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CONTROLLABILITY OF IMPULSIVE NEUTRAL STOCHASTIC INTEGRO-DIFFERENTIAL SYSTEMS DRIVEN BY FBM WITH UNBOUNDED DELAY

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This paper studies the controllability of an impulsive neutral stochastic integro-differential systems with infinite delay driven by fractional Brownian motion in separable Hilbert space. The controllability result is obtained by using stochastic analysis and a fixed-point strategy. Illustrating the obtained abstract results, an example is considered at the end of the paper.

1. Introduction

One of the basic qualitative behaviours of a dynamical system is the controllability. Many fundamental problems of control theory such as pole-assignment, stabilizability and optimal control my be solved under the assumption that the system is controllable. The problem of controllability is to show the existence of control function, which steers the solution of the system from its initial state to final state, where the initial and final states may vary over the entire space. Conceived by Kalman, the controllability concept has been studied extensively in the fields of finite-dimensional systems, infinite-dimensional systems, hybrid systems, and behavioural systems. If a system cannot be controlled completely

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then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details the reader may refer to [10, 11, 23, 26, 28, 29] and references therein.

On the other hand, the properties of long/short-range dependence are widely used in describing many phenomena in fields like hydrology and geophysics as well as economics and telecommunications. As extension of Brownian motion, fractional Brownian motion (fBm) is a self-similar Gaussian process which has the properties of long/short-range dependence. However, fractional Brownian motion is neither a semimartingale nor a Markov process (except for the case $H = \frac{1}{2}$ when it reduces to a standard Brownian motion). A general theory for the infinite-dimensional stochastic differential equations driven by a fBm has begun to receive attention by various researchers, (see e.g. [4, 5, 14, 17] and references therein).

The theory of neutral integro-differential equation with infinite delay has been used for modelling the evolution of physical systems, in which the response of the system depends not only on the current state, but also on the past history of the system, for instance, for the description of heat conduction in materials with fading memory, we refer the reader to the papers of Gurtin and Pipkin [9], Nunziato [21], and the references therein related to this matter. Besides, noise or stochastic perturbation is unavoidable and omnipresent in nature as well as in man-made systems. Therefore, it is of great significance to import the stochastic effects into the investigation of impulsive neutral differential equations. As the generalization of the classic impulsive neutral differential equations, impulsive neutral stochastic integro-differential differential equations with infinite delays has become an important area of investigation in recent years stimulated by their numerous applications to problems arising in mechanics, medicine and biology, economics, electronics and telecommunication etc., in which sudden and abrupt changes occur ingenuously, in the form of impulses. On the existence and the controllability for these equations, we refer the reader to [3, 15, 28].

Recently, Park et *al.* [20] investigated the controllability of impulsive neutral integro-differential systems with infinite delay in Banach spaces using Schauder-type fixed point theorem. Arthi et *al.* [3] established the existence and exponential stability for impulsive neutral stochastic integro-differential equations driven by a fractional Brownian motion with finite delays. Very recently, Diop et *al.* [7] proved sufficient conditions for the existence, uniqueness and asymptotic behaviours of mild solutions to a class of neutral stochastic integro-differential equations driven by a fractional Brownian motion with impulsive effects and time-varying delays. Ren et *al.* [25] studied a class of impulsive neutral stochastic functional integro-differential equations with infinite delay driven by fBm.

Moreover, in recent years, there has been an increasing interest in studying the control problem for stochastic systems. Controllability for stochastic systems driven by Brownian motion are well investigated, we refer to [12, 24] and references therein. By contrast, there has not been very much research on the controllability of stochastic systems driven by fBm, more precisely, Chen [6] concerned the approximate controllability for semilinear stochastic equations with fBm, Ahmed [2] investigated the approximate controllability of impulsive neutral stochastic equations driven by fBm, Lakhel [13] concerned the controllability of neutral stochastic functional integro-differential equations driven by fbm. Very recently, Lakhel and Mckibben [15, 16] have discussed the controllability for a class of neutral stochastic evolution integro-differential equations driven by a fBm with finite delay by using the Weiner integral. Moreover, the controllability of neutral impulsive stochastic integro-differential systems with infinite delay driven by a fBm is an untreated topic in the literature so far. Thus, we will make the first attempt to study such problem in this paper.

Inspired by the previously mentioned works, this paper establishes the controllability of impulsive neutral stochastic integro-differential equations of the form

$$\begin{aligned} d[x(t)-g(t,x_t,\int_0^t r_1(t,s,x_s)ds)] &= [Ax(t)+f(t,x_t,\int_0^t r_2(t,s,x_s)ds)+Bu(t)]dt \\ &\quad +\sigma(t)dB^H(t), t\in I:=[0,T],\ t\neq t_k, \\ \Delta x|_{t=t_k} = x(t_{k^+})-x(t_{k^-}) &= I_k(x(t_{k^-})),\ k=1,\ldots,m,\ m\in\mathbb{N} \\ x(t)=\varphi(t)\in L_2^0(\Omega,\mathcal{B}_h),&\ for\, a.e.\, t\in(-\infty,0]. \end{aligned} \tag{1.1}$$

Here, A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X; B^H is a fractional Brownian motion with Hurst parameter $H>\frac{1}{2}$ on a real and separable Hilbert space Y; and the control function $u(\cdot)$ takes values in $L^2([0,T],U)$, the Hilbert space of admissible control functions for a separable Hilbert space U; and B is a bounded linear operator from U into X.

The history $x_t:(-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$, belongs to an abstract phase space $\mathcal{B}_h$ defined axiomatically, and $f,g:[0,T]\times\mathcal{B}_h\times X\to X$, $r_1,r_2:D\times\mathcal{B}_h\to X$, $\sigma:[0,T]\to\mathcal{L}_2^0(Y,X)$, are appropriate functions and will be specified later, where $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below) and $D=\{(s,t)\in I\times I: s<t\}$. Moreover, the fixed moments of time t_k satisfy $0<t_1<t_2<\ldots<t_m<T$; $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at time t_k respectively. $\Delta x(t_k)$ denotes the jump

in the state x at time t_k with $I(.) : X \longrightarrow X$ determining the size of the jump.

