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# MALTITUDES, ANTICENTERS AND ALTITUDES OF CYCLIC POLYGONS AND POLYHEDRA

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In this paper we extend to cyclical polygons the concepts of maltitude and anti-center of a cyclic quadrilateral and to cyclical polyhedra the concepts of Monge plane and Monge point of a tetrahedron. In addition, we find several properties of these new concepts. The research is conducted through the study of m-point systems, starting from the results on the centroids and the medians of these systems.

## 1. Introduction

In [4, 6, 10] it is studied the problem of concurrency of the maltitudes of a quadrilateral; in particular in [10] it is proved that this happens if and only if the quadrilateral is cyclic. The common point of the maltitudes is called anticenter. It also proved that the circumcenter, the centroid and the anticenter of a cyclic quadrilateral are collinear and the straight line that contains them is called Eulers line, as in triangles. In [1, 10] an analogous study is conducted on the tetrahedra and instead of the notions of maltitude and anticenter of a cyclic quadrilateral the notions of Monge plane and Monge point of a tetrahedron are considered. Also, the straight line containing the circumcenter, the centroid and the Monge point of a tetrahedron is called the Eulers line of the tetrahedron.

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The aim of the paper is to extend to cyclic polygons the concept of maltitude and anticenter of a cyclic quadrilateral and to extend to cyclic polyhedra the concept of Monge plane and of Monge point of a tetrahedron.

The research is conducted by studying m-system of points [7], starting from the results on the centroids and on the medians of these systems (paragraph 2).

In paragraphs 2-7 we consider plane m-systems, i.e. systems constituted by m points of a plane, and the results are interpreted for polygons. In particular, in paragraphs 3 and 4 we define the notions of n-maltitude and of n-anticenter of a cyclic m-system, that is a system whose m points lie on a circle, and we obtain a general theorem on cyclic polygons (theorem 4.2), that contains, as particular cases, the results found in [4, 6, 10]; in paragraph 5 we study the system of n-anticenters of a cyclic m-system and we find a result on cyclic polygons with m sides, with m4 (theorem 5.2), that generalize the property on the quadrilateral of the orthocenters of a cyclic quadrilateral [2, 7]; in paragraph 6 we introduce the notion of n-altitude of a cyclic m-system, with m $\geq$ 6 and, in particular, we find a theorem of concurrency of n-altitudes of a cyclic polygon (theorem 6.6). in paragraph 7 we apply the concept of n-anticenter to the study of some geometric loci. In the end, in paragraph 8, we observe that all definitions and all theorems of the previous paragraphs that hold for plane systems, still hold, with appropriate changes, for three-dimensional systems, i.e. for systems of points of the three dimensional space, and, then, for cyclic polyhedra. In particular, the concepts of 1-maltitude and of 1-anticenter of a cyclic polyhedra generalize the ones of Monge plane and Monge point of a tetrahedron, respectively [1, 9].

## 2. Centroid and medians of a polygon

We give some definitions and theorems, contained mostly in [8], that will be useful in this paper.

A set $S$ of m points of the plane is called *m-system* or *system of order m*.

An m-system $S$, with m$\geq$3, is said *cyclic* if all points of $S$ lie on a circle. The centre O of the circumference is called *circumcentre* of $S$. In particular, any 3-system is cyclic.

A nonempty subset $S'$ of an m-system $S$, with k points, is called *k-subsystem* of $S$ or *subsystem of order k* of $S$. There exist $\binom{m}{k}$ different subsystems of order k.

Two subsystems $S'$ and $S''$ of an m-system $S$ are said to be *complementary* if they are a partition of $S$, i.e. if $S' \cup S'' = S$ and $S' \cap S'' = \emptyset$. We say that also $S'$ is the complementary of $S''$ and $S''$ is the complementary of $S'$. Note that if $S'$ is a k-subsystem, $S''$ is an (m-k)-subsystem.

We call A$_i$ (i = 1, 2, , m) the points of an m-system $S$ and with $x_i$ the position vector of A$_i$ with respect to a fixed point P of the plane. We call *centroid* of $S$

the point $H_0$ of the plane whose position vector space with respect to P is:

$$x = \frac{1}{m}\sum_{i=1}^{m} x_i.$$

It is proved, in [8], that the point $H_0$ does not depend from the choice of P.

