanesh [10] (also see [14]).

The main goal of this paper is to extend the notions of identity-summand graph of semirings, identity-summand graph of semirings based on a co-ideal and zero-divisor graph of lattices (rings) based on an ideal and investigate the interplay between some lattice-theoretic properties of L and graph-theoretic properties of its associated graph, identity-summand graph of a lattice based on a semiprime filter F. Besides some new results and examples, we take many results from [12, 13, 25, 28] and discuss them in a parallel fashion and more general setting.

Beck [4] conjectured that $\chi(G) = w(G)$, for the zero-divisor graph of commutative rings, but Anderson and Naseer [1] gave an example of a commutative local ring R with 32 elements for which $\chi(G) > w(G)$. A form of Beck's conjecture is proved for the zero divisor graph of a poset with 0 by Halaš and Jukl

[21], for the zero divisor graph of a poset having the smallest element 0 with respect to an ideal by Joshi [24] under the assumption that the corresponding zero divisor graph does not contain an infinite clique and by V. Joshi and A. Khiste [25] for the complement of the zero divisor graph of a lattice L based on the semiprime ideal under the assumption that L is a finite Boolean lattice such that $|L| = 2^n$. Therefore, we raise the following question.

Is Becks conjecture true for the identity-summand graph of lattices based on semiprime filters?

One of our motivations to study the identity-summand graph of a lattice based on a semiprime filter is to answer this question.

Let L be a lattice with the least element 0 and the greatest element 1. An element a of L is said to be *identity-summand* if there exists $1 \neq b \in L$ such that $a \vee b = 1$. We define another graph on L, $\Gamma(L)$, with vertices as elements of $I(L)^* = I(L) \setminus \{1\}$ (where $I(L) = \{a \in L : a \vee b = 1 \text{ for some } 1 \neq b \in L\}$), where two distinct vertices a and b are adjacent if and only if $a \vee b = 1$. This definition was motivated from [12]. Let F be a filter of L. Denote by $I_F(L) = \{x \in L : x \vee y \in F \text{ for some } y \in L \setminus F\}$ and $I_F^*(L) = \{x \in L \setminus F : x \vee y \in F \text{ for some } y \in L \setminus F\}$. Clearly $I_F(L) = F \cup I_F^*(L)$. In the present paper, we study the *filter-based identity-summand graph* of L with respect to F, denoted by $\Gamma_F(L)$, with vertices $I_F^*(L)$, and distinct vertices x and y are adjacent if and only if $x \vee y \in F$. In the case $F = \{1\}$, $\Gamma_F(L) = \Gamma(L)$. The basic properties of the graph $\Gamma_F(L)$ are investigated. Here is a brief summary of our paper. In section 2, motivated from the notion of Q-ideals and Q-co-ideals of semirings ([3, 11]) the notion of a partitioning filter of a lattice will be defined and construction process will be presented by which one can build the quotient structure of a lattice modulo a partitioning filter [3]. In fact, the bulk of this section is devoted to stating and proving several theorems in the theory of quotients of lattices which be useful in the later section. Among other things, we show that every semiprime filter F of L is an intersection of all prime filters of L that contains F (Theorem 2.3). In section 3, we completely characterize the diameter and girth of the graph $\Gamma_F(L)$ for such lattices in Theorem 3.3, Theorem 3.5, and Theorem 3.6, respectively. Also it is shown that $\Gamma_F(L)$ is a complete bipartite graph if and only if there exist two prime filters of L such that F is an intersection of these filters (Theorem 3.7). Finally, we collect some basic properties concerning independence number, chromatic number, clique number, and planar property of the graph $\Gamma_F(L)$ in theorem 3.13, Theorem 3.14, and 3.16, respectively. In Theorem 3.15, we answer affirmatively the Beck's conjecture.

In order to make this paper easier to follow, we recall in this section various notions which will be used in the sequel. For a graph Γ by $E(\Gamma)$ and $V(\Gamma)$ we denote the set of all edges and vertices, respectively. We recall that a graph is *connected* if there exists a path connecting any two distinct vertices. A graph Γ is said to be *totally disconnected* if it has no edge. The *distance* between two distinct vertices a and b, denoted by $d(a,b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a,b) = \infty$). The diameter of graph Γ, denoted by diam(Γ), is equal to $\sup\{d(a,b) : a,b \in V(\Gamma)\}$. If a and b are two adjacent vertices of G, then we write $a - b$. A graph is *complete* if it is connected with diameter less than or equal to one. We denote the complete graph on n vertices by K_n. The *girth* of a graph Γ, denoted gr(Γ), is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{g}(\Gamma) = \infty$. The *chromatic number* of Γ, denoted by $\chi(\Gamma)$, is the minimal number of colors which can be assigned to the vertices of Γ in such a way that every two adjacent vertices have different colors. A *complete bipartite* graph with part sizes m and n is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a star graph. A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $w(G)$, is called the *clique number* of G. An *induced subgraph* of a graph G by the set $S \subseteq V(G)$ is a subgraph H of G where vertices are adjacent in H precisely when adjacent in G. In a graph G, a set $S \subseteq V(G)$ is an *independent set* if the subgraph induced by S is totally disconnected. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G [6]. Let G be a graph. The (open) *neighborhood* $N(v)$ of a vertex v of $V(G)$ is the set of vertices which are adjacent to v. For each $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \bigcup S$. A set of vertices S in G is a *dominating set*, if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G ([23]).

A *lattice* is a poset $(L, \leq)$ in which every pair of elements x,y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is *complete* when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any non-empty complete lattice contains the least element 0 and the greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A sublattice K of a lattice L is defined on a non-empty subset K of L with the property that $a,b \in K$ implies that $a \wedge b \in K$ and $a \vee b \in K$. A non-empty subset F of a lattice L is called a *filter* (*ideal*), if for $a \in F$, $b \in L$, $a \leq b$ ($b \leq a$) implies $b \in F$, and $x \wedge y \in F$ ($x \vee y$) for all $x,y \in F$ (so if L is a lattice with 1, then $\{1\}$ is a filter of L). A lattice L with 1 is called *L-domain* if $a \vee b = 1$ ($a,b \in L$), then $a = 1$ or $b = 1$. A proper filter F of L is called *prime* if $x \vee y \in F$, then $x \in F$ or $y \in F$. Let F be a filter of a lattice L. A prime filter P containing a filter F of a lattice L is said to be a *minimal prime filter of* F, if it is minimal among all prime ideals containing F. The set of all minimal prime ideals of F is denoted by $min(F)$. A filter F of a lattice L is said to be *semiprime* if $a \vee b \in F$ and $a \vee c \in F$, then $a \vee (b \wedge c) \in F$ [27]. Dually, we have the concept of *semiprime ideal*. A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). In a distributive lattice, every ideal and every filter is semiprime. For any term not defined here the reader is referred to [7]. A lattice L is called 1-distributive, if $\{1\}$ is a semiprime filter of L. In a lattice L, for each $a \in L$, the notation $[a)$ denotes the set $\{x \in L : x \geq a\}$.

We need the following well-known result.

Proposition 1.1. *[15, Lemma 1.1] Let F be a non-empty subset of a lattice L.*

(a) F is a filter of L if and only if $a \wedge b \in F$ and $a \vee c \in F$ for all $a, b \in F$ and $c \in L$. In particular, if F is a filter of L with 1, then $1 \in F$.

(b) If F is a filter of L with $x \wedge y \in F$ $(x, y \in L)$, then $x, y \in F$.

2. Some basic properties of filters

This section generalizes some well-known results on quotient rings in commutative rings to lattices. Here we list some properties concerning of filters of a lattice which be useful in the later section. From now on, we shall assume, unless otherwise stated, that L is a lattice with the least element 0 and the greatest element 1.

Lemma 2.1. *If $F, F_1, F_2, \cdots, F_n$ are filters of L with F prime, then the following statements are equivalent:*

(a) $F \supseteq F_j$ for some j with $1 \leq j \leq n$;

(b) $F \supseteq \cap_{i=1}^{n} F_i$;

Proof. (a) $\Rightarrow$ (b) is clear. To see (b) $\Rightarrow$ (a), Suppose that, for all j with $1 \leq j \leq n$, $F \not\supseteq F_j$. Then for each such j, there exists $a_j \in F_j \setminus F$; but then $a_1 \vee a_2 \vee \cdots \vee a_n \in \vee_{j=1}^{n} F_j \setminus F$(because F is a prime filter), and this contradicts the statement of (b). □

Lemma 2.2. *If F is a semiprime filter of L, then $(F : a) = \{r \in L : r \vee a \in F\}$ is a semiprime filter of L for all $a \in L$. In particular, the set $(1 : a) = \{r \in L : r \vee a = 1\}$ is a filter of L for all $a \in L$, provided that L is 1-distributive.*

Proof. Let $b \in L$ and $c \in (F : a)$. Since F is a filter, $b \vee c \vee a \in F$. Therefore $b \vee c \in (F : a)$. Now, let $b, c \in (F : a)$. Then $a \vee b \in F$ and $a \vee c \in F$. Hence

$(b \wedge c) \vee a \in F$, because F is semiprime. Hence $(F : a)$ is a filter. We will show $(F : a)$ is semiprime. Assume that $b \vee c \in (F : a)$ and $c \vee d \in (F : a)$. Therefore $c \vee b \vee a \in F$ and $c \vee d \vee a \in F$. Since F is semiprime, we have $(a \vee c) \vee (d \wedge b) = a \vee (c \vee (b \wedge b)) \in F$. Therefore $(F : a)$ is semiprime. The other implication is clear, because $\{1\}$ is a semiprime filter. □

Proposition 2.3. *Let L be a lattice and F a proper semiprime filter of L. The following hold:*

(1) There exists a prime filter P of L such that $F \subseteq P$.