The outline of this paper is as follows: In Section 2 we introduce some notations, concepts, and basic results about fractional Brownian motion, the Weiner integral defined in general Hilbert spaces, phase spaces and properties of analytic semigroups and the fractional powers associated to its generator. In Section 3, we derive the controllability of impulsive neutral stochastic integro-differential systems driven by a fractional Brownian motion. Finally, in Section 4, we conclude with an example to illustrate the applicability of the general theory.

2. Preliminaries

For details of the topics addressed in this section, we refer the reader to [19, 22] and the references therein.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A standard fractional Brownian motion (fBm) $\{\beta^H(t), t \in \mathbb{R}\}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with continuous sample paths such that

$$R_H(t,s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \qquad s,t \in \mathbb{R}. \quad (2.1)$$

Remark 2.1. In the case $H > \frac{1}{2}$, it follows from [19]that the second partial derivative of the covariance function

$$\frac{\partial R_H}{\partial t \partial s} = \alpha_H |t-s|^{2H-2},$$

where $\alpha_H = H(2H-2)$, is integrable, and we can write

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} du dv. \quad (2.2)$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$. where $\lambda_n \geq 0$ $(n = 1,2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1,2...)$ is a complete orthonormal basis in Y.

We define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where β_n^H are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t),x\rangle\langle B^H(s),y\rangle = R(s,t)\langle Q(x),y\rangle \text{ for all } x,y\in Y \text{ and } t,s\in[0,T]$$

In order to define Weiner integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product

$$\langle \varphi, \psi\rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$$

is a separable Hilbert space.
Let $\phi(s); s\in[0,T]$ be a function with values in $\mathcal{L}_2^0(Y,X)$, such that

$$\sum_{n=1}^{\infty} \|K^*\phi Q^{\frac{1}{2}} e_n\|^2_{\mathcal{L}_2^0} < \infty.$$

The Weiner integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s)dB^H(s) = \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda_n}\phi(s)e_n d\beta_n^H(s). \tag{2.3}$$

We conclude this subsection by stating the following result which is critical in the proof of our result, see for example [5]

Lemma 2.1. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$, then (2.3) is well-defined as an X-valued random variable and*

$$\mathbb{E}\|\int_0^t \psi(s)dB^H(s)\|^2 \le 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

It is known that the study of theory of differential equation with infinite delays depends on a choice of the abstract phase space. We assume that the phase space $\mathcal{B}_h$ is a linear space of functions mapping $(-\infty,0]$ into X, endowed with a norm $\|.\|_{\mathcal{B}_h}$. We shall introduce some basic definitions, notations and lemma which are used in this paper. First, we present the abstract phase space $\mathcal{B}_h$. Assume that $h : (-\infty,0] \longrightarrow [0,+\infty)$ is a continuous function with $l = \int_{-\infty}^0 h(s)ds < +\infty$.

We define the abstract phase space $\mathcal{B}_h$ by

$$\begin{aligned}\mathcal{B}_h = \quad & \{\psi : (-\infty, 0] \longrightarrow X \text{ for any } \tau > 0, (\mathbb{E}\|\psi\|^2)^{\frac{1}{2}} \text{ is bounded and measurable} \\ & \text{function on } [-\tau, 0] \text{ and } \textstyle\int_{-\infty}^{0} h(t) \sup_{t\leq s\leq 0}(\mathbb{E}\|\psi(s)\|^2)^{\frac{1}{2}} dt < +\infty\}.\end{aligned}$$

If we equip this space with the norm

$$\|\psi\|_{\mathcal{B}_h} := \int_{-\infty}^{0} h(t) \sup_{t\leq s\leq 0} (\mathbb{E}\|\psi\|^2)^{\frac{1}{2}} dt,$$

then it is clear that $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space.

We now consider the space $\mathcal{B}_{DI}$ (D and I stand for delay and impulse, respectively) given by

$$\begin{aligned}\mathcal{B}_{DI} \quad & = \{x : (-\infty, T] \to X : x|I_k \in \mathcal{C}(I_k, X) \text{ and } x(t_k^+), x(t_k^-) \text{ exist with} \\ & x(t_k^-) = x(t_k), k = 1, 2, ..., m, x_0 = \varphi \in \mathcal{B}_h \text{ and } \textstyle\sup_{0\leq t\leq T} \mathbb{E}(\|x(t)\|^2) < \infty\},\end{aligned}$$

where $x|I_k$ is the restriction of x to the interval $I_k = (t_k, t_{k+1}]$, $k = 1, 2, ..., m$. The function $\|.\|_{\mathcal{B}_{DI}}$ to be a semi-norm in $\mathcal{B}_{DI}$, it is defined by

$$\|x\|_{\mathcal{B}_{DI}} = \|x_0\|_{\mathcal{B}_h} + \sup_{0\leq t\leq T} (\mathbb{E}(\|x(t)\|^2))^{\frac{1}{2}}.$$

The following lemma is a common property of phase spaces.

Lemma 2.2. *[18] Suppose $x \in \mathcal{B}_{DI}$, then for all $t \in [0, T]$, $x_t \in \mathcal{B}_h$ and*

$$l(\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}} \leq \|x_t\|_{\mathcal{B}_h} \leq l \sup_{0\leq s\leq t} (\mathbb{E}\|x(s)\|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$.

Next, we introduce some notations and basic facts about the theory of semigroups and fractional power operators. Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq 0}$, of bounded linear operators on X. The theory of strongly continuous is thoroughly discussed in [22] and [8]. It is well-known that there exist $M \geq 1$ and $\lambda \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\lambda t}$ for every $t \geq 0$. If $(S(t))_{t\geq 0}$ is a uniformly bounded, analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^{\alpha}$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^{\alpha}$. Furthermore, the subspace $D(-A)^{\alpha}$ is dense in X, and the expression

$$\|h\|_{\alpha} = \|(-A)^{\alpha} h\|$$

defines a norm in $D(-A)^{\alpha}$. If X_{α} represents the space $D(-A)^{\alpha}$ endowed with the norm $\|.\|_{\alpha}$, then the following properties hold (see [22], p. 74).