Let $S'$ be a k-subsystem of $S$, with $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$ (where $\lfloor \frac{m}{2} \rfloor$ is the maximum integer contained in $\frac{m}{2}$), and let $H_0'$ be its centroid. Let $S''$ be the subsystem complementary of $S'$ and let $H_0''$ be its centroid. We call *median* of $S$ relative to $S'$ the segment $H_0'H_0''$.

The following theorem holds ([7], p. 122):

**Theorem 2.1.** *Each median of an m-system S passes through the centroid $H_0$ of S. Moreover, $H_0$ divides the median $H_0'H_0''$ relative to a k-subsystem $S'$ of S in two parts such that:*

$$H_0'H_0 = \frac{m-k}{k} H_0H_0''. \tag{1}$$

From Theorem 2.1 it follows that the correspondence between the system of centroids and the (m-k)-subsystems of $S$ and the system of the centroids of the k-subsystems of $S$, that associates to the centroid $H_0''$ of an (m-k)-subsystem $S''$ of $S$ the centroid $H_0'$ of the k-subsystem $S'$ complementary to $S''$, is the homothety of coefficient $-\frac{m-k}{k}$ with centre in the centroid $H_0$ of $S$. Therefore, from 2.1, the following corollaries hold:

**Corollary 2.2.** *In the homothety of ratio $-\frac{m-k}{k}$ and with centre in the centroid $H_0$ of an m-system S, the centroid of any k-subsystem $S'$ of S is the correspondent of the centroid of the complementary (m-k)-subsystem of $S'$.*

**Corollary 2.3.** *The segment $H_{01}'H_{02}'$ joining the centroids of two k-subsystems $S_1'$, $S_2'$ of an m-system S is parallel to the segment $H_{01}''$ $H_{02}''$ joining the centroids of the (m-k)-subsystems complementary to $S_1'$, $S_2'$, and moreover it is:*

$$H_{01}'\, H_{02}' = \frac{m-k}{k}\, H_{01}''\, H_{02}''$$

**Corollary 2.4.** *If S is an m-sub-system with m=2k, the centroids of any two complementary k-subsystems of S are symmetric with respect to the centroid $H_0$ of S.*

Given a polygon $P$, we associate to it the system $S$ whose points are the vertices of $P$. We observe that, given a polygon $P$ the associated system is uniquely determined, while in correspondence with an m-system $S$ there may exist up to $(m-1)!$ different polygons (convex, concave or crossed) having $S$ as associated system.

For example, given the system $S = \{A_1, A_2, A_3, A_4\}$, if any three points $A_i$ they are not collinear, there exist 6 different quadrilaterals with $S$ as associated system: $A_1A_2A_3A_4$, $A_1A_2A_4A_3$, $A_1A_3A_2A_4$, $A_1A_3A_4A_2$, $A_1A_4A_2A_3$, $A_1A_4A_3A_2$.

Let $P$ be a polygon and let $S$ be its associated system.

The 1-subsystems of $S$ determine the vertices of $P$, the 2-subsystems of $S$ determine the sides and the diagonals of $P$, the 3-subsystems of $S$ determine the sub-triangles of $P$ (i.e., triangles with vertices three vertices of $P$), and so on.

We call *median* of $P$, each median of the system associated $S$ and centroid of $P$ the centroid of $S$. The centroid of the 2-subsystems of $S$ are the mid-points of the sides and of the diagonals of $P$. The medians of a triangle are its usual medians, the medians of a quadrilateral relative to the sides and to the diagonals are the bimedians [3]. The usual centroid of a triangle or of a quadrilateral is the centroid of the system associated $S$.

Theorem 2.1 and its corollary can easily be formulated in terms of polygons.

## 3. 1-maltitudes, 1-anticenter and Eulers line of a cyclic polygon

Let $S = \{A_1, A_2, ..., A_m\}$ be any m-system, with $m \geq 3$.

Consider the $\binom{m}{2}$ subsystems of order 2 of $S$, $S' = \{A_i, A_j\}$, for i, j = 1, 2, , m and i $\neq$ j. Let $M_{ij}$ be the midpoint of the segment $A_iA_j$ and let $H_0^{(ij)}$ be the centroid of the subsystem $S''$ complementary to $S'$. For theorem 2.1, the medians $M_{ij}H_0^{(ij)}$ meet in the centroid $H_0$ of $S$ and it is:

$$M_{ij}H_0 = \frac{m-2}{2} H_0 H_0^{(ij)} \tag{2}$$

The straight line passing through $H_0^{(ij)}$ and perpendicular to $A_iA_j$ is called *1-maltitude* relative to the subsystem $S'$. Then, $S$ has $\binom{m}{2}$ 1-maltitudes.