(2) If $\{F_\alpha\}_{\alpha \in \Lambda}$ is the set of all prime filters of L containing F, then $F = \cap_{\alpha \in \Lambda} F_\alpha$. Moreover, if $F_1, \ldots, F_n$ are the only distinct minimal prime filters L containing F, then $\cap_{i=1}^{n} F_i = F$ and $F \neq \cap_{1 \le i \le n, i \neq j} F_i$, for each $1 \le j \le n$.

Proof. (1) Set $\sum = \{H : H$ is a proper semiprime filter of L containing $F\}$. As $F \in \sum$, $\sum \neq \emptyset$. Also, the relation of inclusion, $\subseteq$, is a partial order on $\sum$. Let Λ be a non-empty totally ordered subset of $\sum$. Then $S = \cup_{H \in \Lambda} H$ is a proper semiprime filter of L such that $F \subseteq S$. Hence the partially ordered set $(\sum, \subseteq)$ has a maximal element P, by Zorn's Lemma. We claim that P is a prime filter. Let $a \vee b \in P$ for $a, b \in L$ and $a \notin P$. Then $P \subsetneq (P : b)$. Since P is a maximal semiprime filter that contains F and $(P : b)$ is a semiprime filter by Lemma 2.2, we have $(P : b) = L$. Therefore $b \in P$ and so P is prime.

(2) It is enough to show that $\cap_{\alpha \in \Lambda} F_\alpha \subseteq F$. Let $x \in \cap_{\alpha \in \Lambda} F_\alpha$ with $x \notin F$. Set $\Delta = \{K : K$ is a filter of L containing $F, x \notin K\}$. Since $F \in \Delta$, $\Delta \neq \emptyset$. Moreover, the relation of inclusion, $\subseteq$, is a partial order on Δ. Let Θ be a non-empty totally ordered subset of Δ. Then $T = \cup_{K \in \Theta} K$ is a proper filter of L such that $T \supseteq F$ and $x \notin T$. So it follows from Zorn's Lemma that the partially ordered set $(\Delta, \subseteq)$ has a maximal element F'. Since $x \notin F'$, $F' \neq L$. We show that F' is prime. Let $a \vee b \in F'$ such that $a \notin F'$. Then $a \in (F' : b)$ gives $F' \subsetneq (F' : b)$. Thus $x \in (F' : b)$ by maximality of F'. Hence $b \in (F' : x)$. If $c \in F'$, then $c \vee (a \wedge x) \in F'$ since F' is a filter, so $c \in (F' : a \wedge x)$. Thus $F' \subseteq (F' : a \wedge x)$. If $(F' : a \wedge x) \neq F'$, then $x \in (F' : a \wedge x)$ by maximality of F'; hence $x \vee (a \wedge x) = x \in F'$ (because $a \wedge x \le x$), a contradiction. So $(F' : a \wedge x) = F'$. We claim that $(F' : a \wedge x) = (F' : a) \cap (F' : x)$. If $r \in (F' : a \wedge x)$, then $a \wedge x \in (F' : r)$; so $a, x \in (F' : r)$ by Proposition 1.1(b). Thus $r \in (F' : x) \cap (F' : a)$ and $(F' : a \wedge x) \subseteq (F' : a) \cap (F' : x)$. For the reverse of inclusion let $r \in (F' : a) \cap (F' : x)$. Then $a, x \in (F' : r)$, so $a \wedge x \in (F' : r)$ since it is a filter, and so $r \in (F' : a \wedge x)$. Thus $b \in (F' : a) \cap (F' : x) = F'$. So F' is prime, which implies $x \in F'$ that is a contradiction, as required.

Clearly, $\cap_{i=1}^{n} F_i = F$. To see the other statement, suppose $F = \cap_{1 \le i \le n, i \neq j} F_i$ for some $1 \le j \le n$. Since for each $i \neq j$, $F_i \not\subseteq F_j$, there is $x_i \in F_i$ such that $x_i \notin F_j$. As $\vee_{i \neq j} x_i \in \cap_{1 \le i \le n, i \neq j} F_i \subseteq F_j$, it is clear that $x_i \in F_j$ for some $i \neq j$, that is a contradiction. So there exists $1 \le i \le n$, such that $F_i \subseteq F_j$ and this leads to a contradiction. Thus $F \neq \cap_{1 \le i \le n, i \neq j} F_i$ for each $1 \le j \le n$. □

Example 2.4. Let $\mathbb{N}$ be the set of natural numbers and $L = F \cup \{\mathbb{N}\}$, where $F = \{X \subseteq \mathbb{N} : \text{ X is finite}\}$. Then L is a distributive lattice. Every (resp. prime) filter of L has the form $[X) = \{H \in L : X \subseteq H\}$ where $X \in L$ (resp. $[\{a\}) = \{H \in L : a \in H\}, a \in \mathbb{N}$). Since L is distributive, every filter of L is semiprime. By Proposition 2.3, every semiprime filter is an intersection of prime filters. For each $X \in L$, we show $[X) = \cap_{a \in X}[\{a\})$. Let $T \in [X)$. Then $X \subseteq T$, and so $a \in T$ for each $a \in X$. Therefore $T \in \cap_{a\in X}[\{a\})$, and so $[X) \subseteq \cap_{a\in X}[\{a\})$. Now, let $T \in \cap_{a\in X}[\{a\})$. Then $a \in T$ for each $a \in X$. Thus $X \subseteq T$ and so $T \in [X)$. Therefore $[X) = \cap_{a\in X}[\{a\})$ (note that if $X,Y \in L$, then $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$).

In the next example we show that the condition "F is semiprime" is not superfluous in Proposition 2.3.

Example 2.5. Let L be the lattice $N_5 = \{0,a,b,c,1\}$, with the relations $b < a$, $a \wedge c = 0$ and $b \vee c = 1$. Then $b \vee c \in [a)$ and $b \vee a \in [a)$. But $b \vee (a \wedge c) \notin [a)$. Therefore $[a)$ is not semiprime. Also $[b)$ is the unique prime filter that $[a)$ is contained in it. Therefore $[a)$ is not an intersection of prime filters.

Theorem 2.6. *Let F be a semiprime filter of a lattice L and $I = L \setminus I_F(L)$. Then I is semiprime ideal of L.*

Proof. Let $x \in S$ and $y \leq x$. If $y \notin I$, then $y \in I_F(L)$. Thus $y \vee z \in F$ for some $z \notin F$. This gives $x \vee y \vee z = x \vee z \in F$. This contradicts the fact that $a \in I$. Let $a,b \in I$. Suppose, to the contrary, that $a \wedge b \notin I$, that is, $a \wedge b \in I_F(L)$. Therefore there exists $z \notin F$ such that $(a \wedge b) \vee z \in F$. Therefore $a \wedge b \in (F : z)$. By Proposition 1.1(b), $a \in (F : z)$ and $b \in (F : z)$. This gives $a,b \in I_F(L)$, a contradiction. Thus we have $a \wedge b \in I$. This proves that I is an ideal of L.

Now, we prove that I is semiprime. Let $x \wedge y, x \wedge z \in I$. We prove that $x \wedge (y \vee z) \in I$. If $x \wedge (y \vee z) \notin I$, then $x \wedge (y \vee z) \in I_F(L)$. therefore $(x \wedge (y \vee z)) \vee p \in F$ for some $p \notin F$. Hence $x \vee p \in F$ and $(y \vee z) \vee p \in F$. If $y \vee p \in F$, then $p \vee (y \wedge x) \in F$, since F is semiprime. Therefore $y \wedge x \in I_F(L)$, a contradiction. So $y \vee p \notin F$. As $x \vee (y \vee p) \in F$ and $z \vee (y \vee p) \in F$, we have $(x \wedge z) \vee (y \vee p) \in F$. This gives $x \wedge z \in I_F(L)$, a contradiction. Therefore $x \wedge (y \vee z) \in I$ and I is semiprime. □

In the following, the notion of a Q-filter will now be defined and a construction process will be presented by which one can build the quotient structure of a lattice with respect to a Q-filter.

Definition 2.7. A filter F of L is called a *partitioning filter* (= Q-filter or $F(Q)$-filter) if there exists a sublattice Q of L such that

(1) $L = \cup\{q \wedge F : q \in Q\}$;

(2) If $q_1, q_2 \in Q$, then $(q_1 \wedge F) \cap (q_2 \wedge F) \neq \emptyset$ if and only if $q_1 = q_2$.