Lemma 2.3. *Suppose that A, X_α, and $(-A)^\alpha$ are as described above.*

(i) For $0 < \alpha \le 1$, *X_α is a Banach space.*

(ii) If $0 < \beta \le \alpha$, *then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.*

(iii) For every $0 < \alpha \le 1$, *there exists $M_\alpha > 0$ such that*

$$\|(-A)^\alpha S(t)\| \le M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \ \lambda > 0.$$

3. Controllability Result

Before stating and proving the main result, we give the definition of mild solutions for equation (1.1).

Definition 3.1. An X-valued process $\{x(t) : t \in (-\infty, T]\}$ is a mild solution of (1.1) if

1. $x(t)$ is measurable for each $t > 0$, $x(t) = \varphi(t)$ on $(-\infty, 0]$, $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \ldots, m$, the restriction of $x(.)$ to $[0, T] - \{t_1, t_2, \ldots, t_m\}$ is continuous,
2. for every $0 \le s < t$ the function $AS(t-s)g(s, x_s, \int_0^s r_1(s, \tau, x_\tau)d\tau)$ is integrable such that the following integral equation is satisfied

$$\begin{aligned} x(t) &= S(t)(\varphi(0) - g(0, \varphi, 0)) + g(t, x_t, \textstyle\int_0^t r_1(t, s, x_s)ds) \\ &+ \textstyle\int_0^t AS(t-s)g(s, x_s, \int_0^s r_1(s, \tau, x_\tau)d\tau)ds + \int_0^t S(t-s)Bu(s)ds \\ &+ \textstyle\int_0^t S(t-s)f(s, x_s, \int_0^s r_2(s, \tau, x_\tau)d\tau)ds + \int_0^t S(t-s)\sigma(s)dB^H(s) \\ &+ \textstyle\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \quad \mathbb{P} - a.s. \end{aligned} \tag{3.1}$$

Definition 3.2. The impulsive neutral stochastic functional integro-differential equation (1.1) is said to be controllable on the interval $(-\infty, T]$ if for every initial stochastic process φ defined on $(-\infty, 0]$, there exists a stochastic control $u \in L^2([0, T], U)$ such that the mild solution $x(\cdot)$ of (1.1) satisfies $x(T) = x_1$, where x_1 and T are the preassigned terminal state and time, respectively.

In order to establish the controllability of (1.1), we impose the following conditions on the data of the problem:

$(\mathcal{H}.1)$ A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq 0}$, of bounded linear operators on X. Further, $0 \in \rho(A)$, and there exist constants M, $M_{1-\beta}$ such that

$$\|S(t)\|^2 \leq M \quad \text{and} \quad \|(-A)^{1-\beta}S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}, \text{ for all } t \in [0,T]$$

(see Lemma 2.3).

$(\mathcal{H}.2)$ The mapping $g : I \times \mathcal{B}_h \times X \to X$ satisfies the following conditions

(i) The function $r_1 : D \times \mathcal{B}_h \to X$ satisfies the following condition. There exists a constant $k_1 > 0$, for $x_1, x_2 \in \mathcal{B}_h$ such that

$$\mathbb{E}\| \int_0^t [r_1(t,s,x_1) - r_1(t,s,x_2)]ds\|^2 \leq k_1 \|x_1 - x_2\|^2_{\mathcal{B}_h}, \quad (t,s) \in D,$$

and $\overline{k}_1 = \sup_{(t,s)\in D} \| \int_0^t r_1(t,s,0)ds\|^2$.

(ii) There exist constants $0 < \beta < 1$, $k_2 > 0$ such that the function g is X_β-valued and for $x_1, x_2 \in \mathcal{B}_h$, $y_1, y_2 \in X$ and satisfies for all $t \in [0,T]$

$$\mathbb{E}\|(-A)^\beta g(t,x_1,y_1) - (-A)^\beta g(t,x_2,y_2)\|^2$$

$$\leq k_2[\|x_1 - x_2\|^2_{\mathcal{B}_h} + \mathbb{E}\|y_1 - y_2\|^2],$$

$$\lim_{t \longrightarrow s} \mathbb{E}\|(-A)^\beta g(t,x_1,y_1) - (-A)^\beta g(s,x_2,y_2)\|^2 = 0,$$

and $\overline{k}_2 = \sup_{t\in[0,T]} \|(-A)^{-\beta} g(t,0,0)\|^2$.

$(\mathcal{H}.3)$ The mapping $f : I \times \mathcal{B}_h \times X \to X$ satisfies the following Lipschitz conditions

(i) There exist positive constants $k_3, \overline{k}_3$ for $t \in [0,T]$, $x_1, x_2 \in \mathcal{B}_h$, $y_1 y_2 \in X$ such that

$$\mathbb{E}\|f(t,x_1,y_1) - f(t,x_2,y_2)\|^2 \leq k_3[\|x_1 - x_2\|^2_{\mathcal{B}_h} + \mathbb{E}\|y_1 - y_2\|^2],$$

and $\overline{k}_3 = \sup_{t\in[0,T]} \|f(t,0,0)\|^2$.

(ii) The function $r_2 : D \times \mathcal{B}_h \to X$ satisfies the following condition. There exists a constant $k_4 > 0$, for $x_1, x_2 \in \mathcal{B}_h$ such that

$$\mathbb{E}\| \int_0^t [r_2(t,s,x_1) - r_2(t,s,x_2)]ds\|^2 \leq k_4 \|x_1 - x_2\|^2_{\mathcal{B}_h}, \quad (t,s) \in D,$$

and $\overline{k}_4 = \sup_{(t,s)\in D} \| \int_0^t r_2(t,s,0)ds\|^2$.

