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MULTIPARTITE GRAPH DECOMPOSITION: CYCLES AND CLOSED TRAILS

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This paper surveys results on cycle decompositions of complete multipartite graphs (where the parts are not all of size 1, so the graph is not K_n), in the case that the cycle lengths are "small". Cycles up to length n are considered, when the complete multipartite graph has n parts, but not hamilton cycles. Properties which the decompositions may have, such as being gregarious, are also mentioned.

1. Introduction and definitions.

A great deal of work has been done on edge-disjoint decompositions of complete graphs and of complete multipartite graphs, where the decomposition is into isomorphic copies of some "small" graph G. This graph G may be itself a complete graph K_k — in which case the decompositions of complete graphs or of complete multipartite graphs are, respectively, a balanced incomplete block design (of index 1), or a group divisible design with block size k. A vast amount of work has also been carried out when the "small" graph G is a cycle — again in both cases, when the graph being decomposed is either complete, or complete multipartite. Furthermore, some results also include decompositions into closed trails rather than cycles.

In this paper I shall concentrate on reviewing the case of a complete multipartite graph when the decomposition is into copies of a fixed length cycle;

also certain decompositions into closed trails are considered here. Particular properties which such a decomposition may have will also be considered.

Let us begin with some basic definitions, which knowledgeable readers may skip over!

A complete multipartite graph $G = K(a_1, a_2, \ldots, a_n)$ has its vertices grouped into n partite sets, of sizes $a_1, \ldots, a_n$. There is an edge between any two vertices in different partite sets, but no edge between any two vertices in the same partite set. If $a_i = m$ for each i, $1 \leq i \leq n$, we write $K_{n(m)}$ and refer to this graph as the complete *equipartite* graph having n parts of size m. Of course if $a_i = 1$ for all i, then G is the complete graph K_n.

A k-cycle, written $(x_1, x_2, \ldots, x_k)$, consists of k distinct vertices $x_1, \ldots$ $\ldots, x_k$, and k edges $\{x_i, x_{i+1}\}$, $1 \leq i \leq k-1$, and $\{x_k, x_1\}$. A k-cycle system of a simple graph G is an edge-disjoint decomposition of G into copies of k-cycles; equivalently, it can be regarded as a partition of the edge set $E(G)$ into k-cycles.

A k-trail is a closed path of length k, where the vertices of the trail are not necessarily distinct. Of course for $k = 3, 4, 5$, a k-trail is also a k-cycle, but when $k \geq 6$ this is not necessarily so; for instance a 6-trail could be a *bowtie* (that is, two triangles with a common vertex).

From a design-theoretic perspective, an edge-disjoint decomposition of the complete multipartite graph $K(a_1, a_2, \ldots, a_n)$ into k-cycles can be regarded as a group-divisible k-cycle system or design (a C_k-GDD), with n groups, of sizes a_i, $1 \leq i \leq n$. And in the case $k = 3$, when a 3-cycle is also a block or a triple, we can talk about a 3-GDD of type $a_1 a_2 \ldots a_n$. Necessary and sufficient conditions for existence of a 3-GDD of arbitrary type $a_1 a_2 \ldots a_n$ are not known, although in the uniform case (of type a^n) they are, and also in a very few 'almost' uniform cases; see Section 2.1.

In the following, Section 2 surveys existence results on complete multipartite graph decompositions into cycles (including 3-cycles which are also of course complete K_3 blocks), and into closed trails. In Section 3, three extra properties are considered: *resolvable* cycle decompositions, *coloured* cycle decompositions, and so-called *gregarious* cycle decompositions of complete multipartite graphs. Section 4 mentions some work which has been done on *packing* complete multipartite graphs with cycles, when a complete edge-disjoint decomposition is not possible, and the final section includes some open problems. To limit the content, the part sizes in the graph $K(a_1, \ldots, a_n)$ will not all be 1, and so we do not deal here with decompositions of K_n. Moreover, for actual proofs, the reader is referred to the original papers.

2. Existence results.

2.1. 3-cycles, many parts

Since a 3-cycle is certainly not bipartite, any complete multipartite graph K having an edge-disjoint decomposition into 3-cycles must necessarily have at least three parts. It is well-known that if the graph K has precisely three parts, then a decomposition into 3-cycles exists if and only if these three parts are all of the same size. Indeed, it is well-known that such a decomposition of $K_{n,n,n}$ into 3-cycles is equivalent to the existence of a latin square L of order n: index the vertices in the three parts by the rows, the columns and the entries of L; then each filled cell in L corresponds to one triangle (or 3-cycle) in the decomposition of $K_{n,n,n}$.

In the case of $K_{n(m)}$, with n parts of size m, Hanani [28] gave necessary and sufficient conditions for existence of a decomposition into 3-cycles; he essentially proved the following.

Theorem 1. (Hanani [28]) *There is a* 3*-cycle decomposition of* $K_{n(m)}$ *if and only if* $n \geq 3$*, the degree* $m(n-1)$ *of any vertex is even, and the number of edges* $m^2\binom{n}{2}$ *is divisible by* 3.

A more accessible one page proof of this appears as Theorem 3.4 in Colbourn and Rosa's "Triple Systems" [21].