Assume that F is a filter of a lattice $(L,\leq)$ and let Q be a sublattice of $(L,\leq)$. Set $L/F := \{q\wedge F : q\in Q\}$. We set up a partial order $\leq_Q$ on L/F as follows: for each $q\wedge F, q'\wedge F\in L/F$, we write $(q\wedge F)\leq_Q(q'\wedge F)$ if and only if $q\leq q'$. It is straightforward to check that $(L/F,\leq_Q)$ is a poset. The notation below (Proposition 2.8) will be kept in this paper.

Proposition 2.8. *Assume that F is a filter of a lattice $(L,\leq)$ and let Q be a sublattice of $(L,\leq)$. Then $(L/F,\leq_Q)$ is a lattice.*

Proof. It suffices to show that every pair of elements $q_1\wedge F$, $q_2\wedge F\in L/F$ has a least upper bound (called the join of $q_1\wedge F$ and $q_2\wedge F$, and written $(q_1\wedge F)\vee_Q(q_2\wedge F)$) and a greatest lower bound (called the meet of $q_1\wedge F$ and $q_2\wedge F$, and written $(q_1\wedge F)\wedge_Q(q_2\wedge F)$). Set $X=\{q_1\wedge F, q_2\wedge F\}$. By the definition of $\leq_Q$, $(q_1\vee q_2)\wedge F$ is an upper bound for the set X. If $q_3\wedge F$ is any upper bound of X, it is easy to see that $(q_1\vee q_2)\wedge F\leq_Q q_3\wedge F$. Thus $(q_1\wedge F)\vee_Q(q_2\wedge F)=(q_1\vee q_2)\wedge F$. Similarly, $(q_1\wedge F)\wedge_Q(q_2\wedge F)=(q_1\wedge q_2)\wedge F$. □

Proposition 2.9. *Let F be a Q-filter of L. If $x\in L$, Then $x\wedge F\subseteq q\wedge F$ for some a unique $q\in Q$.*

Proof. Since $\{q\wedge F\}_{q\in Q}$ is a partition of L, there exists $q\in Q$ such that $x\in q\wedge F$. If $y\in x\wedge F$, there exists $a\in F$ such that $y=x\wedge a$. Since $x\in q\wedge F$, there exists $b\in F$ such that $x=q\wedge b$; hence $y=x\wedge a=q\wedge a\wedge b\in q\wedge F$. Thus $x\wedge F\subseteq q\wedge F$. The uniqueness follows from part (2) of Definition 2.7. □

Remark 2.10. Let F be a Q-filter of L .

(1) By Proposition 2.9, $1\wedge F=F\subseteq q\wedge F$ for some $q\in Q$. It follows that $1=q\wedge a$ for some $a\in F$; hence $q=1\in Q$. Also, it is easy to see that $1\wedge F=F$ is the greatest element of L/F.

(2) If $0\in F$, then $x=0\vee x\in F$ for all $x\in L$; so $F=L$.

(3) If $a\in F\cap Q$, then $a\in(a\wedge F)\cap(1\wedge F)$; so $a=1$. Thus $Q\cap F=\{1\}$.

(4) By the definition of $\leq_Q$, it is easy to see that if L is distributive, then L/F is distributive.

If F is a filter of L, then it is possible that F can be considered to be a Q-filter with respect to many different sublattices Q of L. However, the following proposition implies that the structure L/F is "essentially independent" of the choice of Q.

Proposition 2.11. *Let F be a partitioning filter of L with respect to two sublattices Q_1 and Q_2 of L. Then $L/F_{(Q_1)}$ and $L/F_{(Q_2)}$ are equal as sets. Moreover, $L/F_{(Q_1)}\cong L/F_{(Q_2)}$.*

Proof. Let $q_1 \wedge F \in L/F_{(Q_1)}$. Since $q_1 \in L$, there exists a unique $q_2 \in Q_2$ such that $q_1 \wedge F \subseteq q_2 \wedge F$ by Proposition 2.9. Again there exists a unique $t_1 \in Q_1$ such that $q_2 \wedge F \subseteq t_1 \wedge F$. It follows that $q_1 \wedge F = q_2 \wedge F = t_1 \wedge F \in L/F_{(Q_2)}$. Thus $L/F_{(Q_1)} \subseteq L/F_{(Q_2)}$. Likewise, $L/F_{(Q_2)} \subseteq L/F_{(Q_1)}$.

Define $\psi : L/F_{(Q_1)} \to L/F_{(Q_2)}$ by $\psi(q \wedge F) = q' \wedge F$, where q' is the unique element of Q_2 such that $q \wedge F \subseteq q' \wedge F$. Clearly, ψ is well-defined. Then it is not hard to see that ψ is a lattice isomorphism. □

Example 2.12. Let $D = \{1,2,3\}$. Then the set $L = \{X : X \subseteq D\}$ forms a distributive complete lattice under set inclusion with the greatest element D and the least element $\emptyset$. An inspection will show that $F = \{D, \{1,2\}\}$ is a $F(Q)$-filter, where $Q = \{\{3\}, \{1,3\}, \{2,3\}, D\}$, F is not prime since $\{1\} \vee \{2\} \in F$ but $\{1\}, \{2\} \notin F$, and $L/F = \{F, \{3\} \wedge F, \{1,3\} \wedge F, \{2,3\} \wedge F\}$. Moreover, If $F' = \{\{1\}, \{1,2\}, \{1,3\}, D\}$, then F' is a prime $F'(Q)$-filter, where $F'(Q) = \{D, \{2,3\}\}$ and $L/F' = \{F', \{2,3\} \wedge F'\}$ (note that if $x, y \in L$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$).

Theorem 2.13. *Let F be a Q-filter of L.*

(1) If F' is a filter of L with $F \subseteq F'$, then $F'/F = \{a \wedge F : a \in F' \cap Q\}$ is a filter of L/F.

(2) If K is a filter of L/F, then $K = F'/F$ for some filter F' of L.

(3) If F' is a filter of L with $F \subseteq F'$, then F' is a prime filter of L if and only if F'/F is a prime filter of L/F.

(4) F is a prime filter of L if and only if L/F is a L-domain.

(5) If P is a filter of L with $F \subseteq P$, then $P/F \in \min(L/F)$ if and only if $P \in \min(L)$.

Proof. (1) As $1 \in F' \cap Q$, $1 \wedge F \in F'/F$; so $F'/F \neq \emptyset$. Let $q_1 \wedge F, q_2 \wedge F \in F'/F$ and $q \wedge F \in L/F$, where $q_1, q_2 \in F' \cap Q$ and $q \in Q$; so $q_1 \wedge q_2, q_1 \vee q \in F' \cap Q$ since Q is a sublattice and F' is a filter. Now we have $(q_1 \wedge F) \wedge_Q (q_2 \wedge F) = (q_1 \wedge q_2) \wedge F \in F'/F$ and $(q_1 \wedge F) \vee_Q (q \wedge F) = (q_1 \vee q) \wedge F \in F'/F$. Thus F'/F is a filter of L/F.

(2) Let $F' = \{\ r \in L\ :\ \exists q \in Q\ \ s.t\ \ r \in q \wedge F\ ,\ q \wedge F \in K\}$. If $a \in F$, then $a \in F = 1 \wedge F \in K$, so $F \subseteq F'$. Let $r, s \in F'$ and $t \in L$. By Proposition 2.9 and the definition of F', $r \in q_1 \wedge F \in K$ and $s \in q_2 \wedge F \in K$ for some $q_1, q_2 \in Q$; hence $r \wedge s \in (q_1 \wedge F) \wedge_Q (q_2 \wedge F) = (q_1 \wedge q_2) \wedge F \in K$. Thus $r \wedge s \in F'$. Similarly, $r \vee t \in F'$. Therefore F' is a filter of L by Proposition 1.1(a). Finally, it is easy to see that $K = F'/F = \{q \wedge F : q \in F' \cap Q\}$.

(3) Assume that F' is a prime filter of L and let $q_1 \wedge F, q_2 \wedge F \in L/I$ be such that $(q_1 \wedge F) \vee_Q (q_2 \wedge F) = (q_1 \vee q_2) \wedge F \in F'/F$, where $q_1, q_2 \in Q$. Then there exist $q_3 \in F' \cap Q$ and $a \in F \subseteq F'$ such that $q_1 \vee q_2 = q_3 \wedge a \in F'$ by Proposition

1.1(a). Then F' prime gives $q_1 \in F'$ or $q_2 \in F'$; thus $q_1 \wedge F \in F'/F$ or $q_2 \wedge F \in F'/F$. The other implication is similar.

(4) Let F be a prime filter of L and let $q_1 \wedge F, q_2 \wedge F$ be elements of L/F such that $(q_1 \wedge F) \vee_Q (q_2 \wedge F) = (q_1 \vee q_2) \wedge F = 1 \wedge F = F$. Then $(q_1 \vee q_2) \wedge a = (q_1 \wedge a) \vee (q_2 \wedge a) \in F$ for all $a \in F$. Since F is a prime filter, either $q_1 \wedge a \in F$ or $q_2 \wedge a \in F$; hence $(q_1 \wedge F) \cap (1 \wedge F) \neq \emptyset$ or $(q_2 \wedge F) \cap (1 \wedge F) \neq \emptyset$. This implies that $q_1 \wedge F = 1 \wedge F$ or $q_2 \wedge F = 1 \wedge F$. The other side is similar.