$(\mathcal{H}.4)$ The impulses functions I_k for $k = 1,2,...,m$, satisfies the following condition. There exist positive constants M_k, $\widetilde{M}_k$ such that $\|I_k(x) - I_k(y)\|^2 \leq M_k\|x-y\|^2$ and $\|I_k(x)\|^2 \leq \widetilde{M}_k$ for all $x, y \in \mathcal{B}_h$.

$(\mathcal{H}.5)$ The function $\sigma : [0,\infty) \to \mathcal{L}_2^0(Y,X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \;\; \forall T > 0.$$

$(\mathcal{H}.6)$ The linear operator W from U into X defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds$$

has an inverse operator W^{-1} that takes values in $L^2([0,T],U) \setminus kerW$, where

$$kerW = \{x \in L^2([0,T],U) : \; Wx = 0\}$$

(see [10]), and there exists finite positive constants M_b, M_w such that $\|B\|^2 \leq M_b$ and $\|W^{-1}\|^2 \leq M_w$.

$(\mathcal{H}.7)$ There exists a constant $\omega > 0$ such that

$$\begin{aligned} \omega = \;\; & 10l^2(1+4MM_bM_wT^2)[(c_1^2 + \tfrac{(M_{1-\beta}T^{\beta})^2}{2\beta-1})k_2(1+2k_1) \\ & + MT^2k_3(1+k_4) + mM\textstyle\sum_{k=1}^m M_k] < 1, \end{aligned}$$

and $c_1 = \|(-A)^{-\beta}\|$.

The main result of this chapter is the following.

Theorem 3.3. Suppose that $(\mathcal{H}.1) - (\mathcal{H}.7)$ hold. Then, the system (1.1) is controllable on $(-\infty, T]$ provide that

$$7l^2(1+8MM_bM_wT^2)\{8(c_1^2 + \frac{(M_{1-\beta}T^{\beta})^2}{2\beta-1})k_2(1+2k_1) + 8MT^2k_3(1+2k_4)\} < 1. \quad (3.2)$$

Proof. Transform the problem(1.1) into a fixed-point problem. To do this, using the hypothesis $(\mathcal{H}.6)$ for an arbitrary function $x(\cdot)$, define the control u_x by

$$\begin{aligned} u_x(t) &= W^{-1}\{x_1 - S(T)(\varphi(0) - g(0,x_0,0)) - g(T,x_T,\textstyle\int_0^T r_1(T,s,x_s)ds)) \\ &- \textstyle\int_0^T AS(T-s)g(s,x_s,\int_0^s r_1(s,\eta,x_\eta)d\eta)ds \\ &- \textstyle\int_0^T S(T-s)f(s,x_s,\int_0^s r_2(s,\eta,x_\eta)d\eta)ds \\ &- \textstyle\int_0^T S(T-s)\sigma(s)dB^H(s)\}(t) - \sum_{0<t_k<T} S(T-t_k)I_k(x(t_k^-))\}(t). \end{aligned}$$

To formulate the controllability problem in the form suitable for application of the Banach fixed point theorem, put the control $u(.)$ into the stochastic control system (3.1) and obtain a non linear operator Π on $\mathcal{B}_{DI}$ given by

$$\Pi(x)(t) = \begin{cases} \varphi(t), \quad \text{if } t \in (-\infty,0], \\ S(t)(\varphi(0) - g(0,\varphi,0)) + g(t,x_t,\int_0^t r_1(t,s,x_s)ds) \\ + \int_0^t AS(t-s)g(s,x_s,\int_0^s r_1(s,\eta,x_\eta)d\eta)ds \\ + \int_0^t S(t-s)Bu_x(s)ds + \int_0^t S(t-s)f(s,x_s,\int_0^s r_2(s,\eta,x_\eta)d\eta)ds \\ + \int_0^t S(t-s)\sigma(s)dB^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \text{ if } t \in [0,T]. \end{cases}$$

Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator Π. Clearly, $\Pi x(T) = x_1$, which means that the control u steers the system from the initial state φ to x_1 in time T, provided we can obtain a fixed point of the operator Π which implies that the system in controllable.

Let $y : (-\infty,T] \longrightarrow X$ be the function defined by

$$y(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty,0], \\ S(t)\varphi(0), & \text{if } t \in [0,T], \end{cases}$$

then, $y_0 = \varphi$. For each function $z \in \mathcal{B}_{DI}$, set

$$x(t) = z(t) + y(t).$$

It is obvious that x satisfies the stochastic control system (3.1) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) = \quad & g(t, z_t + y_t, \textstyle\int_0^t r_1(t, s, z_s + y_s) ds) - S(t) g(0, \varphi, 0) \\ & + \textstyle\int_0^t AS(t-s) g(s, z_s + y_s, \int_0^s r_1(s, \eta, z_\eta + y_\eta) d\eta) ds \\ & + \textstyle\int_0^t S(t-s) B u_{z+y}(s) ds + \int_0^t S(t-s) f(s, z_s + y_s, \int_0^s r_2(s, \eta, z_\eta + y_\eta) d\eta) ds \\ & + \textstyle\int_0^t S(t-s) \sigma(s) dB^H(s) + \sum_{0<t_k<t} S(t-t_k) I_k(z(t_k^-) + y(t_k^-)), \text{ if } t \in [0, T], \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} u_{z+y}(t) \quad & = W^{-1}\{x_1 - S(T)(\varphi(0) - g(0, z_0 + y_0, 0)) \\ & - g(T, z_T + y_T, \textstyle\int_0^T r_1(T, s, z_s + y_s) ds) \\ & - \textstyle\int_0^T AS(T-s) g(s, z_s + y_s, \int_0^s r_1(s, \eta, z_\eta + y_\eta) d\eta) ds \\ & - \textstyle\int_0^T S(T-s) f(s, z_s + y_s, \int_0^s r_2(s, \eta, z_\eta + y_\eta) d\eta) ds \\ & - \textstyle\int_0^T S(T-s) \sigma(s) dB^H(s)\}(t) - \sum_{0<t_k<T} S(T-t_k) I_k(z(t_k^-) + y(t_k^-))\}(t). \end{aligned}$$