When the size of parts in the complete multipartite graph are allowed to differ, there are a very few papers in the case of block size three; the general situation of a 3-cycle decomposition of $K(a_1, a_2, \ldots, a_n)$ remains open. In 1992, Colbourn, Hoffman and Rees [19] showed that the "obvious" necessary conditions for existence of a 3-cycle decomposition of $K_{n(a),b}$, the complete multipartite graph with n groups of size a and one of size b, are always sufficient:

Theorem 2. (Colbourn, Hoffman and Rees [19]) *Let the edge-set of* $K_{n(a),b}$ *be non-empty. Then there is a* 3*-cycle decomposition of* $K_{n(a),b}$ *if and only if*

(i) when $n = 2$ *there are only three parts, and so* $a = b$*;*
(ii) $b \leq a(n-1)$*;*
(iii) any vertex in a part of size a has even degree, i.e. $a(n-1) + b$ *is even;*
(iv) any vertex in the part of size b has even degree, i.e. an is even;
(v) the number of edges, $a^2\binom{n}{2} + anb$*, is divisible by* 3.

Here condition (ii) is necessary because a vertex x in a part of size a must be in a 3-cycle with each of the vertices in the part of size b, and there are at most $a(n-1)$ other vertices to complete this 3-cycle containing vertex x. Of course the difficult part is showing sufficiency of these conditions!

The second result on 3-cycle decompositions of complete multipartite graphs having different sized parts appeared in 1995 (Colbourn, Cusack and Kreher [18]). This paper [18] deals with two different part sizes, a and 1, so with the graph $K_{n(a),t(1)}$, having n parts of size a and t parts of size 1. (From a design theoretic perspective, this can be regarded as a complete graph on $na+t$ points, having n holes of size a.) Again, the "obvious" necessary conditions are shown to be sufficient.

Theorem 3. (Colbourn, Cusack and Kreher [18]) *There is a* 3*-cycle decomposition of* $K_{n(a),t(1)}$ *where* a, n, t *are positive integers, if and only if*

(i) a *is odd, and* $n+t$ *is odd;*
(ii) if $n = 1$ *then* $t - 1 \geq a$*;*
(iii) if $n = 2$ *then* $t \geq a$*;*
(iv) the total number of edges, $\binom{t}{2} + nat + \binom{n}{2}a^2$*, is divisible by* 3.

The necessity here is straightforward: (i) follows because every vertex must have even degree; (ii) follows because each vertex in a part of size 1 must be in a 3-cycle with a vertex in the part of size a; for (iii), consider a vertex x in one part of size a — this must appear in a 3-cycles with the a vertices in the second part of size a, and to complete these 3-cycles we require $t \geq a$. Again, it is sufficiency which proves difficult!

Colbourn [17] gives six necessary conditions for a 3-cycle decomposition of $K(a_1, a_2, \ldots, a_n)$, and shows sufficiency for order at most 60. The fourth condition listed in [17] has the following consequence for the case of four groups:

Lemma 1. *A* 3*-cycle decomposition of* $K(a_1, a_2, a_3, a_4)$*, with* $a_1 \geq a_2 \geq a_3 \geq a_4$*, cannot exist unless at least three of the parts have the same size.*

A simple direct proof follows from considering the four types of triples (3-cycles). Let A_i denote the vertices in the part of size a_i. If there are α triples which miss part A_4, β triples which miss part A_3, γ triples which miss part A_2 and δ triples which miss part A_1, then

$$\begin{array}{rclcrcl} \alpha+\beta & = & a_1a_2, & \qquad & \gamma+\delta & = & a_3a_4, \\ \alpha+\delta & = & a_1a_3, & & \beta+\gamma & = & a_2a_4, \\ \alpha+\gamma & = & a_2a_3, & & \beta+\delta & = & a_1a_4. \end{array}$$

These imply that $a_1a_2 + a_3a_4 = a_1a_3 + a_2a_4 = a_2a_3 + a_1a_4$, and so

$$(a_1 - a_4)(a_2 - a_3) = 0 \quad \text{and} \quad (a_1 - a_3)(a_2 - a_4) = 0.$$

So $a_1 = a_4$ or $a_2 = a_3$, and also $a_1 = a_3$ or $a_2 = a_4$. Hence at least three of the parts have the same size.

Work in the paper [20] by Colbourn et al. includes results (couched in the language of triple systems with holes) on $K(a_1, a_2, 1, 1, \ldots, 1)$ in certain cases, and a recent paper [11] by Bryant and Horsley nicely completes the determination of sufficiency of the conditions for existence of a 3-cycle system of $K(a_1, a_2, 1, 1, \ldots, 1)$. These are that the degrees are all even, the number of edges is a multiple of 3, and there are sufficient parts of size 1 to enable completion of a_1 triangles from a point in the part of size a_2 joining each of the points in the part of size a_1, where $a_1 \geq a_2$ (that is, there are at least a_1 parts of size 1).

Other general results for 3-cycles (or K_3 decompositions) for arbitrary complete multipartite graphs remain open.

