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A NOTE ON THE SPECTRUM OF DIAGONAL PERTURBATION OF WEIGHTED SHIFT OPERATOR

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This note provides a complete description of the spectrum of diagonal perturbation of weighted shift operator acting on a separable Hilbert space.

1. Introduction

When working with infinite-dimensional linear dynamical systems, the spectral theory is often helpful in determining of the asymptotic properties, as a stability, cyclicity, hypercyclicity, chaoticity... (see [3, 4, 8, 10, 16–18, 23, 24]). On the other hand the shift operator and his perturbation are a natural and appropriate setting for testing those concepts (see [9, 11, 13, 22]).

Throughout this paper X will denote a separable complex Hilbert space with an orthonormal basis $\{e_i\}_i \subset X$.

Let $\mathscr{B}(X)$ denote the algebra of all bounded linear operators acting on X. The norm induced by the inner product on X and the associated operator norm on $\mathscr{B}(X)$ are both denoted by $\|\cdot\|$. On the other hand, for $T \in \mathscr{B}(X)$, we denote by $\sigma(T)$, $\rho(T)$ and $r(T)$ the spectrum, the resolvent and the spectral radius of T respectively. Recall that $\sigma(T)$ is a non-empty compact subset of

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$\mathbb{C}$, $r(T) \leq \|T\|$ and $r(T) = \lim \left\|T^k\right\|^{\frac{1}{k}} = \inf \left\|T^k\right\|^{\frac{1}{k}}$. If T is invertible, the inverse is denoted by T^{-1} and we have $\sigma(T^{-1}) = \left\{\frac{1}{\lambda} : \lambda \in \sigma(T)\right\}$. Moreover, $\frac{1}{r(T^{-1})} = \inf\{|\lambda| : \lambda \in \sigma(T)\}$ (see [6, 7, 12, 14, 15, 19, 20]).

A bounded linear operator S on X is called a weighted shift operator with weight sequence $\{w_i\}_i \subset \mathbb{C}$, if

$$Se_i = w_i e_{i+1}. \tag{1}$$

If the index i runs over the set $\mathbb{N}$ of non-negative integers, then S is called a unilateral weighted shift and it is called a bilateral weighted shift when i runs over the set $\mathbb{Z}$ of all integers. Such operators have been studied by many authors in relation to various applications (theory of dynamical systems, integro-functional, differential-functional, functional and difference equations, nonlocal boundary value problems, nonclassical boundary value problems for certain partial differential equations, the general theory of operator theory, etc. see [1, 2, 12, 20**?**]). In [5, 6, 25], it is shown that if S is bounded, then there exists $0 \leq r^- \leq r^+$ such that the spectrum $\sigma(S)$ of S is given by

$$\sigma(S) = \left\{\lambda \in \mathbb{C} : r^- \leq |\lambda| \leq r^+\right\}.$$

In this work, we propose to extend this type of result to the case of the perturbed operator $S + D$, where D is a diagonal operator.

2. The spectrum of perturbed weighted shift

Through the following results, we provide a complete description of the spectrum of diagonal perturbation of unilateral and bilateral weighted shift operators.

2.1. The spectrum of perturbed unilateral weighted shift

Let $T \in \mathscr{B}(X)$ be a diagonal perturbation of a unilateral weighted shift S on X. That is

$$T := S + D, \tag{2}$$

where D is a diagonal operator on X with diagonal entries $(d_i)_{i\in\mathbb{N}}$; i.e. $De_i := d_i e_i$ for all $i \in \mathbb{N}$.

Theorem 2.1. *Let $T \in \mathscr{B}(X)$ be the operator given by (2) and set*

$$R_T(\lambda) = \lim_{k\to\infty} \left[\sup_{i\in\mathbb{Z}} \left| \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} (d_{i+m} - \lambda)} \right| \right]^{\frac{1}{k}}, \ (\lambda \in \mathbb{C}). \tag{3}$$

Then

$$\sigma(T) = \{\lambda \in \mathbb{C} : R_T(\lambda) \geq 1\}. \tag{4}$$

Proof. Let $\lambda \notin \sigma(T)$ then $T - \lambda I$ be invertible. For $i \in \mathbb{N}$, set $x_i = \sum_{j\in\mathbb{N}} a_j^i e_j = (T - \lambda I)^{-1} e_i$ we have

$$\begin{cases} w_{i-1}a_{i-1}^i + (d_i - \lambda)a_i^i = 1, \\ \\ w_{j-1}a_{j-1}^i + d_j a_j^i = 0, \qquad \text{if } j \neq i \end{cases} \tag{5}$$

and

$$\begin{cases} w_i a_i^{i+1} + (d_i - \lambda)a_i^i = 1, \\ \\ w_i a_j^{i+1} + d_i a_j^i = 0, \qquad \text{if } j \neq i. \end{cases} \tag{6}$$