(5) If $P/F \in \min(L/F)$, then P is a prime filter of L by (3). Suppose that $F \subsetneqq J \subseteq P$, where J is a prime filter; so J/F is a prime ideal of L/F; hence $J/F = P/F$ by minimality of P/F. Then $P = J$. The other implication is similar. □

3. Basic structure of $\Gamma_F(L)$

We continue to use the notation already established, so L is a lattice with 0 and 1. In this section we study the connectedness and the diameter and girth of the graph $\Gamma_F(L)$, when F is a semiprime filter. Also we investigate some basic properties concerning chromatic number, clique number, and planar property of this graph.

Proposition 3.1. *The following hold:*

(1) $\Gamma_F(L) = \emptyset$ if and only if F is prime;

(2) If F is a Q-filter, then $\Gamma(L/F) = \emptyset$ if and only if F is prime.

(3) If F is a Q-filter of L and $\Gamma(L/F) \neq \emptyset$, then $\Gamma(L/F)$ has at least two vertices.

(4) If $a \in L$ is a vertex of $\Gamma_F(L)$ which is adjacent to every other vertex, then $(F : a)$ is a maximal element of the set $\sum = \{(F : x) : x \in L \setminus F\}$ with respect to inclusion. Moreover, $(F : a)$ is a prime filter of L.

Proof. (1) This follows directly from the definitions.

(2) By Theorem 2.13(3), F is prime if and only if L/F is a L-domain. Therefore F is prime if and only if $\Gamma(L/F) = \emptyset$.

(3) Since $(q \wedge F) \vee_Q (q \wedge F) = q \wedge F$, $\Gamma(L/F)$ has no loop, so it has more than one vertex.

(4) Suppose, on the contrary, $(F : a)$ is not maximal. So there is $x \in L \setminus F$ such that $(F : a) \subset (F : x)$. Since a is adjacent to every other vertex in $\Gamma_F(L)$, $x \vee a \in F$, which gives $x \in (F : a) \subset (F : x)$. So $x \vee x = x \in F$, a contradiction. Let $x \vee y \in (F : a)$ be such that $x \notin (F : a)$; so $x \vee a \notin F$. As $(F : a) \subseteq (F : x \vee a)$ (since F is a filter) and $(F : a)$ is maximal in $\sum$, we have $(F : a) = (F : x \vee a)$. Now $x \vee y \vee a \in F$ gives $y \in (F : a \vee x) = (F : a)$. Thus $(F : a)$ is prime. □

The next several results investigate the relationship between $\Gamma(L/F)$ and $\Gamma_F(L)$.

Theorem 3.2. *Let F be a Q-filter of L and let $x,y \in I_F(L)$ such that $x \in q_1 \wedge F$ and $y \in q_2 \wedge F$, for some $q_1, q_2 \in Q$. Then*

(1) x is adjacent to y in $\Gamma_F(L)$ if and only if $q_1 \wedge F$ is adjacent to $q_2 \wedge F$ in $\Gamma(L/F)$ and $q_1 \neq q_2$. In particular, each element of $q_1 \wedge F$ is adjacent to each element of $q_2 \wedge F$ in $\Gamma_F(L)$.

(2) If $q_1 \wedge F \in I^*(L/F)$, then all the distinct elements of $q_1 \wedge F$ are not adjacent to each other in $\Gamma_F(L)$.

(3) $w(\Gamma_F(L)) = w(\Gamma(L/F))$.

Proof. (1) If x is adjacent to y in $\Gamma_F(L)$, then $x \vee y \in 1 \wedge F = F$. Since $(q_1 \wedge F) \vee_Q (q_2 \wedge F) = (q_1 \vee q_2) \wedge F$, $x \vee y \in (q_1 \vee q_2) \wedge F \cap (1 \wedge F)$, so $(q_1 \wedge F) \vee_Q (q_2 \wedge F) = 1 \wedge F$. Thus $q_1 \wedge F$ is adjacent to $q_2 \wedge F$ in $\Gamma(L/F)$ (this shows that $q_1 \vee q_2 = 1$). We show $q_1 \neq q_2$. Suppose, on the contrary, $q_1 = q_2$. Since $q_1 \wedge F$ and $q_2 \wedge F$ are adjacent, we have $F = 1 \wedge F = (q_1 \wedge F) \vee_Q (q_2 \wedge F) = (q_1 \vee q_1) \wedge F = q_1 \wedge F$, a contradiction (since $x, y \notin F$). Thus $q_1 \neq q_2$. Conversely, let $q_1 \wedge F$ be adjacent to $q_2 \wedge F$ in $\Gamma(L/F)$, so $(q_1 \wedge F) \vee_Q (q_2 \wedge F) = (q_1 \vee q_2) \wedge F = 1 \wedge F = F$. Then $x \vee y \in (q_1 \vee q_2) \wedge F = F$; hence x is adjacent to y in $\Gamma_F(L)$. Now we show each two elements of $q_1 \wedge F$ and $q_2 \wedge F$ are adjacent to each other. Let $q_1 \wedge c \in q_1 \wedge F$ and $q_2 \wedge d \in q_2 \wedge F$, where $c, d \in F$. We observe that $q_1 \wedge c, q_2 \wedge d \notin F$ by Proposition 1.1(b) and $q_1, q_2 \notin F$. As $q_1 \vee q_2 = 1 \in F$ and $q_2 \vee c \in F$, $q_2 \vee (q_1 \wedge c) \in F$, because F is semiprime. Also, semiprime property of F and $d \vee (q_1 \wedge c) \in F$ implies that $(q_1 \wedge c) \vee (q_2 \wedge d) \in F$. Hence $q_1 \wedge c$ is adjacent to $q_2 \wedge d$ in $\Gamma_F(L)$.

(2) Since $x \notin F$, $q_1 \notin F$. Let $q_1 \wedge a, q_1 \wedge b \in q_1 \wedge F$ be distinct, where $a, b \in F$. Clearly, $q_1 \wedge a, q_1 \wedge b \notin F$. If $(q_1 \wedge a) \vee (q_1 \wedge b) \in F$, then $(q_1 \wedge a) \vee (q_1 \wedge b) \leq q_1 \wedge (a \vee b)$ gives $q_1 \wedge (a \vee b) \in F$. So $q_1 \in F$ by Proposition 1.1(b) that is a contradiction. So none of elements of $q_1 \wedge F$ are adjacent to each other in $\Gamma_F(L)$.

(3) Assume that $\{x_i\}_{i \in J}$ is a clique in $\Gamma_F(L)$ and let q_i be the unique element of Q such that $x_i \in q_i \wedge F$ $(i \in J)$. Then $\{q_i \wedge F\}_{i \in J}$ is a clique in $\Gamma(L/F)$. Hence $w(\Gamma(L/F)) \geq w(\Gamma_F(L))$. Now, let $\{q_i \wedge F\}_{i \in K}$ be a clique in $\Gamma(L/F)$. Then $\{q_i\}_{i \in K}$ is a clique in $\Gamma_F(L)$. Thus $w(\Gamma_F(L)) \geq w(\Gamma(L/F))$, as needed. □

Theorem 3.3. *Let F be a nonprime filter of L. Then $\Gamma_F(L)$ is connected with* $\mathrm{diam}(\Gamma_F(L)) \in \{2,3\}$.

Proof. Let x and y be distinct elements of $I_F(L)$. If $x \vee y \in F$, then $x - y$ is a path in $\Gamma_F(L)$. So we can assume that $x \vee y \notin F$. Since $x, y \notin F$, there exist $a, b \in L \setminus (F \cup \{x,y\})$ such that $x \vee a \in F$ and $y \vee b \in F$. If $a = b$, then $x - a - y$ is a path in $\Gamma_F(L)$. If $a \neq b$ and $a \vee b \in F$, then $x - a - b - y$ is a path. So suppose that $a \vee b \notin F$. We claim that $x \neq a \vee b$. If $x = a \vee b$, then $x \vee y = (x \vee a) \vee (y \vee b) \in F$ that is a contradiction. Thus $x \neq a \vee b$. Similarly, $y \neq a \vee b$. Since F is a filter, $x - a \vee b - y$ is a path. Thus $\Gamma_F(L)$ is connected with diam$(\Gamma_F(L)) \leq 3$. Since $\Gamma_F(L)$ has no loop, it suffices to show that $\Gamma_F(L)$ is not complete. Assume that $\Gamma_F(L)$ is complete and let $a,b,c \in I_F(L)$ be distinct elements. Then $a \vee c, a \vee b, b \vee c \in F$, so $b,c \in (F:a)$; hence $b \wedge c \in (F:a)$ since $(F:a)$ is a filter by Lemma 2.2. If $b \wedge c \in F$, then $b,c \in F$ by Proposition 1.1(b), a contradiction. So $b \wedge c \notin F$. If $b \wedge c = c$, then $c \leq b$ and $c \vee b = b \in F$, a contradiction. So $b \wedge c \neq c$. Since $\Gamma_F(L)$ is complete, $(c \wedge b) \vee c = c \in F$ which is a contradiction. Thus $\Gamma_F(L)$ is not complete (so diam$(\Gamma_F(L)) \neq 1$). Thus diam$(\Gamma_F(L)) \in \{2,3\}$. □

By Theorem 3.3, we have $\Gamma_F(L)$ is not complete for a semiprime filter F of a lattice L. The following example shows that if F is not a semiprime filter of L, then we may have $\Gamma_F(L)$ is complete.