2.2. Even length cycles

An oft-quoted paper of Dominique Sotteau's [41] deals with complete bipartite graph decompositions, into cycles of some (necessarily even) fixed length. She showed:

Theorem 4. (Sotteau [41]) *The complete bipartite graph $K_{a,b}$ can be decomposed into cycles of length $2k$ if and only if a and b are even, $a \geq k$, $b \geq k$, and $2k$ divides ab.*

No such general result is known when the complete multipartite graph has more than two parts, even when the restriction is to cycles of even length. In [14], Cavenagh and Billington list certain necessary conditions for a $2k$-cycle decomposition of the complete multipartite graph $K(a_1, a_2, \ldots, a_n)$ to exist, and show sufficiency for 4-, 6- and 8-cycles. If we let $K(a_1, a_2, \ldots, a_n)$ satisfy $a_1 \geq a_2 \geq \ldots \geq a_n$, then these conditions are (see [14], p. 50):

(i) $2k$ must divide the total number of edges;
(ii) the total number of vertices is at least $2k$;
(iii) half the degree of a vertex in the smallest sized part, $\frac{1}{2}\sum_{i=1}^{n-1} a_i$, must be less than or equal to the total number of cycles (which of course is the total number of edges divided by $2k$); i.e. $k\sum_{i=1}^{n-1} a_i \leq \sum_{1\leq i<j\leq n} a_i a_j$;
(iv) the number of vertices *not* in the largest part must be at least k, i.e. $\sum_{i=2}^{n} a_i \geq k$;
(v) the degree of each vertex is even;
(vi) the a_i are all of the same parity; if this parity is odd, then n is odd.

Condition (iv) above is not independent, but is implied by other conditions, and (v) and (vi) are in fact equivalent. However (iii) above is independent of the other conditions listed. This is illustrated by the complete multipartite graph $K(6,6,2)$ which fails only condition (iii) in the case $2k = 12$. (There is no 12-cycle decomposition of $K(6,6,2)$.)

Any graph $K(a_1, a_2, \ldots, a_n)$ satisfying conditions (i)–(vi) above is called $2k$-*sufficient*. It is easy to see that if each a_i is odd, then $n \equiv 1 \pmod 4$ if k is odd and $n \equiv 1 \pmod 8$ if k is even.

A $2k$-*small* graph $K(a_1, \ldots, a_n)$ is a $2k$-sufficient graph such that for all i, $1 \le i \le n$, there is no positive integer a_i' with

$$a_i' < a_i, \quad a_i' \equiv a_i \begin{cases} \pmod{2k} & \text{if } k \text{ is odd} \\ \pmod{k} & \text{if } k \text{ is even} \end{cases}$$

such that $K(a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n)$ is also $2k$-sufficient. Existence of a decomposition of any such $2k$-small graph into $2k$-cycles would suffice, since if $K(a_1, \ldots, a_n)$ has an edge-disjoint decomposition into $2k$-cycles, then so does $K(a_1, \ldots, a_{i-1}, a_i + 2k, a_{i+1}, \ldots, a_n)$, and if k is even, so does $K(a_1, \ldots, a_{i-1}, a_i + 2, a_{i+1}, \ldots, a_n)$, for any i, $1 \le i \le n$.

However, $2k$-small graphs can be quite large! For instance, $K(180, 20, 2)$ is 40-small (the graph $K(160, 20, 2)$ fails (iii) above).

If one could find $2k$-cycle decompositions of all $2k$-small graphs, $2k \ge 10$, this would show sufficiency of conditions (i)–(vi) above for a $2k$-cycle decomposition of $K(a_1, \ldots, a_n)$ to exist.

2.3. Odd length cycles

Very little is known about decompositions of a complete multipartite graph $K(a_1, \ldots, a_n)$ into k-cycles where k is odd. For 3-cycles, see Subsection 2.1 above. For 5-cycles, even the tripartite graph case remains open (see the next section), although the equipartite case was solved in [8], along with the *lambda*-fold equipartite graph case. In two recent preprints [37], [38], decompositions in the equipartite case $K_n(m)$ for 7-cycles and also for p-cycles, where $p \ge 11$ is a prime, are given, when the obvious necessary conditions hold.

Returning to the non-equipartite case, there are some partial results for 5-cycle decompositions of complete tripartite graphs, which we consider in the next section.

2.4. Tripartite graph decompositions: cycles

Suppose the three parts of the tripartite graph under consideration have the same size; so consider the graph $K_{n,n,n}$. Cavenagh [12] gave necessary and sufficient conditions for $K_{n,n,n}$ to have an edge-disjoint decomposition into k-cycles. These conditions are the obvious necessary ones: the number of vertices, $3n$, must be at least k, and the number of edges, $3n^2$, must be a multiple of k.

The proof of sufficiency splits into the cycle length k being a multiple of 3, or not. The latter case uses a result which enables a closed *k-trail* decomposition of $K_{m,m,m}$ in which any vertex occurs at most ℓ times, to yield a *k-cycle* decomposition of $K_{\ell m,\ell m,\ell m}$.

In the case of the tripartite graph $K_{r,s,t}$ when the three parts have possibly different arbitrary sizes, necessary and sufficient conditions for a decomposition into k-cycles is not known in general. As remarked above, for 3-cycles, the three parts must have the same size. For $2k$-cycles, $k \geq 2$, an exact decomposition will only be possible if all three parts have even size (and satisfy (i)–(vi) in Section 2.2 above). It is still an open problem to verify that the necessary conditions in Section 2.2 are sufficient for existence of a decomposition into $2k$-cycles, $2k \geq 10$, even when there are only three parts.