The first equation of (5) implies that

$$a_i^i = \frac{1 - a_{i-1}^i w_{i-1}}{d_i - \lambda}, \tag{7}$$

for all $i \in \mathbb{N}$. Since $d_0 \neq \lambda$ (otherwise $T - \lambda I$ would be non-invertible) and $w_{-1} = 0$, then

$$a_0^0 = \frac{1}{d_0 - \lambda}, \tag{8}$$

The first equation of (6) implies that

$$a_{i-1}^i = \frac{1 - (d_{i-1} - \lambda)a_{i-1}^{i-1}}{w_{i-1}}. \tag{9}$$

for all $i \in \mathbb{N}$. From (7)-(9), we have

$$a_i^i d_i = 1, \text{ for every } i \in \mathbb{N}.$$

Consequently, for all $i \in \mathbb{N}$ and $k \geq 0$, we have

$$a_{i+k}^i = \left\langle (T - \lambda)^{-1} e_i, e_{i+k} \right\rangle = (-1)^k \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} (d_{i+m} - \lambda)},$$

with the assumption that $\prod_{m=0}^{-1} w_{i+m} = 1$. Cauchy-Schwarz inequality provides the inequality

$$\left| \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} (d_{i+m} - \lambda)} \right| \leq \left\| (T - \lambda I)^{-1} \right\|, \text{ for every } i \in \mathbb{N} \text{and } k \geq 0.$$

By passing to the supremum over $i \in \mathbb{N}$, we get

$$\sup_{i\in\mathbb{N}} \left| \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} (d_{i+m} - \lambda)} \right| \leq \left\| (T - \lambda I)^{-1} \right\|, \text{ for every } k \geq 0. \tag{10}$$

Taking the kth root and letting $k \longrightarrow \infty$ in (10), we get $R_T(\lambda) \leq 1$. But the equality is excluded by the spectrum compactness. So we have

$$R_T(\lambda) < 1. \tag{11}$$

So, $\lambda \notin \{\lambda \in \mathbb{C} : R_T(\lambda) \geq 1\}$ and then

$$\{\lambda \in \mathbb{C} : R_T(\lambda) \geq 1\} \subset \sigma(T).$$

Conversely, in order to show that

$$\sigma(T) \subset \{\lambda \in \mathbb{C} : R_T(\lambda) \geq 1\},$$

we take $\lambda \in \mathbb{C}$, such that (11) is verified and we show that $\lambda \notin \sigma(T)$. Suppose that $R_T(\lambda) < 1$ and let F be a linear operator on X, defined by

$$F := \sum_{k=0}^{\infty} F_k, \tag{12}$$

such as, for all $k \in \mathbb{N}$, F_k is an operator on X given by

$$F_k e_i = a_{i+k}^i e_{i+k}, \ i = 0, 1, 2, ... \tag{13}$$

and

$$a_{i+k}^i = (-1)^n \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} (d_{i+m} - \lambda)}, \tag{14}$$

with the assumption that $\prod_{m=0}^{-1} w_{i+m} = 1$. Since $\|F_k\| = \sup_{i\in\mathbb{N}} \left|a_{i+k}^i\right|$ then the inequality (11), implies that the operator F is well defined, $\|F\| < \infty$ and $\lim_{k\to\infty} a_{i+k}^{i+1} = \lim_{k\to\infty} a_{i+k}^i = 0$. From (12)-(14), and for all $i \in \mathbb{N}$, we have $(F \circ (T - \lambda I)) e_i = ((T - \lambda I) \circ F) e_i = e_i$, then $\lambda \notin \sigma(T)$. The proof of theorem is now complete. □

2.2. The spectrum of perturbed bilateral weighted shift

Let $T \in \mathscr{B}(X)$ be a diagonal perturbation of bilateral weighted shift S on X. That is

$$T := S + D, \tag{15}$$

where D is a diagonal operator with diagonals entries $(d_i)_{i\in\mathbb{Z}}$; i.e., $De_i := d_i e_i$ for all $i \in \mathbb{Z}$.