Example 3.4. Let L be the lattice N_5 (Example 2.5) and $F = [a)$. By Example 2.5 F is not semiprime. It can be seen that $I_F^*(L) = \{b,c\}$ and $\Gamma_F(L)$ is complete.

Theorem 3.5. *Let F be a filter of L. Then*

(1) If $\Gamma_F(L)$ *contains a cycle, then* gr$(\Gamma_F(L)) \leq 4$*;*

(2) If F is a Q-filter such that $\Gamma(L/F)$ *and* $\Gamma_F(L)$ *contains a cycle, then* gr$(\Gamma_F(L)) =$ gr$(\Gamma(L/F))$*. Moreover, If* $\Gamma(L/F)$ *has only two vertices* $q_1 \wedge F$ *and* $q_2 \wedge F$ *with* $|q_i \wedge F| \geq 2$ $(i = 1,2)$*, then* gr$(\Gamma_F(L)) = 4$*;*

(3) The only cycle graph with respect to F is $K_{2,2}$*.*

Proof. (1) Suppose that $\Gamma_F(L)$ contains a cycle. Hence gr$(\Gamma_F(L)) \leq 7$. Suppose that gr$(\Gamma_F(L)) = n$, where $n \in \{5,6,7\}$ and let $x_1 - x_2 - ... - x_n - x_1$ be a cycle of minimum length. Since x_1 is not adjacent to x_3, $x_1 \vee x_3 \notin F$. If $x_1 \vee x_3 \neq x_i$ for each $1 \leq i \leq n$, then $x_2 - x_3 - x_4 - x_1 \vee x_3 - x_2$ is a 4-cycle, a contradiction. Therefore $x_1 \vee x_3 = x_i$ for some $1 \leq i \leq n$. If $x_1 \vee x_3 = x_1$ (resp. $x_1 \vee x_3 = x_3$), then $x_1 - x_2 - x_3 - x_4 - x_1$ (resp. $x_1 - x_2 - x_3 - x_n - x_1$) is a 4-cycle, a contradiction. If $x_1 \vee x_3 = x_2$ (resp. $x_1 \vee x_3 = x_4$), then $x_2 - x_3 - x_4 - x_2$ (resp. $x_2 - x_3 - x_4 - x_2$) is a 3-cycle that is a contradiction. If $x_1 \vee x_3 = x_n$, then $x_2 - x_3 - x_4 - x_n - x_2$ is a 4-cycle which is a contradiction. Thus, every case leads to a contradiction; hence gr$(\Gamma_F(L)) \leq 4$.

(2) Assume that gr$(\Gamma_F(L)) = n$ and let $x_1 - x_2 - ... - x_n - x_1$ be a cycle in $\Gamma_F(L)$. Since F is a Q-filter, there exist unique elements $q_i \in Q$ $(1 \leq i \leq n)$ such that $x_i \in q_i \wedge F$. By Theorem 3.2, $q_1 \wedge F - q_2 \wedge F - \cdots - q_n \wedge F - q_1 \wedge F$ is a cycle in $\Gamma(L/F)$; thus gr$(\Gamma(L/F)) \leq$ gr$(\Gamma_F(L))$. Now suppose that gr$(\Gamma(L/F)) = m$ and let $q_1 \wedge F - q_2 \wedge F - \cdots - q_m \wedge F - q_1 \wedge F$ be a cycle of length m in $\Gamma(L/F)$. Then $q_1 - q_2 - \cdots - q_m - q_1$ is a cycle of length m in $\Gamma_F(L)$ by Theorem 3.2, so gr$(\Gamma_F(L)) \leq$ gr$(\Gamma(L/F))$. Thus gr$(\Gamma_F(L)) =$ gr$(\Gamma(L/F))$. Finally, let $\Gamma(L/F)$ has only two vertices $q_1 \wedge F$ and $q_2 \wedge F$; we show that gr$(\Gamma_F(L)) = 4$. Let

$x,y \in I_F(L)$. If x,y are adjacent, then $x \in q_i \wedge F$ and $y \in q_j \wedge F$, where $i \neq j \in \{1,2\}$, and if x,y are not adjacent, then either $x,y \in q_1 \wedge F$ or $x,y \in q_2 \wedge F$ by Theorem 3.2. Also, as $q_1 \wedge F$ and $q_2 \wedge F$ are adjacent in $\Gamma(L/F)$, every element of $q_1 \wedge F$ and $q_2 \wedge F$ are adjacent in $\Gamma_F(L)$ by Theorem 3.2. Hence $\Gamma_F(L)$ is complete bipartite with two parts $q_1 \wedge F$ and $q_2 \wedge F$. Since $|q_i \wedge F| \geq 2$ for $i = 1,2$, $\mathrm{gr}(\Gamma_F(L)) = 4$.

(3) By Theorem 3.3, there is no 3-cycle graph. By (1), there is no cycle graph with five or more vertices. So the only cycle graph is $K_{2,2}$. □

For a graph G and vertex $x \in V(G)$, the degree of x, denoted by $\deg(x)$, is the number of edges of G incident with x.

Theorem 3.6. *Let F be a nonprime filter of L. Then*

(1) $\mathrm{gr}(\Gamma_F(L)) = \infty$ *if and only if* $\Gamma_F(L)$ *is a star graph;*

(2) $\mathrm{gr}(\Gamma_F(L)) = 4$ *if and only if* $\Gamma_F(L)$ *is bipartite but not a star graph;*

(3) $\mathrm{gr}(\Gamma_F(L)) = 3$ *if and only if* $\Gamma_F(L)$ *contains an odd cycle;*

(4) If $\mathrm{gr}(\Gamma_F(L)) = 4$, *then there is no end vertex (i.e, vertex with degree 1) in* $\Gamma_F(L)$.

Proof. (1) Assume that $\mathrm{gr}(\Gamma_F(L)) = \infty$ and $\Gamma_F(L)$ is not a star graph. So $|I_F(L)| \geq 4$ since $\Gamma_F(L)$ is not complete by Theorem 3.3. As $\Gamma_F(L)$ is connected, there exists a vertex $x \in I_F(L)$ such that $\deg(x) \geq 2$. Then $\Gamma_F(L)$ is not a star graph gives there exists a path of the form $a - x - b - c$ in $\Gamma_F(L)$ for some $a,b,c \in I_F(L)$. If a is adjacent to c, then $a - x - b - c - a$ is a cycle in $\Gamma_F(L)$, a contradiction. If a is not adjacent to c, then $a \vee c \vee x \in F$ gives $x - a \vee c - b - x$ is a cycle which is a contradiction. Thus $\Gamma_F(L)$ is a star graph. The other implication is clear.

(2) Let $\mathrm{gr}(\Gamma_F(L)) = 4$. So $\Gamma_F(L)$ is not a star graph by (1). It is known that a graph is bipartite if and only if it contains no odd cycle [6, Theorem 4.7]. Thus it suffices to show that $\Gamma_F(L)$ has no odd cycle. Assume that $x_1 - x_2 - \cdots - x_n - x_1$ is an odd cycle of minimal length n in $\Gamma_F(L)$. Since $\mathrm{gr}(\Gamma_F(L)) = 4$, $n \geq 5$. As $\mathrm{gr}(\Gamma_F(L)) \neq 3$, x_2 is not adjacent to x_4, and so $x_2 \vee x_4 \notin F$. Since F is a filter, $x_2 \vee x_4 \vee x_1 \in F$. It follows that $x_1 - x_2 \vee x_4 - x_5 - \cdots - x_n - x_1$ is an odd cycle of length $n-2$ in $\Gamma_F(L)$, a contradiction. Hence $\Gamma_F(L)$ is a bipartite graph. Conversely, let $\Gamma_F(L)$ be bipartite which is not a star graph. Therefore $\Gamma_F(L)$ has no odd cycle, and so $\mathrm{gr}(\Gamma_F(L)) \neq 3$. By (1), $\mathrm{gr}(\Gamma_F(L)) \neq \infty$. Therefore $\mathrm{gr}(\Gamma_F(L)) = 4$.

(3) If $\mathrm{gr}(\Gamma_F(L)) = 3$, then we are done. Conversely, let $\mathrm{gr}(\Gamma_L(L)) \neq 3$. If $\mathrm{gr}(\Gamma_F(L)) = 4$, then (2) and [6, Theorem 4.7] make a contradiction. If $\mathrm{gr}(\Gamma_F(L)) = \infty$, then $\Gamma_F(L)$ is a star graph which is a contradiction.