When the cycle length $2k+1$ is odd and greater than 3, the problem of determining necessary and sufficient conditions for a decomposition of $K_{r,s,t}$ into $2k+1$-cycles remains open. Indeed, even in the case of 5-cycles, determination of a 5-cycle decomposition of $K_{r,s,t}$, whenever the "obvious" necessary conditions hold, is incomplete. This problem was considered by Mahmoodian and Mirzakhani [36], where the necessary conditions for a decomposition of $K_{r,s,t}$ $(r \leq s \leq t)$ into 5-cycles were listed:

(i) r, s and t are all even or all odd;
(ii) $rs + rt + st$ is divisible by 5;
(iii) $t \leq 4rs/(r+s)$.

It is easy to see that these conditions are necessary: condition (i) ensures that every vertex has even degree; condition (ii) ensures that the total number of edges is a multiple of 5; and condition (iii) follows from the fact that the number of edges between the two smallest parts (of sizes r and s) must be greater than or equal to the total number of 5-cycles (which is one fifth of the total number of edges).

Mahmoodian and Mirzakhani [36] deal with the case when r, s, t are all 0 (mod 5) and and they offer a prize of 100 000 Iranian Rials for the proving of the sufficiency of these three conditions. Cavenagh and Billington [15], [13] deal with further cases, so the current state of play is that the necessary conditions (i), (ii), (iii) above are sufficient when two (or more) of the partite sets have the same size, or when all parts have even size. So the remaining open case is when all partite sets are odd, and of three different sizes. It is likely that the method used in [15] will work for this open case, but it will be long and tedious! This method basically exploits the connection between a tripartite graph and a kind of latin rectangle. The edges in the graph $K_{r,s,t}$, where $r \leq s \leq t$, can be represented by the entries in the so-called "latin rectangle" shown in Figure

2.1. The entries in $[A\ B]$ are row latin in the t symbols; the entries in $\begin{bmatrix} A \\ C \end{bmatrix}$ are column latin. Each filled cell in A corresponds to a triangle in $K_{r,s,t}$; each entry in B (and note that B need not be column latin) corresponds to an edge between parts of sizes r and t, while each entry in C corresponds to an edge between parts of size r and s.

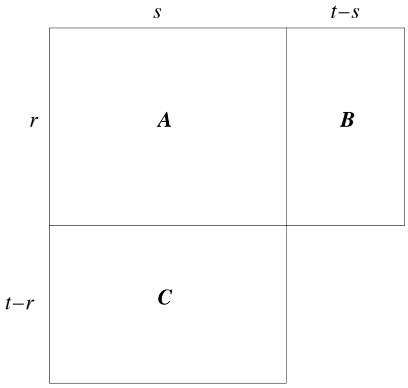

Figure 2.1

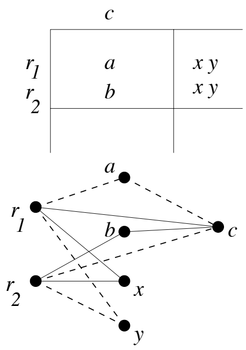

Figure 2.2

Figure 2.2 gives an example of how some of the entries in a latin representation, consisting of two triangles and four further edges, can be "traded" to give two 5-cycles. The edges are listed explicitly in the table below.

Edges from entries in latin representation	Edges reconfigured as 5-cycles
(r_1, a, c), (r_2, b, c), r_1c, r_1d, r_2c, r_2d	(r_1, c, b, r_2, c), (r_1, a, c, r_2, d)

By judicious partitioning of a suitable latin representation into various trades like the one illustrated in Figure 2.2 (but often considerably more complicated!), results on decomposition of tripartite graphs $K_{r,s,t}$ into various cycles is achieved.

A precurser of this method was used in [2], where $K_{r,s,t}$ is decomposed into specified numbers of 3- and 4-cycles; it was fully exploited in the 5-cycle papers [15], [13].

2.5. Tripartite graph decompositions: closed trails

Some recent work by Billington and Cavenagh [3] has dealt with the decomposition of a complete tripartite graph with equal sized parts into any number of any length closed trails. Balister [1] showed that a complete graph K_n (when n is odd) or $K_n - F$ (where F is a 1-factor in the case n is even) can be decomposed into circuits of lengths $m_1, m_2, \ldots, m_t$ whenever $m_i \geq 3$ $(1 \leq i \leq t)$ and $\sum_{i=1}^{t} m_i$ equals the number of edges in K_n (n odd) or in $K_n - F$ (n even). The paper [3] does likewise for the graph $K_{n,n,n}$. In particular, the following is proved.

Theorem 5. ([3]) *The complete tripartite graph $K_{n,n,n}$ has an edge-disjoint decomposition into closed trails of (not necessarily distinct) lengths $m_1, m_2, \ldots,$ $\ldots, m_t$ if and only if $m_i \geq 3$ for $1 \leq i \leq t$ and $\sum_{i=1}^{t} m_i = 3n^2$.*

The method used to show sufficiency of the obvious necessary conditions involves a back-circulant latin square of order n, which itself represents the edges of $K_{n,n,n}$ as a set of n^2 triangles (with row, column, entry being the index sets of the three parts in $K_{n,n,n}$). By judicious use of "trades", working, generally speaking, down pairs of columns (or three columns in one instance when n is odd), the latin square is partitioned up into closed trails of the required lengths. On the whole, these trails consist of linked cycles of lengths 3, 4 and 5 (depending upon their length modulo 3), although some cases require further refinement (see [3] for details).