Lemma 2.2. *If T is invertible, then at least one of the following two inequalities holds*

$$R_T^+ = \lim_{k\to\infty}\left[\sup_{i\in\mathbb{Z}}\left|\frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} d_{i+m}}\right|\right]^{\frac{1}{k}} \leq 1 \tag{16}$$

or

$$R_T^- = \lim_{k\to\infty}\left[\sup_{i\in\mathbb{Z}}\left|\frac{\prod_{m=1}^{k-1} d_{i-m}}{\prod_{m=1}^{k} w_{i-m}}\right|\right]^{\frac{1}{k}} \leq 1. \tag{17}$$

Proof. Let T be invertible and set $x_i = \sum_{j\in\mathbb{Z}} a_j^i e_j = T^{-1}e_i$. Thus, we have

$$\begin{cases} w_{j-1}a_{j-1}^i + d_j a_j^i = 1, \text{ if } j = i, \\ \\ w_{j-1}a_{j-1}^i + d_j a_j^i = 0, \text{ otherwise.} \end{cases} \tag{18}$$

and

$$\begin{cases} w_i a_j^{i+1} + d_i a_j^i = 1, \quad \text{if } j = i, \\ \\ w_i a_j^{i+1} + d_i a_j^i = 0, \quad \text{otherwise.} \end{cases} \tag{19}$$

The first equation of (18) and of (19) implies that

$$a_i^i d_i = a_0^0 d_0, \tag{20}$$

for all $i \in \mathbb{Z}$. From (18), we get, for all $i \in \mathbb{Z}$ and $k > 0$,

$$a_{i-k}^i = \langle T^{-1}e_i, e_{i-k}\rangle = (-1)^{k+1}\frac{(1 - d_i a_i^i)\prod_{m=1}^{k-1} d_{i-m}}{\prod_{m=1}^{k} w_{i-m}},$$

assuming that $\prod_{m=1}^{0} d_{i-m} = 1$. Cauchy-Schwarz inequality gives us

$$\left| \frac{(1-d_0a_0^0)\prod_{m=1}^{k-1} d_{i-m}}{\prod_{m=1}^{k} w_{i-m}} \right| \leq \left\| T^{-1} \right\|, \text{ for every } i \in \mathbb{Z} \text{ and } k \geq 0. \tag{21}$$

Consequently, for all $i \in \mathbb{Z}$ and $k > 0$, we have

$$a_{i+k}^{i} = \left\langle T^{-1}e_i\,, e_{i+k} \right\rangle = (-1)^k \frac{d_ia_i^i \prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} d_{i+m}},$$

Cauchy-Schwarz inequality provides the inequality

$$\left| d_0a_0^0 \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} d_{i+m}} \right| \leq \left\| T^{-1} \right\|, \text{ for every } i \in \mathbb{Z} \text{ and } k > 0. \tag{22}$$

From the first equation of (18), for all $i \in \mathbb{Z}$, either $w_{i-1}a_{i-1}^{i}$ or $d_ia_i^i$ is not zero. Thus, we can distinguish two cases:

1st case: $d_ia_i^i \neq 0$, from (20) and by taking the supremum over i in (21), we get

$$\sup_{i\in\mathbb{Z}} \left| \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} d_{i+m}} \right| < \infty, \text{ for every } k > 0. \tag{23}$$

2nd case: $w_{i-1}a_{i-1}^{i} \neq 0$, from (20) and by taking the supremum over i in (22) , we get

$$\sup_{i\in\mathbb{Z}} \left| \frac{\prod_{m=1}^{k-1} d_{i-m}}{\prod_{m=1}^{k} w_{i-m}} \right| < \infty, \text{ for every } k > 0. \tag{24}$$

We conclude, by taking the kth root and letting $k \longrightarrow \infty$ in (23) and (24). □

In the folow, we give a converse of the previous lemma.

Lemma 2.3. *If $R_T^+ < 1$ or $R_T^- < 1$ then T is invertible.*

Proof. Note that for all $k>0$

$$\left[\sup_{i\in\mathbb{Z}}\left|\frac{\prod_{m=0}^{k} d_{i-m}}{\prod_{m=0}^{k} w_{i-m}}\right|\right]^{-1} = \inf_{i\in\mathbb{Z}}\left|\frac{\prod_{m=0}^{k} w_{i-m}}{\prod_{m=0}^{k} d_{i-m}}\right| \leq \sup_{i\in\mathbb{Z}}\left|\frac{\prod_{m=0}^{k} w_{i+m}}{\prod_{m=0}^{k} d_{i+m}}\right|, \tag{25}$$

then only one of inequality $R_T^+ < 1$ or $R_T^- < 1$ can be satisfied.