Set

$$\mathcal{B}_{DI}^0 = \{z \in \mathcal{B}_{DI} : z_0 = 0\};$$

for any $z \in B_{DI}^0$, we have

$$\|z\|_{\mathcal{B}_{DI}^0} = \|z_0\|_{\mathcal{B}_h} + \sup_{t \in [0,T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t \in [0,T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}}.$$

Then, $(\mathcal{B}_{DI}^0, \|.\|_{\mathcal{B}_{DI}^0})$ is a Banach space. Define the operator $\Phi : \mathcal{B}_{DI}^0 \longrightarrow \mathcal{B}_{DI}^0$ by

$$(\Phi z)(t) = \begin{cases} 0 \text{ if } t \in (-\infty, 0], \\ g(t, z_t + y_t, \int_0^t r_1(t, s, z_s + y_s) ds) - S(t) g(0, \varphi, 0) \\ + \int_0^t AS(t-s) g(s, z_s + y_s, \int_0^s r_1(s, \eta, z_\eta + y_\eta) d\eta) ds \\ + \int_0^t S(t-s) f(s, z_s + y_s, \int_0^s r_2(s, \eta, z_\eta + y_\eta) d\eta) ds \\ + \int_0^t S(t-s) B u_{z+y}(s) ds + \int_0^t S(t-s) \sigma(s) dB^H(s) \\ + \sum_{0<t_k<t} S(t-t_k) I_k(z(t_k^-) + y(t_k^-)), \quad \text{if } t \in [0, T], \end{cases} \tag{3.4}$$

Set

$$\mathcal{B}_k = \{z \in \mathcal{B}^0_{DI} : \|z\|^2_{\mathcal{B}^0_{DI}} \leq k\}, \qquad \text{for some } k \geq 0,$$

then $\mathcal{B}_k \subseteq \mathcal{B}^0_{DI}$ is a bounded closed convex set, and for $z \in \mathcal{B}_k$, we have

$$\begin{aligned}\|z_t + y_t\|_{\mathcal{B}_{DI}} &\leq 2(\|z_t\|^2_{\mathcal{B}_{DI}} + \|y_t\|^2_{\mathcal{B}_{DI}}) \\ &\leq 4(l^2 \sup_{0\leq s\leq t} \mathbb{E}\|z(s)\|^2 + \|z_0\|^2_{\mathcal{B}_h} \\ &+ l^2 \sup_{0\leq s\leq t} \mathbb{E}\|y(s)\|^2 + \|y_0\|^2_{\mathcal{B}_h}) \\ &\leq 4l^2(k + M\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|^2_{\mathcal{B}_h} \\ &:= r^*.\end{aligned}$$

From our assumptions, using the fact that $(\sum_{i=1}^n a_i)^2 \leq n\sum_{i=1}^n a_i^2$ for any positive real numbers a_i, $i = 1,2,...,n$, we have

$$\begin{aligned}\mathbb{E}\|u_{z+y}\|^2 &\leq 8M_w\{\|x_1\|^2 + M\mathbb{E}\|\varphi(0)\|^2 + 2Mc_1^2[k_2\|y\|^2_{\mathcal{B}_h} + \overline{k}_2] \\ &+ 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})[k_2(1+2k_1)r^* + 2k_2\overline{k}_1 + \overline{k}_2] + 2MT^2[k_3(1+2k_4)r^* \\ &+ 2k_3\overline{k}_4 + \overline{k}_3] + 2MT^{2H-1}\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds + mM\sum_{k=1}^m \widetilde{M}_k\} := \mathcal{G}.\end{aligned} \tag{3.5}$$

Noting that

$$\begin{aligned}\mathbb{E}\|u_{z+y} - u_{v+y}\|^2 &\leq 4M_w\{(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})k_2(1+2k_1) + MT^2k_3(1+2k_4) \\ &+ mM\sum_{k=1}^m M_k\}\mathbb{E}\|z_t - v_t\|^2_{\mathcal{B}_h}.\end{aligned} \tag{3.6}$$

It is clear that the operator Π has a fixed point if and only if Φ has one, so it turns to prove that Φ has a fixed point. Since all functions involved in the operator are continuous therefore Φ is continuous. The proof will be given in following steps.

Step 1: We claim that there exists a positive number k, such that $\Phi(x) \in \mathcal{B}_k$ whenever $x \in \mathcal{B}_k$. If it is not true, then for each positive number k, there is a function $z^k(.) \in \mathcal{B}_k$, but $\Phi(z^k) \notin \mathcal{B}_k$, that is $\mathbb{E}\|\Phi(z^k)(t)\|^2 > k$ for some $t \in [0,T]$. However, on the other hand, we have

$$
\begin{aligned}
k \quad &< \mathbb{E}\|\Phi(z^k)(t)\|^2 \\
&\leq 7\{2Mc_1^2(k_2\|y\|_{\mathcal{B}_h}^2 + \overline{k}_2) + 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1}[k_2(1+2k_1)r^* + 2k_2\overline{k}_1 + \overline{k}_2] \\
&+ MM_bT^2\mathcal{G} + 2MT^2(k_3(1+2k_4)r^* + 2k_3\overline{k}_4 + \overline{k}_3) + 2MT^{2H-1}\textstyle\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \\
&+ M\textstyle\sum_{k=1}^m \widetilde{M}_k\} \\
&\leq 7(1+8MM_bM_wT^2)\{2Mc_1^2(k_2\|y\|_{\mathcal{B}_h}^2 + \overline{k}_2) + 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1}[k_2(1+2k_1)r^* \\
&+ 2k_2\overline{k}_1 + \overline{k}_2] + MM_bT^2\mathcal{G} + 2MT^2[k_3(1+2k_4)r^* + 2k_3\overline{k}_4 + \overline{k}_3] \\
&+ 2MT^{2H-1}\textstyle\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds + M\sum_{k=1}^m \widetilde{M}_k\} + 8MM_bM_wT^2(\|x_1\|^2 + M\mathbb{E}\|\varphi(0)\|^2) \\
&\leq \overline{K} + 7(1+8MM_bM_wT^2)\{2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1}k_2(1+2k_1)r^* \\
&+ 2MT^2(k_3(1+2k_4)r^*\},
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{K} \quad &= 7(1+8MM_bM_wT^2)\{2Mc_1^2(k_2\|y\|_{\mathcal{B}_h}^2 + \overline{k}_2) + 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1}(2k_2\overline{k}_1 + \overline{k}_2)) \\
&+ 2MT^2(2k_3\overline{k}_4 + \overline{k}_3) + 2MT^{2H-1}\textstyle\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds + mM\sum_{k=1}^m \widetilde{M}_k\} \\
&+ 8MM_bM_wT^2(\|x_1\|^2 + M\mathbb{E}\|\varphi(0)\|^2)\}
\end{aligned}
$$