Two straightforward observations help with the proof. One is that a collection of 3-cycles in $K_{n,n,n}$ arising from entries in a latin square of order n will form a connected circuit provided that the entries can be ordered with adjacent ones being (i) in the same row, or (ii) in the same column, or (iii) the same symbol. Another is that any set of integers (being potential circuit lengths), $P = \{x_1, x_2, \ldots, x_m\}$ satisfying $\sum_{i=1}^{m} x_i = 3n^2$, can be partitioned into subsets P_j, each containing at most three of the x_i, so that the numbers in each P_j sum to a multiple of 3, and no subset of P_j sums to a multiple of 3.

For example, if $P = \{4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 7, 8, 9, 10, 10, 17\}$ (these numbers sum to 3×6^2), in the case of the graph $K_{6,6,6}$, a possible partition of P is given by

$$\{4, 4, 4\}, \{4, 4, 4\}, \{9\}, \{4, 4, 10\}, \{5, 5, 5\}, \{7, 8\}, \{10, 17\}.$$

Then a backcirculant latin square of order 6 is appropriately partitioned up so that connected entries give rise to trails of lengths in P_j, for each set P_j in the partition of P. For instance, consider $\{7, 8\}$ above; trails consisting of $C_3 \cup C_4$ and $C_3 \cup C_5$ have lengths 7 and 8, and arise from cells (the five 3-cycles) such as those in Figure 2.3 below.

There is an obvious corollary to Theorem 5, since a closed trail of length less than 6 is a cycle:

	c_1	c_2
r_1	1	2
r_2	2	3
r_3	3	

$(r_1, 1, c_1, 2)$, $(r_3, 3, c_1)$, which form a 7-trail;
$(r_1, c_1, r_2, 2, c_2)$, $(r_2, 3, c_2)$, which form an 8-trail.

Figure 2.3

Corollary 5.1. *Let α, β, γ be non-negative integers such that $3\alpha + 4\beta + 5\gamma = 3n^2$. Then there exists an edge-disjoint decomposition of $K_{n,n,n}$ into α 3-cycles, β 4-cycles and γ 5-cycles.*

ADCT graphs

If a connected even graph G contains a closed trail of length m_i, for $1 \le i \le t$, and $\sum_{i=1}^{t} m_i = |E(G)|$, then G is said to be *arbitrarily decomposable into closed trails* (ADCT) if G has an edge-disjoint decomposition into closed trails of lengths m_i, $1 \le i \le t$.

Using this terminology, Balister [1] showed that K_n (n odd) and $K_n - F$ (n even, F a 1-factor) are both ADCT, and Billington and Cavenagh [3] showed that $K_{n,n,n}$ is ADCT. In the bipartite case, the graph $K_{r,s}$ with r and s both even was shown by Horňák and Woźniak [32] to be ADCT (here, necessarily, each trail length m_i must be even). It can be easily verified that if the general tripartite graph $K_{r,s,t}$ is ADCT, then the partite sizes are 1, 1, 3 or 1, 1, 5, or else $r = s = t$. Hence [3] completes the determination of ADCT tripartite graphs.

3. Extra properties on the decomposition.

3.1. Resolvable cycle decompositions of complete multipartite graphs

The requirement of *resolvability* imposed on a cycle decomposition means that the cycles in the decomposition are able to be partitioned into *resolution classes*, where each resolution class contains, once each, all the vertices in the whole graph. In graph theoretic terms, this means that the cycle decomposition forms a 2-factorization of the complete multipartite graph, where the 2-factors consist of a number of k-cycles.

The following example illustrates an extra property which a decomposition may have, which will be reviewed in Sections 3.3 and 3.4.

Example 3.1. Two resolvable 4-cycle decompositions of $K(2, 2, 2, 2)$.

Figure 3.1(a) shows a resolvable 4-cycle decomposition of $K(2, 2, 2, 2)$, with the three parallel classes (or 2-factors), while Figure 3.1(b) shows one which is not only resolvable but also *gregarious*, in that each cycle has its vertices in different parts (see Section 3.3).

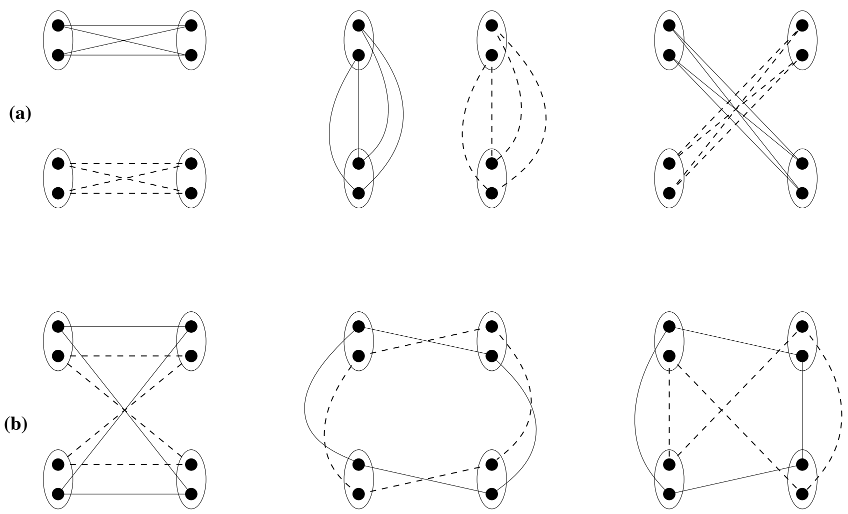

Figure 3.1

Given Sotteau's result [41] for bipartite graphs, Example 3.1(a) perhaps seems more "natural", and is certainly the easier one to find! We shall meet Example 3.1(b) in Section 3.4; note that each 4-cycle has all its vertices in different parts of the equipartite graph $K(2, 2, 2, 2)$.