**Theorem 3.1.** *The 1-maltitudes of an m-system S, with m≥3, are concurrent in a point if and only if S is cyclic.*

*Proof.* For (2), the homothety with ratio $-\binom{m-2}{2}$ and centre $H_0$ maps the 1-maltitudes of $S$ in the perpendicular bisectors of the segments $A_iA_j$ (in fact it maps $H_0^{(ij)}$ into $M_{ij}$ and the 1-maltitude into a parallel line through $H_0^{(ij)}$ , that is the perpendicular bisector of $A_iA_j$). Therefore, the 1-maltitudes of $S$ are concurrent in a point if and only if the perpendicular bisectors of the segments $A_iA_j$ are concurrent in a point (an homothetic transformation maps the intersection point of two or more lines into the intersection point of the corresponding lines), i.e. if an only if $S$ is cyclic.

$\square$

In a cyclic m-system $S$, with m$\geq$3, we call *1-anticentre* of $S$ the common point of the 1-maltitudes and we denote it with $H_1$.

**Theorem 3.2.** *In a cyclic m-system S with O, $H_0$, $H_1$, as circumcentre, centroid and 1-anticentre respectively, $H_0$ is the point of the segment $OH_1$ such that:*

$$OH_1 = \frac{m}{m-2} OH_0 \tag{3}$$

*Proof.* For (2), the homothety with ratio $-\frac{m-2}{2}$ and centre $H_0$ maps the 1-maltitudes of $S$ in the perpendicular bisectors of the segments $A_iA_j$ (Figure 1 refers to case m=5),therefore it transforms $H_1$ in O. It follows that:

$$OH_0=\frac{m-2}{2}\ H_0H_1,$$

then, since $OH_1=OH_0+H_0H_1$, it easily follows (3). □

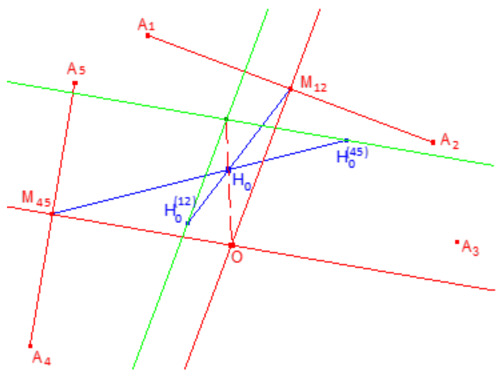


Figure 1: case m=5

In particular, the points O, $H_0$, $H_1$ are collinear. We call *Eulers line* of $S$ the straight line joining them (figure 2 refers to the case m=5).

Let $P$ be a polygon and $S$ its associated system.

We call $1-maltitude$ of $P$ each 1-maltitude of $S$. Note that the definition of 1-maltitude of polygon generalize the one of height of a triangle and of maltitude of a quadrilateral [4]. Theorems 3.1 and 3.2 can be formulated in terms of polygons. For example, theorem 3.1 can be formulated as it follows:

**Theorem 3.3.** *The 1-maltitudes of a polygon P are concurrent in a point if and only if P is cyclic.*

If $P$ is a cyclic polygon, we call $1-anticenter$ and $Eulersline$ of $P$ the 1-anticenter and the Eulers line of $S$. The definition of 1-anticenter of a cyclic polygon generalizes the one of orthocenter of a triangle and of anticenter of a

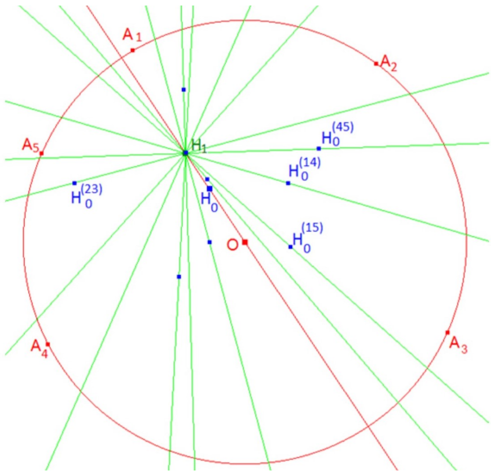


Figure 2: m=5

cyclic quadrilateral [4, 6, 10]. The definition of Eulers line of a cyclic polygon generalizes the one of Eulers line of a triangle [3] and of cyclic quadrilateral [4].