(4) First we show that if $a - b - c - d$ is a path in $\Gamma_F(L)$ such that the edge $b - c$ is not contained in a 3-cycle and a,b,c,d are vertices, then the vertices a

and d are distinct and are adjacent to each other. Clearly $a \neq d$. If $a \vee d \notin F$, then F is a filter and $a \vee b \in F$ gives $(a \vee d) \vee b \in F$; hence $a \vee d \in I_F(L)$. Thus $a \vee d - b - c - a \vee d$ is a 3-cycle, a contradiction. Now let a be an end vertex in $\Gamma_F(L)$ and b be a vertex in $\Gamma_F(L)$ such that a and b are adjacent. Since $\mathrm{gr}(\Gamma_F(L)) < \infty$, $\Gamma_F(L)$ is not a star graph by (1). By Theorem 3.3, $\Gamma_F(L)$ is connected, hence there is a path $a - b - c - d$ in $\Gamma_F(L)$. Since $\mathrm{gr}(\Gamma_F(L)) = 4$, the edge $b - c$ is not contained in a 3-cycle. By the above considerations, $a \neq d$ and a, d are adjacent to each other which is contradiction. □

The connectivity of a graph G, denoted by $k(G)$, is defined to be the minimum number of vertices that are necessary to remove from G in order to produce a disconnected graph.

Theorem 3.7. *Let F be a filter of L.*

(1) $\Gamma_F(L)$ is a complete bipartite graph if and only if there exist two noncomporable (with respect to inclusion) prime filters F_1 and F_2 of L such that $F = F_1 \cap F_2$.

(2) If F is a Q-filter and $\Gamma(L/F)$ is the graph on only two vertices $q_1 \wedge F, q_2 \wedge F$, then the following hold:

(a) $\Gamma_F(L)$ is a complete bipartite graph and $k(\Gamma_F(L)) = \min\{|q_1 \wedge F|, |q_2 \wedge F|\}$.

(b) $F = F_1 \cap F_2$, where $F_1 = (q_1 \wedge F) \cup F$ and $F_2 = (q_2 \wedge F) \cup F$ are prime filters of L.

Proof. (1) Suppose that $F = F_1 \cap F_2$ for some prime filters F_1 and F_2 of L; we show that $\Gamma_F(L)$ is a complete bipartite graph with two parts $V_1 = F_1 \setminus F$ and $V_2 = F_2 \setminus F$. Let $a, b \in L \setminus F$ with $a \vee b \in F$; so either $a \in F_1 \setminus F$ and $b \in F_2 \setminus F$ or $a \in F_2 \setminus F$ and $b \in F_1 \setminus F$ since F_1, F_2 are prime filters. Let $a, b \in I_F(L)$ such that $a \in F_2 \setminus F$ and $b \in F_1 \setminus F$. Then $a \vee b \in F_1 \cap F_2 = F$; hence a, b are adjacent. Now we show that each two elements of V_i are not adjacent. Let $x, y \in V_1$ (so $x, y \notin F$). If $x \vee y \in F$, then $x \vee y \in F_2$ gives $x \in F_2$ or $y \in F_2$. Since $x, y \in V_1 \subset F_1$, $x \in F_1 \cap F_2 = F$ or $y \in F_1 \cap F_2 = F$, a contradiction. Similarly, each two elements of V_2 are not adjacent. So $\Gamma_F(L)$ is complete bipartite with two parts V_1 and V_2.

Conversely, let V_1, V_2 be two parts of $\Gamma_F(L)$. Set $F_1 = V_1 \cup F$ and $F_2 = V_2 \cup F$. One can easily see that $F = F_1 \cap F_2$. First we show that F_1, F_2 are filters of L. Let $x, y \in F_1$; we show that $x \wedge y \in F_1$. If $x, y \in F$, then F is filter gives $x \wedge y \in F \subseteq F_1$. So we may assume that either $x \notin F$ or $y \notin F$. If $x, y \in V_1$, then we have $x \vee z \in F$ and $y \vee z \in F$ for each $z \in V_2$ since $\Gamma_F(L)$ is complete bipartite; so $x, y \in (F : z)$ gives $x \wedge y \in (F : z)$ (so $(x \wedge y) \vee z \in F$) since it is a filter. If $x \wedge y \in F$, then $x \in F$ and $y \in F$ by Proposition 1.1(b) which is a contradiction. Thus $x \wedge y \notin F$, and so $x \wedge y \in I_F(L)$. Since $(x \wedge y) \vee z \in F$ for each $z \in V_2$, $x \wedge y \in V_1 \subseteq F_1$. If $x \in V_1$ and $y \in F$, then $x \vee z, y \vee z \in F$ for each $z \in V_2$ and $x \wedge y \notin F$. As F is a semiprime filter, $(x\wedge y)\vee z\in F$ which gives $x\wedge y\in V_1\subseteq F_1$. Thus $x\wedge y\in F_1$. Now suppose that $x\in F_1$ and $r\in L$; we show that $x\vee r\in F_1$. If $x\in F$, then $x\vee r\in F\subseteq F_1$. If $x\in V_1$, then $x\vee z\in F$ for each $z\in V_2$. Since F is a filter, $(x\vee r)\vee z\in F$ for each $r\in L$. If $x\vee r\notin F$, then $x\vee r\in V_1\subseteq F_1$ (because $z\in V_2$ and $\Gamma_F(L)$ is bipartite). If $x\vee r\in F$, then $x\vee r\in F_1$. Therefore F_1 is a filter of L by Proposition 1.1(a). Now we claim that F_1 is prime. Let $a\vee b\in F_1$ such that $a,b\notin F_1$; so $a,b\notin F$. If $a\vee b\in F$, then either $a\in V_1$ and $b\in V_2$ or $a\in V_2$ and $b\in V_1$ which is a contradiction since $a,b\notin F_1$. Thus $a\vee b\notin F$. If $a\vee b\in V_1$, then $a\vee b\vee c\in F$ for each $c\in V_2$. If $b\vee c\in F$, then $c\in V_2$ gives $b\in V_1$, a contradiction. Hence $b\vee c\notin F$. By the similar way, $a\vee c\notin F$. Since $a\vee(b\vee c)\in F$ and $a\notin V_1$, we have $a\in V_2$ and $b\vee c\in V_1$. Likewise, $b\in V_2$ and $a\vee c\in V_1$. As $a\vee b\vee c\in F$, $(a\vee c)\vee(b\vee c)\in F$. It shows that two vertices $a\vee c$ and $b\vee c$ of V_1 are adjacent, a contradiction. Thus F_1 is a prime filter of L. Similarly, F_2 is a prime filter of L.

(2) (a) Since $q_1\wedge F$ and $q_2\wedge F$ are the only vertices of $\Gamma(L/F)$ and $\Gamma(L/F)$ has no loop, $(q_1\wedge F)\vee_Q(q_2\wedge F)=(q_1\vee q_2)\wedge F=1\wedge F=F$. Since by Theorem 3.2, all elements of $q_1\wedge F$ and $q_2\wedge F$ are adjacent and none of elements of $q_i\wedge F$ are adjacent together, we get $\Gamma_F(L)$ is a complete bipartite graph and $k(\Gamma_F(L))=\min\{|q_1\wedge F|,|q_2\wedge F|\}$. (b) follows by the proof of (1). □

In the following example it is shown that Theorem 3.7(1) does not hold for non-semiprime filters.

Example 3.8. Let L be the lattice N_5 (Example 2.5) and $F=[a)$. By Example 2.5 F is not semiprime. It can be seen that $I_F^*(L)=\{b,c\}$ and $\Gamma_F(L)$ is complete 2-partite. However, F is not intersection of two prime filters.

In the following, we give a description of a lower bound for the independence number of $\Gamma_F(L)$.

Proposition 3.9. *Let F be a filter of L. The following hold:*

(1) If $\min(F)=\{F_1,...,F_n\}$, then $\alpha(\Gamma_F(L))\geq max\{|I_F(L)\setminus F_1|,...,|I_F(L)\setminus F_n|\}$.

(2) If F is a Q-filter and $\mathcal{I}$ is a maximal independent set in $\Gamma(L/F)$, then $\alpha(\Gamma_F(L))\geq\sum_{q\wedge F\in\mathcal{I}}|q\wedge F|$.

(3) If F is a Q-filter, then $\alpha(\Gamma(L/F))\leq\alpha(\Gamma_F(L))$.

Proof. (1) We show that for each $1\leq i\leq n$, $I_F(L)\setminus F_i$ is an independence set. Let $x,y\in I_F(L)\setminus F_i$. If $x\vee y\in F$, then $x\vee y\in F_i$. So either $x\in F_i$ or $y\in F_i$, a contradiction.

(2) By Theorem 3.2, for each $q\in Q$, $q\wedge F$ is an independence set. Moreover, if $q_1\wedge F$ and $q_2\wedge F$ are not adjacent in $\Gamma(L/F)$, then every element of $q_1\wedge F$

is not adjacent to every element of $q_2 \wedge F$. Therefore $\alpha(\Gamma_F(L)) \geq max\{|q \wedge F| : q \in Q\}$.

(3) It is clear from (2). □

Proposition 3.10. *Let F be a Q-filter of L. The following hold:*

(1) Let S be a nonempty subset of $I_F^*(L)$. If S is a dominating set of $\Gamma_F(L)$, then $T = \{q \wedge F : s \in q \wedge F, s \in S\}$ is a dominating set of $\Gamma(L/F)$.

(2) $\gamma(\Gamma(L/F)) \leq \gamma(\Gamma_F(L))$.