Thus a cycle decomposition of $G = K(a_1, \ldots, a_n)$ is said to be *resolvable* if it forms a partition of the edge-set of G into 2-factors, with each 2-factor a union of cycles. If the cycles are all of the same length k, then this is also called a C_k-factorization of G. In Liu [34], [35], the problem of finding such a C_k-factorization is posed as a generalisation of the famous Oberwolfach problem ([27] Guy 1967 — see Liu), and of the spouse-avoiding variant of this ([33] Huang, Kotzig and Rosa). For the C_k-factorization case, there are n delegations, with a_i people in the ith delegation. These $\sum_{i=1}^{n} a_i$ people are to be seated at a number s of round tables seating $t_1, \ldots, t_s$ people (where each table seats k people for the case of k-cycles), and where $\sum_{i=1}^{s} t_i = \sum_{i=1}^{n} a_i$, for a number (which can be calculated!) of different meals, so that every person sits next to every person *not* in his or her delegation, exactly once.

This is the same as finding a 2-factorization of $K(a_1, \ldots, a_n)$ where each 2-factor consists of s cycles, of lengths $t_1, t_2, \ldots, t_s$.

In 1991, Hoffman and Schellenberg [30] showed that $K_{n(2)}$ (having n parts of size 2) has a C_k-factorization whenever the necessary conditions (even degree, and total number of vertices a multiple of k) hold, except that there is *no* C_3-factorization of $K_{3(2)}$ or of $K_{6(2)}$.

Also in 1991, Piotrowski [39] dealt with the bipartite case, and showed that $K_{m,m}$ has a C_k-factorization if and only if m and k are even and $k|2m$, except that $K_{6,6}$ has no C_6-factorization.

In 1993, Rees [40] proved that $K_{n(m)}$ has a C_3-factorization if and only if the degree, $m(n-1)$, is even, and the total number of vertices, mn, is divisible by 3, except that $K_{2,2,2}$, $K_{6,6,6}$ and $K_{6(2)} = K(2, 2, 2, 2, 2, 2)$ have no C_3-factorization.

In 2000, and in 2003, Liu [34], [35] showed that for $k \geq 3$, $n \geq 3$, the complete equipartite graph $K_{n(m)}$ has a C_k-factorization if and only if the degree is even and $k|nm$, except for the four cases mentioned above: $K_{6,6}$ has no C_6-factorization, and $K_{2,2,2}$, $K_{6,6,6}$ and $K_{6(2)}$ have no C_3-factorization. Thus for the equipartite case, completed by Liu [34], [35], the following holds:

Theorem 6. *When $k \geq 3$ and $n \geq 2$, there is a resolvable k-cycle decomposition of $K_{n(m)}$ (i.e., a C_k-factorization of the complete equipartite graph having n parts of size m) if and only if*

$$k|mn, \qquad m(n-1) \text{ is even}, \qquad k \text{ is even if } n = 2,$$

and there is no resolvable 3-cycle decomposition of $K_{2,2,2}$, $K_{6,6,6}$ or $K_{6(2)}$, nor any resolvable 6-cycle decomposition of $K_{6,6}$.

In the case of different cycle lengths, some (partial) results have also been obtained. Indeed, Liu [35] uses the fact that $K_{n(4)}$ has a $\{C_3, C_5\}$-factorization for $n \geq 3$ and $n \neq 7, 10, 11$, in order to prove his main result on C_t-factorizations. However, the generalised Oberwolfach problem, with delegations of different sizes $a_1, \ldots, a_n$ and/or with different sized tables $t_1, \ldots, t_s$, is clearly very difficult, and remains open.

Recent work by Hoffman and S.H. Holliday (see [31], [29]) looks at the equipartite case minus a 1-factor, and gives a resolution into $2k$-cycles. In particular, they prove (using a delightfully named "cracked easter egg" approach) the following.

Theorem 7. ([31], [29]) *There is a resolvable $2k$-cycle decomposition of $K_{n(m)} - F$, where F is a 1-factor, if and only if m is odd, n is even, and $2k|mn$.*

3.2. Colouring cycle decompositions of complete multipartite graphs

An m-cycle decomposition of a graph G is said to be *equitably k-coloured* if the vertices of G are coloured with k colours $c_1, \ldots, c_k$ in such a way that for each cycle C in the decomposition, the number of vertices coloured c_i differs by at most 1 from the number of vertices coloured c_j, for $1 \leq i, j \leq k$. Most work on equitable colouring has been done for decompositions of complete graphs, or of complete graphs minus a 1-factor, but Waterhouse [42] deals with equitable 2-colourings of complete multipartite graphs into cycles. She shows that a 3-cycle decomposition of $K_{n(m)}$ has an equitable 2-colouring if and only if (besides the usual requirements on the number of edges, even degree, at least three parts) there are only 3 or 4 parts altogether.

She also shows that a 5-cycle decomposition of $K_{n(m)}$ exists with an equitable 2-colouring if and only if the usual conditions (even degree, and number of edges a multiple of 5) hold.