## 4. n-maltitudes and n-anticenter of a cyclic polygon

Let $S$ be a cyclic m-system, with appropriate m.

Let's call $0-maltitude$ of $S$ every median relative to a subsystem $S'$ of order 2 of S. Lets also consider the centroid $\mathrm{H}_0$ of $S$ as its 0-anticenter. Therefore, the 0-matitudes of $S$ are concurrent in the 0-anticenter of $S$.

In paragraph 3 we defined the 1-maltitudes and 1-anticenter of $S$. We define now, by induction, the n-maltitudes and the n-anticenter of $S$, for any n such that 2≤ n ≤ $\lfloor \frac{m-1}{2} \rfloor$. Fixed n, suppose we have defined the (n-1)-maltitudes and the (n-1)-anticenter of $S$. Consider a subsystem of order 2 of $S$, $S' = \{\mathrm{A}_i, \mathrm{A}_j\}$, and let $\mathrm{H}_{n-1}^{(ij)}$ the (n-1)-anticenter of the subsystem $S''$ complementary of $S'$. We call $n-maltitude$ relative to $S'$ the line trough $\mathrm{H}_{n-1}^{(ij)}$ and perpendicular to $\mathrm{A}_i\mathrm{A}_j$. Then $S$ has $\binom{m}{2}$ n-maltitudes.

**Theorem 4.1.** *For any integer n such that 1≤n≤ $\lfloor \frac{m-1}{2} \rfloor$, the n-maltitudes of a cyclic m-system are concurrent in a point $H_n$ of the Eulers line of S; moreover the segment $OH_n$ contains $H_{n-1}$ and it is*

$$OH_n = \frac{m-2n+2}{m-2n} OH_{n-1}. \tag{4}$$

*Proof.* The theorem is true for n = 1, because of theorems 3.1 and 3.2. By induction, suppose the theorem is true up to n-1 and we prove that it is true also for n. Consider a cyclic m-system $S$ and a subsystem of order 2 of $S$, $S' = \{A_i, A_j\}$; let $H^{(ij)}_{n-2}$ and $H^{(ij)}_{n-1}$, be the (n-2)-anticenter and (n-1)-anticenter of the subsystem $S''$ complementary of $S'$ (figure 3 refers to the case m=6 and n=2). $S''$ is a cyclic (m-2)-system, then the points $H^{(ij)}_{n-2}$ and $H^{(ij)}_{n-1}$ lie on the Eulers line of $S''$; moreover, by applying the inductive hypothesis to $S''$, the segment $OH^{(ij)}_{n-1}$ contains $H^{(ij)}_{n-2}$ and it is:

$$OH^{(ij)}_{n-1} = \tfrac{m-2n+2}{m-2n} OH^{(ij)}_{n-2}.$$

It follows that the homothety $\omega$ with ratio $\frac{m-2n+2}{m-2n}$ and center O maps $H^{(ij)}_{n-2}$ in $H^{(ij)}_{n-1}$. Therefore $\omega$ maps the (n-1)-maltitude through $H^{(ij)}_{n-2}$ in the n-maltitude through $H^{(ij)}_{n-1}$. Since $S$ is cyclic, the (n-1)-maltitudes are concurrent in the (n-1)-anticenter $H_{n-1}$ of $S$, then also the n-maltitudes are concurrent in a point on the Eulers line of $S$. We denote that point $H_n$. Moreover $H_{n-1}$ is the point of $OH_n$ such that (4) holds. □

We call $n-anticenter$ of $S$ the point $H_n$. Repeatingly applying (4), with easy calculations, it follows that $H_0$ is the point of the segment $OH_n$ such that:

$$OH_n = \frac{m}{m-2n} OH_0. \tag{5}$$

Let $P$ a cyclic polygon of m sides and let $S$ be its associated system.

For any integer n such that $1 \leq n \leq \lfloor \frac{m-1}{2} \rfloor$, we call $n-maltitude$ of $P$ any n-maltitude of $S$ and $n-anticenter$ of $P$ the n-anticenter of $S$.