Proof. (1) Let S be a dominating set of $\Gamma_F(L)$ and $p \wedge F$ be a vertex of $\Gamma(L/F)$ ($p \in Q$). Then $p \notin F$ and it is a vertex of $\Gamma_F(L)$. Hence there exists $s \in S$ such that $s \vee p \in F$. Let $q \wedge F$ be an element of T such that $s \in q \wedge F$. As $p \vee s \in F$, $(p \wedge F) \vee_Q (q \wedge F) = F$, by Theorem 3.2. Therefore $T = \{q \wedge F : s \in q \wedge F, s \in S\}$ is a dominating set of $\Gamma(L/F)$.

(2) It is clear from (1). □

The following example shows that the converse of Proposition 3.10 is not necessarily true.

Example 3.11. Let L and F be as described in the Example 2.12. Then $I_F(L) = \{\{1\}, \{2\}, \{1,3\}, \{2,3\}\}$ and $L/F = \{F, \{3\} \wedge F, \{1,3\} \wedge F, \{2,3\} \wedge F\}$ gives $\Gamma(L/F)$ has only two vertices $\{1,3\} \wedge F = \{\{1\}, \{1,3\}\}$ and $\{2,3\} \wedge F = \{\{2\}, \{2,3\}\}$. It can be seen that $\{1,3\} \wedge F$ is a dominating set of $\Gamma(L/F))$. However, $\{\{1\}, \{1,3\}\}$ is a dominating set of $\Gamma_F(L)$ and $\Gamma_F(L)$ cannot be dominated by any set of one vertex. Moreover $\gamma(\Gamma(L/F)) < \gamma(\Gamma_F(L))$.

Proposition 3.12. *Let F be a filter of L. Then*

(1) If F is a Q-filter, then $w(\Gamma_F(L)) \leq |Q| - 2$.

(2) If $w(\Gamma_F(L))$ is finite, then L has a.c.c on filters of the form $(F : x)$, where $x \in L$. Moreover, if $(F : x_i)$ and $(F : x_j)$ are distinct maximal elements of $(\Sigma = \{(F : x) : x \in L \setminus F\}, \subseteq)$, then x_i is adjacent to x_j in $\Gamma_F(L)$.

Proof. (1) If $w(\Gamma_F(L)) = \infty$, then there is an infinite clique $C \subseteq I_F(L)$. Since each two distinct elements $x_1, x_2 \in C$ are adjacent, $x_1 \in q_1 \wedge F$, and $x_2 \in q_2 \wedge F$ for some $q_1, q_2 \in Q$ with $q_1 \neq q_2$ by Theorem 3.2, we have $|Q| = \infty$. Assume that $w(\Gamma_F(L)) = n$ and let $x_1, x_2, ..., x_n$ be the vertices of the greatest complete subgraph of $\Gamma_F(L)$. Since F is a Q-filter, there exist unique elements $q_i \in Q$ such that $x_i \in q_i \wedge F$ ($1 \leq i \leq n$). By Theorem 3.2, $q_i \neq 0, q_i \neq 1$, and $q_i \neq q_j$ for each $1 \leq i \neq j \leq n$. Thus $w(\Gamma_F(L)) \leq |Q| - 2$.

(2) Let $(F : x_1) \subseteq (F : x_2) \subseteq \cdots$ be an ascending chain of filters of L, where $x_i \in L$. If $y_i \in (F : x_i) \setminus (F : x_{i-1})$ ($i \geq 2$), then $y_i \vee x_i \in F$ and $y_i \vee x_{i-1} \notin F$. We show that $y_i \vee x_{i-1} \neq y_j \vee x_{j-1}$ for each $i \neq j$. We can assume that $i < j$. Then $y_i \in (F : x_j)$; so $y_i \vee x_j \in F$. If $y_i \vee x_{i-1} = y_j \vee x_{j-1}$, then $y_i \vee x_{i-1} =$

$(y_i \vee x_{i-1}) \vee (y_j \vee x_{j-1}) \in F$ since F is a filter, a contradiction. So for each $i \neq j$, $y_i \vee x_{i-1} \neq y_j \vee x_{j-1}$. Hence $\{y_i \vee x_{i-1}\}_{2 \leq i \leq n}$ is an infinite clique which is a contradiction. Now, if $(F : x_i)$ and $(F : x_j)$ are distinct maximal elements of $\sum$, then by the usual argument, one can show that $(F : x_i)$ and $(F : x_j)$ are prime. We show $x_i \vee x_j \in F$. If not, then $(F : x_i) \subseteq (F : x_i \vee x_j)$ and $(F : x_j) \subseteq (F : x_i \vee x_j)$, and hence $(F : x_i) = (F : x_i \vee x_j) = (F : x_j)$, a contradiction. □

The next theorem investigate the relationship between the chromatic number and clique number of the graph $\Gamma_F(L)$.

Theorem 3.13. *Let F be a filter of L. Then the following are equivalent.*

(1) $\chi(\Gamma_F(L))$ is finite;

(2) $w(\Gamma_F(L))$ is finite;

(3) The filter F is a finite intersection of prime filters.

Proof. (1)⇒ (2) It is known that $w(\Gamma_F(L)) \leq \chi(\Gamma_F(L))$.

(2) ⇒ (3) Assume that $w(\Gamma_F(L)) = n$ and let $\sum = \{(F : x) : x \in L \setminus F\}$. By Proposition 3.12, $(\sum, \subseteq)$ has a maximal element. Let $(F : x_i)$ $(i \in J)$ be the different maximal members of the set $\sum$. By the usual argument, for each $i \in J$, $(F : x_i)$ is prime. Therefore $\{x_i\}_{i \in J}$ is a clique in $\Gamma_F(L)$ by Proposition 3.12(2). Also, $w(\Gamma_F(L))$ is finite gives J is a finite set. Now we show that $F = \cap_{i \in J}(F : x_i)$. Let $x \in L \setminus F$. Then $(F : x) \subseteq (F : x_i)$ for some $i \in J$. We claim that $x \notin (F : x_i)$. Otherwise, $x \vee x_i \in F$, so $x_i \in (F : x) \subseteq (F : x_i)$; hence $x_i = x_i \vee x_i \in F$, a contradiction. Therefore $x \notin (F : x_i)$, and so $x \notin \cap_{i \in J}(F : x_i)$. Hence $\cap_{i \in J}(F : x_i) \subseteq F$. The other side of inclusion is clear, and so we have equality.

(3) ⇒ (1) Let $F = \cap_{i=1}^{n} F_i$, where for each $1 \leq i \leq n$, F_i is a prime filter of L which contains F. If x is adjacent to y in $\Gamma_F(L)$, then there is no $1 \leq i \leq n$ such that $x \notin F_i$ and $y \notin F_i$. Therefore we can label any vertex x of $\Gamma_F(L)$ by $\min\{i : x \notin F_i\}$. This implies that $\chi(\Gamma_F(L)) \leq n$. □

Theorem 3.14. *Let F be a filter of L. The following hold:*

(1) If F is not a prime filter, then $w(\Gamma_F(L)) = |\min(F)|$;

(2) If F is a Q-filter with $|\min(F)|$ finite, then each $P \in \min(F)$ is of the form $P = (F : q)$ for some $q \in Q$;

Proof. (1) First we prove that $|\min(F)|$ is finite if and only if $w(\Gamma_F(L))$ is finite. If $|\min(F)|$ is finite, then F is a finite intersection of prime filters by Proposition 2.3; so by Theorem 3.13, $w(\Gamma_F(L))$ is finite. Now assume that $w(\Gamma_F(L))$ is finite. Hence by Theorem 3.13, $F = \cap_{i=1}^{n} F_i$ for some prime filters F_i of L. Let $\{P_\alpha\}_{\alpha \in \Lambda} = \min(F)$. For each $\alpha \in \Lambda$, $F \subseteq P_\alpha$, so $\cap_{i=1}^{n} F_i \subseteq P_\alpha$ for each $\alpha \in \Lambda$; hence $F_i \subseteq P_\alpha$ for some $1 \leq i \leq n$ by Lemma 2.1. Since F_α is minimal, $F_i = P_\alpha$. This gives Λ is finite, and so $|\min(F)|$ is finite.

Let $|\min(F)| = n$ and $\min(F) = \{F_1, ..., F_n\}$. By Proposition 2.3, there exists $a_j \in (\cap_{1\leq i\leq n, i\neq j} F_i) \setminus F_j$ for each $1 \leq j \leq n$. Since each F_i is a filter, $a_i \vee a_j \in F$; hence $A = \{a_1, a_2, \cdots, a_n\}$ is a clique in $\Gamma_F(L)$, and so $w(\Gamma_F(L)) \geq n$. Now we show that $w(\Gamma_F(L)) \leq n$. The proof is by induction on n. If $n = 2$, then $\Gamma_F(L)$ is a complete bipartite graph by Theorem 3.7(1); hence $w(\Gamma_F(L)) = 2$. Suppose $n > 2$ and the result is true for any integer less than n. Let $\{a_1, a_2, \cdots, a_m\}$ be a clique in $\Gamma_F(L)$; so $a_1 \vee a_j \in F = \cap_{1\leq i\leq n} F_i$. Without loss generality, suppose that $a_1 \notin F_1$ and $a_2, a_3, \cdots, a_m \in F_1$ and $a_2, ..., a_m \notin \cap_{2\leq i\leq n} F_i$. Set $F' = \cap_{2\leq i\leq n} F_i$. Then $\{a_2, a_3, \cdots, a_m\}$ is a clique in $\Gamma_{F'}(L)$. By induction hypothesis, $m-1 \leq n-1$, and so $m \leq n$, as required.