In the case of a 4-cycle decomposition of $K(a_1, \ldots, a_n)$, Waterhouse shows that one with an equitable 2-colouring exists if and only if each a_i is even; similarly, a 6-cycle decomposition with an equitable 2-colouring exists if and only if the necessary conditions for a decomposition hold *and* all the parts have even size.

3.3. Gregarious cycle decompositions

The first mention of a cycle decomposition in a multipartite graph being *gregarious* appeared in [6] in 2003. The word is chosen for its usual meaning of "outgoing", or "reaching out". Basically, a cycle is said to be gregarious if its vertices occur in as many different parts of the multipartite graph as possible; so provided there are as many parts as there are vertices in the cycle, every vertex will appear in a different part of the graph. Figure 3.1 illustrates the difference between a 4-cycle decomposition of $K(2, 2, 2, 2)$ which is gregarious (in (b)) and which is certainly not gregarious (in (a)). Since no "zig-zagging" to and fro between a pair of parts is allowed, even the case of a 4-cycle decomposition is considerably harder, with the condition of being gregarious imposed.

The paper [6] deals with a gregarious 4-cycle decomposition of $K_{r,s,t}$, and gives necessary and sufficient conditions for existence. Since there are only three parts to this graph, each 4-cycle is required to meet all three parts, and necessarily will have two of its four vertices lying in the same part.

Theorem 8. ([6]) *There exists a gregarious 4-cycle decomposition of $K_{r,s,t}$, $r \leq s \leq t$, if and only if*

(i) $r \equiv s \equiv t \equiv 0 \pmod 2$*;*
(ii) $s(r+t) - rt \geq 8$*;*

(iii) $r(s+t) - st \geq 8$ *or* $r(s+t) - st = 0$.

The necessity of conditions *(ii)* and *(iii)* is not immediately obvious. These follow from counting the three types of possible cycles (see Figure 3.2), which yields

$$t \geq \frac{rs}{r+s}, \quad s \geq \frac{rt}{r+t}, \quad r \geq \frac{st}{s+t}.$$

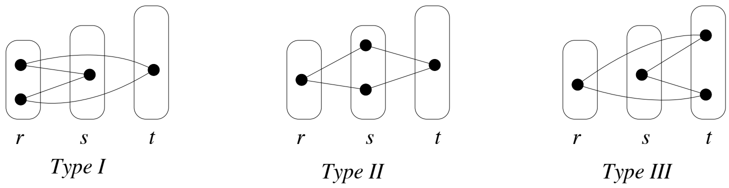

Figure 3.2

Recall that $r \leq s \leq t$. Then if $r \neq s$, we have t bounded above by $rs/(r+s)$, while if $r < s$ then t is unbounded. Also the number of 4-cycles of each type is either 0 or at least 2. So (in order to precisely cover all edges in $K_{r,s,t}$) we have $2 \leq$ number of type II $\leq$ number of type III. The necessity follows.

When at least two of r, s, t are 0 (mod 4), existence of a gregarious decomposition follows from existence of a gregarious *path* decomposition of the tripartite graph with vertex sets of half the size. Then by doubling the number of vertices, each gregarious path gives rise to two gregarious 4-cycles in this tripartite case; see Figure 3.3.

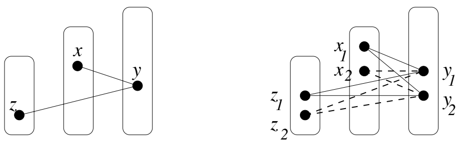

Figure 3.3

However, when two or more of r, s, t are 2 (mod 4), we still use gregarious paths, but the method requires a “latin representation” approach again, related to that described in Section 2.4 above, but with a 3×3 “hole”; see [6] for details!

If we now consider the case of gregarious 4-cycle decompositions of complete multipartite graphs with *more* than three parts, each vertex of every 4-cycle will lie in a different partite set. For four parts, it is straightforward to verify that $K(a_1, a_2, a_3, a_4)$ has a gregarious 4-cycle decomposition if and only if $a_1 = a_2 = a_3 = a_4$ and all a_i are even.

The present state of knowledge on gregarious 4-cycle decompositions includes the following for equipartite and almost equipartite graphs.

Theorem 9. ([7]) *If conditions are right for a* 4*-cycle decomposition of $K_{n(m)}$ or of $K_{n(m),t}$, then a gregarious* 4*-cycle decomposition of these graphs also exists, provided $t \leq m(n-1)/2$.*

When the number of parts n is 1 (mod 8), then a 4-cycle system of K_n can be taken, and the points "blown up" m-fold to obtain a gregarious 4-cycle decomposition of $K_{n(m)}$. Otherwise, if $n \not\equiv 1 (\text{mod } 8)$, necessarily all parts must have even size. So the case $K_{n(2)}$ is dealt with, from which a suitable gregarious decomposition of $K_{n(2m)}$ follows. However the non-equipartite case is less straightforward, and incomplete at the moment, although certain cases (all but one part the same size) have been dealt with in [7].

It is perhaps worth remarking that any group divisible design with block size 5 will give rise to a gregarious 5-cycle decomposition, since of course each block of size 5 will give rise to two 5-cycles with all five vertices in different parts or groups. (The same also applies to p-cycles for any odd prime p.) However, there will be gregarious 5-cycle decompositions of complete multipartite graphs in cases when a K_5-decomposition is not possible.