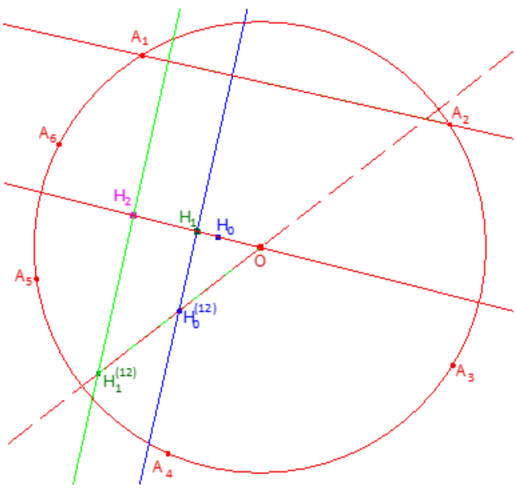


Figure 3: case m=6 and n=2

For cyclic polygons the analogue of theorem 4.1 holds:

**Theorem 4.2.** *For any integer n such that* $1 \leq n \leq \lfloor \frac{m-1}{2} \rfloor$, *the n-maltitudes of a cyclic polygon P with m sides are concurrent in a point* $H_n$ *of the Eulers line of P; moreover the segment* $OH_n$ *contains* $H_{n-1}$ *and (4) holds.*

For theorem 4.2, fixed $P$ the following sequence of points is defined:

$$\mathrm{H}_0, \mathrm{H}_1, \mathrm{H}_2, , \mathrm{H}_n,, \mathrm{H}_{m^*},$$

where $m^* = \lfloor \frac{m-1}{2} \rfloor$. For example, when m=3 and m=4 (triangles and quadrilaterals) the sequence consists only of $\mathrm{H}_0$ and of the 1-anticenter $\mathrm{H}_1$; when m=5 and m=6 (pentagons and hexagons) there are three points: the centroid $\mathrm{H}_0$, the 1-anticenter $\mathrm{H}_1$ and the 2-anticenter $\mathrm{H}_2$; when m=7 and m=8 (heptagons e octagons) there are four points: the centroid $\mathrm{H}_0$, the 1-anticenter $\mathrm{H}_1$, the 2-anticenter $\mathrm{H}_2$ and the 3-anticenter $\mathrm{H}_3$; and so on.

## 5. The polygon of the n-anticenters

Consider any cyclic m-system $S = \{\mathrm{A}_1, \mathrm{A}_2, , \mathrm{A}_m\}$, with m$\geq$4, and let O be its circumcenter. For any integer n such that $0 \leq n \leq \lfloor \frac{m-2}{2} \rfloor$, let $\mathrm{H}_n$ be the n-anticenter of $S$. Let $S'_i = \{\mathrm{A}_i\}$ and let $S''_i$ be its (m-1)-subsystem complementary to it. Let $S_n$ be the system of the n-anticenter $\mathrm{H}''_{ni}$ of the systems $S''_i$ (i = 1, 2, , m). We call *system of the n-anticenters* of $S$ the system $S_n$.

**Theorem 5.1.** *For any integer n such that* $0 \leq n \leq \lfloor \frac{m-2}{2} \rfloor$, *the homothety with ratio* $-(m-1-2n)$ *and center* $H_n$ *maps* $S_n$ *in* $S$.

*Proof.* For $n = 0$ the theorem is true because of corollary 2.2.

Suppose then $n \geq 1$. Consider any 1-subsystem $S'_i$ of $S$ and let $S''_i$ its complementary (m-1)-subsystem. Let $H''_{ni}$ be the n-anticenter of $S''_i$. (figure 4 refers to the case m=7 and n=2). The points O, $\mathrm{H}''_{0i}$ and $\mathrm{H}_{ni}$ lie on the Eulers line $S''_i$ and, for (5), it follows:

$$\mathrm{OH}''_{0i} = \tfrac{m-1-2n}{m-1}\, \mathrm{OH}''_{ni}.$$

It follows that the homothety with ratio $\frac{m-1-2n}{m-1}$ and center O, that we denote by $\omega_1$, maps $\mathrm{H}''_{ni}$ in $\mathrm{H}''_{0i}$. For corollary 2.2, the homothety with ratio $-(m-1)$ and center $\mathrm{H}_0$, that we denote by $\omega_2$, maps $\mathrm{H}''_{oi}$ in $\mathrm{A}_i$.