is independent of k. Dividing both sides by k and taking the limit as $k \longrightarrow \infty$, we get

$$
7l^2(1+8MM_bM_wT^2)\{8(c_1^2 + \frac{(M_{1-\beta}T^\beta)^2}{2\beta - 1})k_2(1+2k_1) + 8MT^2k_3(1+2k_4)\} \geq 1.
$$

This contradicts (3.2). Hence for some positive k,

$$
(\Phi)(\mathcal{B}_k) \subseteq \mathcal{B}_k.
$$

Step 2: Φ is a contraction. Let $t \in [0,T]$ and $z^1, z^2 \in \mathcal{B}_{DI}^0$, we have

$$
\mathbb{E}\|\Phi z^1(t) - \Phi z^2(t)\|^2 \leq 5\mathbb{E}\|g(t, z_t^1 + y_t, \int_0^t r_1(t,s,z_s^1 + y_s)ds)
$$

$$-g(t,z_t^2+y_t,\int_0^t r_1(t,s,z_s^2+y_s)ds)\|^2$$

$$+5\mathbb{E}\|\int_0^t AS(t-s)[g(s,z_s^1+y_s,\int_0^s r_1(s,\eta,z_\eta^1+y_\eta)d\eta)$$

$$-g(s,z_s^2+y_s,\int_0^s r_1(s,\eta,z_\eta^2+y_\eta)d\eta)]ds\|^2$$

$$+5\mathbb{E}\|\int_0^t S(t-s)B[u_{z^1+y}(s)-u_{z^2+y}(s)]ds\|^2$$

$$+5\mathbb{E}\|\int_0^t S(t-s)[f(s,z_s^1+y_s,\int_0^s r_2(s,\eta,z_\eta^1+y_\eta)d\eta)$$

$$-f(s,z_s^2+y_s,\int_0^s r_2(s,\eta,z_\eta^2+y_\eta)d\eta)]ds$$

$$+5\mathbb{E}\|\sum_{0<t_k<T} S(T-t_k)[I_k(z^1(t_k^-)+y(t_k^-))-I_k(z^2(t_k^-)+y(t_k^-))]\|^2$$

On the other hand from $(\mathcal{H}.1)-(\mathcal{H}.7)$ combined with (3.6), we obtain

$$\mathbb{E}\|\Phi z^1(t)-\Phi z^2(t)\|^2 \leq 5(1+4MM_bM_wT^2)[(c_1^2+\tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})k_2(1+2k_1)$$

$$+MT^2k_3(1+k_4)+mM\textstyle\sum_{k=1}^m M_k]\|z_t^1-z_t^2\|^2_{\mathcal{B}_h}$$

$$\leq 10(1+4MM_bM_wT^2)[(c_1^2+\tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})k_2(1+2k_1)$$

$$+MT^2k_3(1+k_4)+mM\textstyle\sum_{k=1}^m M_k]$$

$$\times\{l^2\sup_{0\leq s\leq t}\mathbb{E}\|z^1(s)-z^2(s)\|^2+\|z_0^1-z_0^2\|^2_{\mathcal{B}_h}\}$$

$$\leq \omega\sup_{0\leq s\leq T}\mathbb{E}\|z^1(s)-z^2(s)\|^2) \quad (\text{ since } z_0^1=z_0^2=0)$$

Taking supremum over t,

$$\|\Phi z^1-\Phi z^2\|_{\mathcal{B}^0_{DI}} \leq \omega\|z^1-z^2\|_{\mathcal{B}^0_{DI}},$$

where

$$\omega = \quad 10l^2(1+4MM_bM_wT^2)[(c_1^2+\tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})k_2(1+2k_1)$$

$$+MT^2k_3(1+k_4)+mM\textstyle\sum_{k=1}^m M_k].$$

By condition $(\mathcal{H}.7)$, we have $\omega < 1$, hence Φ is a contraction mapping on $\mathcal{B}_{DI}^{0}$ and therefore has a unique fixed point, which is a mild solution of equation (1.1) on $(-\infty, T]$. Clearly, $(\Phi x)(T) = x_1$ which implies that the system (1.1) is controllable on $(-\infty, T]$. This completes the proof.

□

Remark 3.4. When the impulses disappear, that is $M_k = \tilde{M}_k = 0$, $k = 1, ..., m$ then the system (1.1) reduces to the following neutral stochastic integrodifferential equation:

$$\begin{aligned} d[x(t) - g(t, x_t, \int_0^t r_1(t,s,x_s)ds)] &= [Ax(t) + f(t, x_t, \int_0^t r_2(t,s,x_s)ds) + Bu(t)]dt \\ &\quad + \sigma(t)dB^H(t), t \in I, \\ x(t) = \varphi(t) \in L_2^0(\Omega, \mathcal{B}_h),\ for\, a.e.\, t \in (-\infty, 0], \end{aligned} \tag{3.7}$$

where the operators A, g, f, r_1, r_2 and σ are defined as same as before. Here $\mathcal{C} = \{x : (-\infty, T] \to X : x(t)$ is continuous $\}$, the Banach space of all stochastic processes $x(t)$ from $(-\infty, T]$ into X, equipped with the supremum norm $\|\phi\|_{\mathcal{C}}^2 = \sup_{s\in(-\infty,T]} \mathbb{E}\|\phi(s)\|^2$, for $\phi \in \mathcal{C}$. By using the same technique in Theorem 3.3, we can easily deduce the following corollary.