3.4. Resolvable and gregarious cycle decompositions

Since work on cycle decompositions with the extra property of being gregarious is relatively new, very little has been done on requiring the decomposition to be both resolvable *and* gregarious. The paper [9] is a start in this direction.

In [9], Billington, Hoffman and Rodger investigate n-cycle decompositions of the complete equipartite graph $K_{n(m)}$ which are both gregarious *and* resolvable. The main result is:

Theorem 10. ([9]) *There exists an edge-disjoint decomposition of $K_{n(m)}$ into n-cycles which is both gregarious and resolvable if and only if m is not odd when n is even, and $(m, n) \neq (2, 3), (6, 3)$.*

A gregarious resolvable decomposition of $K_{3(2)}$ or $K_{3(6)}$ into 3-cycles is not possible (for these two cases would imply existence of a pair of orthogonal latin squares of orders 2 and 6!).

Liu's [34], [35] resolvable decompositions (or 2-factorizations!) of $K_{n(m)}$ are not generally gregarious, so his results did not help in [9].

4. Maximum Packings.

When the necessary conditions for an edge-disjoint decomposition of $K(a_1, \dots, a_n)$ into k-cycles fail, it is natural to ask for a *packing* of this complete multipartite graph with k-cycles. A k-cycle packing of $G = K(a_1, \dots, a_n)$ is a set of edge-disjoint k-cycles in G. The packing is *maximum* if its number of cycles is not less than the number of cycles in any other packing of G with k-cycles. The edges of G not occurring in any k-cycle are referred to as the *leave* of the packing. Obviously a maximum packing will have a minimum leave.

Not much work has been done on packing complete multipartite graphs K with cycles, in the case that K is "genuine", that is, when K is *not* a complete graph (all parts of size 1).

For 3-cycles, maximum packings in the equipartite case, $K_{n(m)}$, for all n and m, are dealt with in [10].

In [4], the problem of finding a maximum packing of $K(a_1, \dots, a_n)$ with 4-cycles is completely solved, and the minimum leaves are given. (These minimum leaves can be quite large!) Using this result, a natural generalisation was to determine a maximum packing of the λ-fold graph $\lambda K(a_1, \dots, a_n)$ with 4-cycles; this appears in [5].

For 6-cycles, the problem of finding a maximum packing in the equipartite case has recently been completed [23]. In this paper Fu and Huang find a maximum packing of the complete equipartite graph $K_{m(n)}$ with edge-disjoint 6-cycles, and they give the minimum leaves. (They also find a minimum *covering*, where every edge of $K_{m(n)}$ appears in at least one 6-cycle, and where the "excess" edges used in more than one 6-cycle — sometimes called the padding — form a set as small as possible.)

As remarked in Section 2.2 above, necessary and sufficient conditions are known [14] for the existence of a decomposition (with empty leave) of $K(a_1, \dots, a_n)$ into 4-cycles, 6-cycles and 8-cycles. As far as I am aware, no packing results for cycles in multipartite graphs are known other than those mentioned above for 3-, 4- and 6-cycles.

5. Conclusions and some open problems.

There are several related topics which are not mentioned above, such as coverings, and hamilton decompositions. Decompositions into cycles of different lengths have also been ignored here because of space considerations, as have λ-fold decompositions. See [22], [16] for examples in the bipartite case, and [2] in the tripartite case, with cycles of differing lengths within the one decomposition.

Other papers deal with group divisible designs which allow λ_1 edges between pairs of points in the *same* group, and λ_2 edges between pairs of points in different groups. Cycles of lengths 3 and 4 have been dealt with in this way; see papers [24] and [26] by Fu, Rodger and Sarvate for the 3-cycle case, and Fu and Rodger [25] for the 4-cycle case.

Appended below are some of the open problems mentioned here.

Problem 2.1.
Find necessary and sufficient conditions on $K(a_1, \ldots, a_n)$ for it to have an edge-disjoint decomposition into 3-cycles. (See Colbourn [17] for six conditions which are shown sufficient for orders up to 60.)

Find further partial results in this direction; see [11], [20], [17], [18], [19].

Problem 2.2.
Show that a graph $K(a_1, \ldots, a_n)$ which is $2k$-sufficient has a decomposition into $2k$-cycles, for $2k \geq 10$.

Problem 2.3.
Prove that the necessary conditions for a 5-cycle decomposition of $K_{r,s,t}$ are sufficient in the remaining case, when r, s, t are all odd and all different.

Problem 2.4.
Find necessary and sufficient conditions for $K_{2r,2s,2t}$ to have a decomposition into $2k$-cycles, for any $k \geq 5$. Note that this is really a subset of Problem 2.2 above.

Problem 2.5.
Show that the graph $K_{r,s} - F$, where r, s are odd and F is a (smallest possible) spanning subgraph of odd degree, is ADCT. (See Section 2.5.)

Problem 3.2.
Investigate equitable k-colourings of complete multipartite graph cycle decompositions, for $k > 2$.

Problem 4.1.
Investigate maximum packings of complete multipartite graphs with small cycles. In particular:

(1) Investigate packing $K_{r,s,t}$ with 3- and 4-cycles (see [2] for the case of an empty leave, and a specified number of 3- and 4-cycles).

(2) Consider packing tripartite graphs with 5-cycles. With methods used in [14], [13] this could prove no harder than Problem 2.3!

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