Let F be a linear operator on X to X, defined by

$$F := \sum_{k=0}^{\infty} F_k, \tag{26}$$

such as, for all $k\in\mathbb{N}$, F_k is an operator given by

$$F_k e_i = a_{i+k}^i e_{i+k},\ i = 0,\pm1,\pm2,... \tag{27}$$

and

$$a_{i+k}^i = (-1)^k \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} d_{i+m}} \tag{28}$$

with the assumptions that $\prod_{m=0}^{-1} w_{i+m} = 1$. If $R_T^+ < 1$ then the operator F is well defined, $\|F\| < \infty$ and $\lim_{k\longrightarrow\infty} a_{i+n}^{i+1} = \lim_{k\longrightarrow\infty} a_{i+k}^i = 0$. From (26)-(28), and for all $i\in\mathbb{Z}$, we have$(F\circ T)e_i = (T\circ F)e_i = e_i$, which lead to

$$T\circ F = F\circ T = I,$$

where I denotes the identity operator.

If $R_T^- < 1$, let F' be an operator on X to X, defined by

$$F' := \sum_{k=1}^{\infty} F'_{-k}, \tag{29}$$

and for all $k>0$, F'_{-k} is an operator given by

$$F'_{-k} e_i = a_{i-k}^i e_{i-k},\ i = 0,\pm1,\pm2,... \tag{30}$$

and

$$a_{i-k}^i = (-1)^{k+1} \frac{\prod_{m=1}^{k-1} d_{i-m}}{\prod_{m=0}^{k} w_{i-m}}, \tag{31}$$

with assumptions that $\prod_{m=1}^{0} d_{i-m} = 1$. Note that, the condition $R_T^- < 1$ implies that the operator F' is well defined, $\|F'\| < \infty$ and $\lim_{k\longrightarrow\infty} a_{i-k}^i = 0$. From (29)-(31) and for all $i \in \mathbb{Z}$, we have $(T \circ F')\, e_i = (F' \circ T)\, e_i = e_i$, which leads to

$$T \circ F' = F' \circ T = I,$$

then the claim is proved. □

Theorem 2.4. *Let $T \in \mathscr{B}(X)$ be the operator given by (15) and for any $\lambda \in \mathbb{C}$, $R_T^+(\lambda)$, $R_T^-(\lambda)$ are given by*

$$R_T^+(\lambda) = \lim_{k\to\infty} \left[\sup_{i\in\mathbb{Z}} \left| \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^{k} (d_{i+m} - \lambda)} \right| \right]^{\frac{1}{k}} \tag{32}$$

and

$$R_T^-(\lambda) = \lim_{k\to\infty} \left[\sup_{i\in\mathbb{Z}} \left| \frac{\prod_{m=0}^{k-1} (d_{i-m} - \lambda)}{\prod_{m=0}^{k} w_{i-m}} \right| \right]^{\frac{1}{k}}. \tag{33}$$

(i) If S is an invertible operator, then

$$\sigma(T) = \left\{\lambda \in \mathbb{C} : R_T^+(\lambda) \geq 1 \text{ and } R_T^-(\lambda) \geq 1\right\}; \tag{34}$$

(ii) if S is a non-invertible operator, then

$$\sigma(T) = \left\{\lambda \in \mathbb{C} : R_T^+(\lambda) \geq 1\right\}, \tag{35}$$

Proof. Let $\lambda \in \rho(T) = \left\{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \in \mathscr{B}(X)\right\}$. If we replace d_j by $d_j - \lambda$ in the Lemma 2.2, then we get either $R_T^+(\lambda) \leq 1$ or $R_T^-(\lambda) \leq 1$. But the equality is excluded by spectrum compactness. So we have at least

$$R_T^+(\lambda) < 1 \tag{36}$$

or

$$R_T^-(\lambda) < 1. \tag{37}$$

If S is invertible, then from (25), only one of inequality (36) and (37) can be satisfied. Thus,

$$\left\{\lambda \in \mathbb{C} : R_T^+(\lambda) \geq 1 \;\text{ and } R_T^-(\lambda) \geq 1\right\} \subset \sigma(T).$$

Therefore, if S is non invertible, $\sup_{i\in\mathbb{Z}}\left|\dfrac{\prod_{m=1}^{k-1}(d_{i-m}-\lambda)}{\prod_{m=1}^{k} w_{i-m}}\right|$ is not bounded and we have only $R_T^+(\lambda)<1$. So,

$$\left\{\lambda\in\mathbb{C} : R_T^+(\lambda)\geq 1\right\}\subset\sigma(T).$$

Conversely, in order to show that

$$\sigma(T)\subset\left\{\lambda\in\mathbb{C} : R_T^+(\lambda)\geq 1\,\text{and}\,R_T^-(\lambda)\geq 1\right\},$$