Corollary 3.1. *Suppose that $(\mathcal{H}.1) - (\mathcal{H}.3)$ and $(\mathcal{H}.5) - (\mathcal{H}.7)$ hold. Then, the system (3.7) is controllable on $(-\infty, T]$ provide that the condition (3.2) is satisfied.*

Remark 3.5. The concept of nonlocal initial condition, in many cases, more accurately describes initial behaviour of system than does a classical fixed initial condition, so differential equations with nonlocal problem have been studied extensively in the literatures [1, 29]. However, to the best of our knowledge, no result yet exist on the controllability of impulsive stochastic functional integro-differential equations driven by fBm with non local conditions. Upon making some appropriate assumptions, by employing the ideas and techniques same as in this paper, one can establish the controllability of impulsive stochastic functional integro-differential equations driven by fBm with non local conditions.

Remark 3.6. In this paper, we only consider the additive noise. In the future, we will further study existence, uniqueness and qualitative properties of mild solutions for impulsive neutral stochastic partial inegro-differential equations driven by multiplicative noise, for example when the term $\sigma(t)dB^H(t)$ is replaced by $\sigma(t, x_t)dB^H(t)$ term, it may be one of our interesting directions of the future work.

4. Example

To illustrate the previous result, we consider the following impulsive neutral stochastic partial inegro-differential equation with infinite delays, driven by a fBm of the form

$$\begin{cases} \frac{\partial}{\partial t}[z(t,\xi)-G_1(t,z(t-k,\xi),\int_0^t g_1(t,s,z(s-k,\xi))ds)] \\ = \frac{\partial^2}{\partial^2\xi}z(t,\xi)+F_1(t,z(t-k,\xi),\int_0^t f_1(t,s,z(s-k,\xi))ds) \\ +c(\xi)u(t)+\sigma(t)\frac{dB^H(t)}{dt}, \quad 0\le t\le T, t\neq t_k, 0\le\xi\le\pi \\ \Delta z(t_k,\xi)=z(t_k^+,\xi)-z(t_k^-,\xi)=\int_{-\infty}^{t_k}\alpha_k(t_k^- - s)z(s,\xi)ds, \quad k=1,2,...,m; \\ z(t,0)=z(t,\pi)=0, \qquad 0\le t\le T, \\ z(s,\xi)=\varphi(s,\xi), \ ;-\infty<s\le 0 \quad 0\le\xi\le\pi, \end{cases} \tag{4.1}$$

where $0<t_1<t_2<...<t_m<T$ are fixed numbers, and $\varphi:(-\infty,0]\times[0,\pi]\longrightarrow\mathbb{R}$ is a given continuous stochastic process such that $\|\varphi\|^2_{\mathcal{B}_h}<\infty$. We take $X=Y=U=L^2([0,\pi])$ with norm $\|.\|$ and inner product $<.,.>$. Define the operator $A:D(A)\subset X\longrightarrow X$ given by $A=\frac{\partial^2}{\partial^2\xi}$ with

$$D(A)=\{y\in X: y' \text{ is absolutely continuous}, y''\in X, \quad y(0)=y(\pi)=0\},$$

then we get

$$Ax=\sum_{n=1}^{\infty}n^2<x,e_n>_X e_n, \quad x\in D(A),$$

where $e_n:=\sqrt{\frac{2}{\pi}}\sin nx$, $n=1,2,....$ is an orthogonal set of eigenvector of $-A$.

The bounded linear operator $(-A)^{\frac{3}{4}}$ is given by

$$(-A)^{\frac{3}{4}}x=\sum_{n=1}^{\infty}n^{\frac{3}{2}}<x,e_n>_X e_n,$$

with domain

$$D((-A)^{\frac{3}{4}})=X_{\frac{3}{4}}=\{x\in X,\sum_{n=1}^{\infty}n^{\frac{3}{2}}<x,e_n>_X e_n\in X\}, \text{ and } \|(-A)^{\frac{3}{4}}\|=1.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\ge0}$ in X, and is given by (see [22])

$$S(t)x=\sum_{n=1}^{\infty}e^{-n^2t}<x,e_n>e_n,$$

for $x\in X$ and $t\ge0$. Since the semigroup $\{S(t)\}_{t\ge0}$ is analytic, there exists a constant $M>0$ such that $\|S(t)\|^2\le M$ for every $t\ge0$. In other words, the condition $(\mathcal{H}.1)$ holds.

We choose the phase function $h(s) = e^{4s}$, $s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{4} < \infty$, and the abstract phase space $\mathcal{B}_h$ is Banach space with the norm

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} (\mathbb{E}\|\varphi(\theta)\|^2)^{\frac{1}{2}} ds.$$

To rewrite the initial-boundary value problem (4.1) in the abstract form we assume the following:

For $(t,\phi) \in [0,T] \times \mathcal{B}_h$, where $\phi(\theta)(\xi) = \phi(\theta,\xi)$, $(\theta,\xi) \in (-\infty,0] \times [0,\pi]$, we put $z(t)(\xi) = z(t,\xi)$. The functions $g : [0,T] \times \mathcal{B}_h \times X \to X$ and $f : [0,T] \times \mathcal{B}_h \times X \to X$ are defined by

$$g(t,\phi,\int_0^t r_1(t,s,x_s)ds) = G_1(t,\phi(\theta,\xi),\int_0^t g_1(t,s,\phi(\theta,\xi))ds)$$

$$= \int_{-\infty}^{0} V_1(\theta)\phi(\theta,\xi)d\theta + \int_0^t \int_{-\infty}^{0} U_1(t)U_2(\tau)\phi(\theta,\xi)d\tau ds,$$

$$f(t,\phi,\int_0^t r_2(t,s,x_s)ds) = F_1(t,\phi(\theta,\xi),\int_0^t f_1(t,s,\phi(\theta,\xi))ds)$$

$$= \int_{-\infty}^{0} b_1(t,s,\xi,\phi(s,\xi))ds$$

$$+ \int_0^t \int_{-\infty}^{0} b_2(t)b_3(s,\tau,\xi,\phi(\tau,\xi))d\tau ds,$$

where

1. The function $V_1(\theta) \geq 0$ is continuous in $(-\infty,0]$ and satisfying

$$\int_{-\infty}^{0} V_1^2(\theta)d\theta < \infty, \qquad L_g^1 = \left(\int_{-\infty}^{0} \frac{V_1(s)^2}{h(s)}ds\right)^{\frac{1}{2}} < \infty.$$