(2) Let F be a Q-filter with $|\min(F)| = n$. By Proposition 2.3, $F = \cap_{i=1}^{n} F_i$ where $\min(F) = \{F_1, \cdots, F_n\}$. Then by (1), $w(\Gamma_F(L)) = n$. Let $\{a_1, a_2, \cdots, a_n\}$ be a clique in $\Gamma_F(L)$, where $a_j \in (\cap_{1\leq i\leq n, i\neq j} F_i) \setminus F_j$. By assumption, there exists unique element $q_j \in Q$ such that $a_j \in q_j \wedge F$ for each $1 \leq j \leq n$. Since $\{a_1, a_2, \cdots, a_n\}$ is a clique in $\Gamma_F(L)$, $\{q_1, q_2, \cdots, q_n\}$ is a clique in $\Gamma_F(L)$ by Theorem 3.2. If $a_j = q_j \wedge b_j$ for some $b_j \in F$, then we show that $q_j \in \cap_{1\leq i\leq n, i\neq j} F_i \setminus F_j$. It suffices to show that $q_j \notin F_j$ and there is no $i \neq j$ such that $q_j \notin F_i$. If $q_j \in F_j$, then $x_j = q_j \wedge b_j \in F_j$ since F_j is a filter, a contradiction. So $q_j \notin F_j$. Also, if $q_j \notin F_i$ for some $i \neq j$, then $q_i \vee q_j \notin F_i$ and hence $q_i \vee q_j \notin F$, a contradiction (similarly, as $a_i \notin F_i$, $q_i \notin F_i$). Therefore $q_j \in \cap_{1\leq i\leq n, i\neq j} F_i \setminus F_j$. We claim that $(F : q_j) = F_j$. Let $x \in (F : q_j)$. Then $x \vee q_j \in F$, and so $x \vee q_j \in F_j$; hence $x \in F_j$ since F_j is prime. Thus $(F : q_j) \subseteq F_j$. For the reverse of inclusion, let $x \in F_j$. Then $q_j \in \cap_{1\leq i\leq n, i\neq j} F_i \setminus F_j$ gives $x \vee q_j \in F$. Therefore $F_j \subseteq (F : q_j)$, and we have equality. □

We prove the Beck's conjecture for the identity-summand graph of lattices based on semiprime filters in the following theorem.

Theorem 3.15. *Let F be a filter of L. Then* $\chi(\Gamma_F(L)) = w(\Gamma_F(L))$.

Proof. It is known that $w(\Gamma_F(L)) \leq \chi(\Gamma_F(L))$. Let $w(\Gamma_F(L)) = n$. By Theorem 3.13, $F = \cap_{i=1}^{n} F_i$, where for each i, F_i is a minimal prime filter. By an argument like that in the Theorem 3.13 ((3) $\Rightarrow$ (1)), $\chi(\Gamma_F(L)) \leq n$. Therefore $\chi(\Gamma_F(L)) = w(\Gamma_F(L))$. □

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ [6]. It is natural to ask: for which filter F of L, the $\Gamma_F(L)$ is planar?

Theorem 3.16. *Let F be a filter of L.*

(1) If F is a Q-filter and $\Gamma_F(L)$ is planar, then for each edge $q_1 \wedge F$ - $q_2 \wedge F$ of $\Gamma(L/F)$, $|q_i \wedge F| \leq 2$ for some $1 \leq i \leq 2$;

(2) If F is a Q-filter and $\Gamma_F(L)$ is planar, then $\Gamma(L/F)$ is planar;

(3) If $|\min(F)| \geq 5$, then $\Gamma_F(L)$ is not planar;

(4) If $|\min(F)| = 4$, then $\Gamma_F(L)$ is not planar.

Proof. (1) Assume that $\Gamma_F(L)$ is planar and let $q_1 \wedge F$ and $q_2 \wedge F$ are two vertices of $\Gamma(L/F)$ such that $|q_i \wedge F| \geq 3$ for each $i = 1,2$. Let $V_1 = \{a_1,a_2,a_3\} \subseteq q_1 \wedge F$ and $V_2 = \{b_1,b_2,b_3\} \subseteq q_2 \wedge F$. As $q_1 \wedge F$ and $q_2 \wedge F$ are adjacent in $\Gamma(L/F)$, a_i and b_j are adjacent in $\Gamma_F(L)$ by Theorem 3.2. Then V_1 and V_2 are two parts of a complete bipartite graph as a subgraph of $\Gamma_F(L)$. Hence $\Gamma_F(L)$ is not planar.

(2) Let $\Gamma_F(L)$ be planar. By Theorem 3.2, two vertices $q_1 \wedge F$ and $q_2 \wedge F$ are adjacent in $\Gamma(L/F)$ if and only if q_1 and q_2 are adjacent in $\Gamma_F(L)$. Hence we can take $\Gamma(L/F)$ as a subgraph of $\Gamma_F(L)$. If $\Gamma(L/F)$ is not planar, then $\Gamma_F(L)$ is not planar, a contradiction. So $\Gamma(L/F)$ is planar.

(3) This follows from Theorem 3.14(1).

(4) By Theorem 3.14(1), $w(\Gamma_F(L)) = 4$. Hence there exists $\{a_1,a_2,a_3,a_4\} \subseteq I_F(L)$ such that $\{a_1,\cdots,a_4\}$ forms a clique in $\Gamma_F(L)$. Let $a_{ij} = a_i \wedge a_j$, where $1 \leq i,j \leq 4$, $i \neq j$. Suppose that $1 \leq k \neq i, k \neq j \leq 4$. Since $a_i, a_j \in (F : a_k)$, $a_{ij} \in (F : a_k)$ since it is a filter. If $a_{ij} \in F$, then $a_{ij} \vee a_i \in F$ since F is a filter, so $a_{ij} \vee a_i = a_i \wedge (a_i \vee a_j) = a_i \in F$ which is a contradiction; hence $a_{ij} \in I_F(L)$. We claim that $a_{ij} \notin \{a_1,a_2,a_3,a_4\}$. Assume that $a_{ij} = a_s$ for some $1 \leq s \leq 4$. If $s = i$, then $a_{ij} \vee a_j \in F$. This implies that $a_i \in F$, a contradiction. Similarly, for $s = j$. If $s \neq j$ and $s \neq i$, then $a_{ij} \vee a_s \in F$; hence $a_s \vee a_s = a_s \in F$, a contradiction. Therefore $a_{ij} \notin \{a_1,a_2,a_3,a_4\}$. Let $s \neq k$ and $s,k \in \{1,2,3,4\} - \{i,j\}$. Since $a_{ij} \vee a_s \in F$ and $a_{ij} \vee a_k \in F$, we have $a_s, a_k \in (F : a_{ij})$; thus $a_{sk} \in (F : a_{ij})$. Set $V_1 = \{a_1,a_{13},a_3\}$ and $V_2 = \{a_2,a_{24},a_4\}$. Then V_1 and V_2 are two parts of a complete 2-partite subgraph of $\Gamma_F(L)$. Therefore $\Gamma_F(L)$ is not planar. □

Example 3.17. Let L and F be as described in the Example 2.12. Then $I_F(L) = \{\{1\},\{2\},\{1,3\},\{2,3\}\}$. Moreover, $L/F = \{F, \{3\} \wedge F, \{1,3\} \wedge F, \{2,3\} \wedge F\}$ gives $\Gamma(L/F)$ has only two vertices $\{1,3\} \wedge F = \{\{1\},\{1,3\}\}$ and $\{2,3\} \wedge F = \{\{2\},\{2,3\}\}$. Then $\mathrm{gr}(\Gamma_F(L)) = 4$ by Theorem 3.6(2); hence $\Gamma_F(L)$ is bipartite but not a star graph. By Theorem 3.6, $F_1 = \{D,\{1,2\},\{1\},\{1,3\}\}$ and $F_2 = \{D,\{1,2\},\{2\},\{2,3\}\}$ are prime filters of L with $F = F_1 \cap F_2$.

Remark 3.18. Let F be a filter of L.

(1) If $|\min(F)| = 1$, then by Proposition 2.3, F is a prime filter of L; hence $\Gamma_F(L) = \emptyset$ by Proposition 3.1(1).

(2) If $|\min(F)| = 2$, then $F = F_1 \cap F_2$ for some prime filters F_1 and F_2 by Proposition 2.3. Hence by Theorem 3.7, $\Gamma_F(L)$ is $K_{n,m}$ for some integer n and

m, where $|F_1 \setminus F| = n$ and $|F_2 \setminus F| = m$. If $n, m \geq 3$, then $K_{3,3}$ is a subgraph of $\Gamma_F(L)$ and so $\Gamma_F(L)$ is not planar.

(3) If $|\min(F)| \geq 4$, then by Theorem 3.16, $\Gamma_F(L)$ is not planar.

(4) It is not entirely clear for us which semiprime filter F of a lattice L with $|\min(F)| = 3$, the $\Gamma_F(L)$ is planar.

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