we take $\lambda\in\mathbb{C}$, such that

$$R_T^+(\lambda)<1 \tag{38}$$

or

$$R_T^-(\lambda)<1 \tag{39}$$

and we show that $\lambda\notin\sigma(T)$. From Lemma 2.3, $T-\lambda I$ is invertible and there will exist an operator $(T-\lambda I)^{-1}\in\mathscr{B}(X)$ such that

$$I=(T-\lambda I)^{-1}(T-\lambda I)=(T-\lambda I)(T-\lambda I)^{-1}. \tag{40}$$

Therefore, $\lambda\notin\sigma(T)$. Similarly one can show that if S is not invertible, then

$$\sigma(T)\subset\left\{\lambda\in\mathbb{C} : R_T^+(\lambda)\geq 1\right\}.$$

The proof of theorem is now complete. □

Remark 2.5. In the previous theorem, if we take $d_i=0$ for all $i\in\mathbb{Z}$, then we obtain a result already shown in [5, 6, 21, 25] about the spectrum of the operator S. That is

$$\sigma(S)=\{\lambda\in\mathbb{C} : q(S)\leq|\lambda|\leq r(S)\}.$$

where $q(S)=0$ if S is not invertible and $q(S)=\dfrac{1}{r(S^{-1})}$ if S is invertible.

3. Remark about the spectrum of perturbed bilateral weighted n-shift

For a strictly positive integer n, we define the bilateral weighted n-shift operator in X by

$$S_n e_i=w_i^n e_{i+n},\ i=0,\pm1,\pm2,...$$

The sequence $\{w_i^n\}_{i\in\mathbb{Z}}\subset\mathbb{C}$ represents the weights of the operator S_n. It is clear that the weighted 1-shift coincide with weighted shift (in the usual sense, see [26]).

Remark 3.1. Let $T_n \in \mathscr{B}(X)$ be a diagonal perturbation of bilateral weighted n-shift S_n on X. That is

$$T_n := S_n + D, \tag{41}$$

where D is a diagonal operator defined in (15). For every $j \in \{0, ..., n-1\}$ and $i \in \mathbb{Z}$, let that $e_i^j := e_{j+in}$, $w_i^j := w_{j+in}^n$ and $S_n^j e_i^j = w_i^j e_{i+1}^j$. Where S_n^j is the restriction of S_n on X_j, the S_n-invariant closed linear subspace spanned by $\left\{e_i^j : i \in \mathbb{Z}\right\}$. Note that

$$X = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1}$$

and

$$S_n = S_n^0 \oplus S_n^1 \oplus \cdots \oplus S_n^{n-1}$$

Also, since each S_n^j is a weighted 1-shift, then the spectra of S_n is the union of the spectra of all S_n^j, $j = 0, ..., n-1$ (see [14]). In particular,

$$\sigma(S_n) = \sigma(S_n^0) \cup \sigma(S_n^1) \cup \cdots \cup \sigma(S_n^{n-1}).$$

Moreover, if we denote by D^j the restriction of D to X_j, then

$$T_n = \left(S_n^0 + D^0\right) \oplus \left(S_n^1 + D^1\right) \oplus \cdots \oplus \left(S_n^{n-1} + D^{n-1}\right)$$

and thus

$$\sigma(T_n) = \sigma(S_n^0 + D^0) \cup \sigma(S_n^1 + D^1) \cup \cdots \cup \sigma(S_n^{n-1} + D^{n-1})$$

Furthermore, for $j \in \{0, ..., n-1\}$, $R_j^+(\lambda)$ and $R_j^-(\lambda)$ are given by

$$R_j^+(\lambda) = \lim_{k\to\infty}\left[\sup_{i\in\mathbb{Z}}\left|\frac{\prod_{m=0}^{k-1} w_{j+(i+m)n}^n}{\prod_{m=0}^{k}(d_{j+(i+m)n} - \lambda)}\right|\right]^{\frac{1}{k}} \tag{42}$$

and

$$R_j^-(\lambda) = \lim_{k\to\infty}\left[\sup_{i\in\mathbb{Z}}\left|\frac{\prod_{m=0}^{k-1}(d_{j+(i-m)n} - \lambda)}{\prod_{m=0}^{k} w_{j+(i-m)n}^n}\right|\right]^{\frac{1}{k}}. \tag{43}$$

By Theorem 2.4, if S_n^j is invertible operator, then we have

$$\sigma(S_n^j + D^j) = \left\{\lambda \in \mathbb{C} : R_j^+(\lambda) \geq 1 \text{ and } R_j^-(\lambda) \geq 1\right\},$$

and if S_n^j is non-invertible operator then

$$\sigma(S_n^j + D^j) = \left\{\lambda \in \mathbb{C} : R_j^+(\lambda) \geq 1\right\}.$$

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