2. The functions $U_1, U_2 : \mathbb{R} \to \mathbb{R}$ are continuous, and

$$L_g^2 = \left(\int_{-\infty}^{0} \frac{U_2(s)^2}{h(s)}ds\right)^{\frac{1}{2}} < \infty.$$

3. The function b_2 is continuous and $b_i : \mathbb{R} \to \mathbb{R}$, $i = 1,3$ are continuous and there exist continuous functions p_j $j = 1,2,3,4$ such that

$$|b_1(t,s,x,y)| \leq p_1(t)p_2(s)|y|, \qquad (t,s,x,y) \in \mathbb{R}^4,$$

$$|b_3(t,s,x,y)| \leq p_3(t)p_4(s)|y|, \qquad (t,s,x,y) \in \mathbb{R}^4,$$

with $L_1^b = \left(\int_{-\infty}^0 \frac{p_2(s)^2}{h(s)} ds\right)^{\frac{1}{2}} < \infty$ and $L_2^b = \left(\int_{-\infty}^0 \frac{p_4(s)^2}{h(s)} ds\right)^{\frac{1}{2}} < \infty$. Moreover, g and f are bounded linear operators with $\mathbb{E}\|g\|_X^2 \leq \tilde{L}_g$, $\mathbb{E}\|f\|_X^2 \leq \tilde{L}_f$, where $\tilde{L}_g = \left[L_g^1 + T\|U_1\|_\infty L_g^2\right]^2$ and $\tilde{L}_f = \left[\|p_1\|_\infty L_1^b + \|b_2\|_\infty \|p_3\|_{L^1} L_2^b\right]^2$.

4. The functions $\alpha_i : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and $L_{I_k} = \int_{-\infty}^0 \frac{\alpha_k(s)^2}{h(s)} ds$ where $k = 1, ..., m$ are finite.

The above system (4.1) can be written in the abstract form (1.1).
Further, we assume that the following conditions hold:

(5) $B : U \longrightarrow X$ is a bounded linear operator defined by

$$Bu(t)(\xi) = c(\xi)u(t),\ 0 \leq \xi \leq \pi,\ u \in L^2([0,T],U).$$

(6) The linear operator $W : L^2([0,T],U) \longrightarrow X$ given by

$$Wu(\xi) = \int_0^T S(T-s)c(\xi)u(t)ds,\ \ 0 \leq \xi \leq \pi,$$

W is a bounded linear operator but not necessarily one-to-one. Let

$$KerW = \{x \in L^2([0,T],U),\ Wx = 0\}$$

be the null space of W and $[KerW]^\perp$ be its orthogonal complement in $L^2([0,T],U)$. Let $\widetilde{W} : [KerW]^\perp \longrightarrow Range(W)$ be the restriction of W to $[KerW]^\perp$, $\widetilde{W}$ is necessarily one-to-one operator. The inverse mapping theorem says that $\widetilde{W}^{-1}$ is bounded since $[KerW]^\perp$ and $Range(W)$ are Banach spaces. So that W^{-1} is bounded and takes values in $L^2([0,T],U) \setminus KerW$, hypothesis $(\mathcal{H}.6)$ is satisfied.

(7) In order to define the operator $Q : Y := L^2([0,\pi],\mathbb{R}) \longrightarrow Y$, we choose a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, set $Qe_n = \lambda_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.$$

Define the fractional Brownian motion in Y by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta^H(t)e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\beta_n^H\}_{n\in\mathbb{N}}$ is a sequence of one-dimensional fractional Brownian motions mutually independent. Let us assume the function $\sigma : [0,+\infty) \to \mathcal{L}_2^0(L^2([0,\pi]), L^2([0,\pi]))$ satisfies

$$\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds < \infty,\ \ \forall T > 0.$$

Then the condition $(\mathcal{H}.5)$ is satisfied.

Thus, it is easy to verify that all the assumptions on Theorem 3.3 are fulfilled and hence, the system (4.1) is controllable on $(-\infty, T]$.

5. Conclusion and future work

In this paper, a class of neutral impulsive stochastic functional integro-differential equations with infinite delay driven by fractional Brownian motion has been studied. First, we establishes a set of sufficient conditions for the controllability of impulsive neutral stochastic functional integro-differential equations by using stochastic analysis and a fixed-point strategy. Further, an illustrative example is provided to demonstrate the effectiveness of the theoretical result.

There are other issues which require further study. First, one can consider the qualitative behaviour of neutral stochastic integro-differential equations with infinite delay driven by fBm , for example, the transportation inequalities for the law of the mild solution, and invariant measures. Second, we will investigate the optimal control problem for impulsive stochastic partial integro-differential equations with infinite delay driven by fBm.

We conclude this paper with an open question: As we stated in the Introduction, the properties and theories of retarded stochastic differential equations driven by fBm are in the first stage of studying and few literatures study the qualitative properties. Moreover, basically all the works are based on Hurst parameter $H \in (\frac{1}{2}, 1)$ of fBms, one problem is that how to investigate the existence and uniqueness and stability behaviour of mild solutions for neutral impulsive stochastic integro-differential equations with infinite delay driven by fBms with Hurst parameter $H \in (0, \frac{1}{2})$.

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