Let $\omega = \omega_1\omega_2$; $\omega$ is an homothety with ratio $-(m-1-2n)$ that maps $\mathrm{H}''_{ni}$ in $\mathrm{A}_i$, therefore maps $S_n$ in $S$. Let A be the center of $\omega$. In order to finish the proof, lets prove that $A \equiv \mathrm{H}_n$.

Since A is the center of $\omega$, it is $\omega_2(\omega_1(A)) = A$, and then, denoted $\omega_1(A) = B$, it is $\omega_2(B) = A$.

The points O, $H_0$, A and B are collinear, therefore A and B lie on the Eulers line of S. From $\omega_1(A) = B$ and $\omega_2(B) = A$ it follows, respectively:

$$OB = \frac{m-1-2n}{m-1} OA \tag{6}$$

$$AH_0 = (m-1)BH_0 \tag{7}$$

Note that $\frac{m-1-2n}{m-1} < 1$ and $m-1 > 1$, then $OB < OA$ and $AH_0 > BH_0$. It follows that B lies between O and A and that B lies between $H_0$ and O (see figure 4). Then, from (7) it follows:

$$OB = OH_0 - BH_0 = OH_0 - \frac{1}{m-1} AH_0.$$

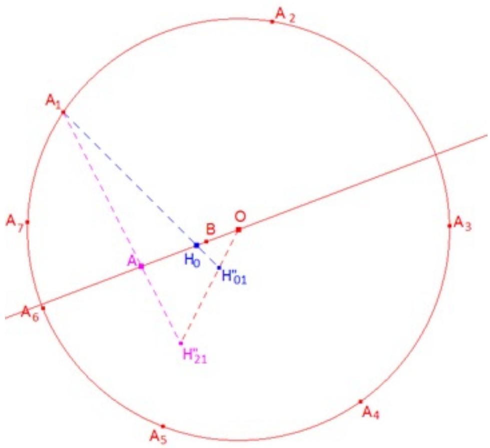


Figure 4: B lies between $H_0$ and O

Then, for (6), it is:

$$OA = \frac{m-1}{m-1-2n} OB = \frac{m-1}{m-1-2n} OH_0 - \frac{1}{m-1-2n} AH_0 =$$
$$= \frac{m-1}{m-1-2n} OH_0 + \frac{1}{m-1-2n} OH_0 - \frac{1}{m-1-2n} OH_0 - \frac{1}{m-1-2n} AH_0 =$$
$$= \frac{m}{m-1-2n} OH_0 - \frac{1}{m-1-2n} OA.$$

Therefore, it follows:

$$\frac{m-2n}{m-1-2n} OA = \frac{m}{m-1-2n} OH_0,$$

and then:

$$OA = \frac{m}{m-2n} OH_0.$$

From this last equality and from (5), it follows that A≡$H_n$.

□

Observe that *if m=2k, for n=*$\lfloor \frac{m-2}{2} \rfloor$ *=k-1 it is m-1-2n=1*. Then, *if m=2k,* $S_{k-1}$ *is the symmetric of S with respect to* $H_{k-1}$ .

From theorem 5.1 it follows that *for any integer n such that* $0 \leq n \leq \lfloor \frac{m-2}{2} \rfloor$ *the system* $S_n$ *is cyclic; its circumcenter* $O_n$ *lies on the Eulers line of S and* $H_n$ *is the point of the segment* $OO_n$ *such that:*

$$OH_n = (m-1-2n)O_nH_n.$$

Moreover, $H_n$ *is the n-anticenter of the system* $S_n$ (figure 5 refers to case m=7 and n=2).

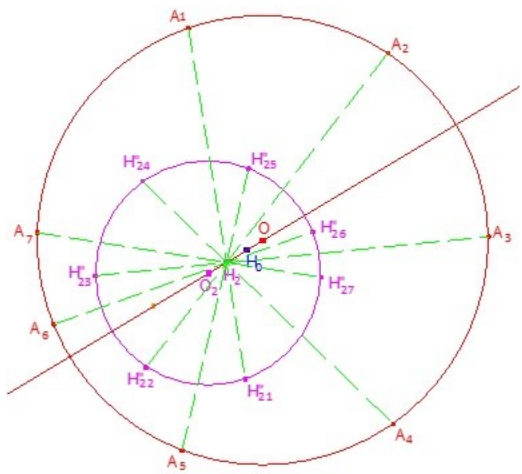

Figure 5: The case m=7, n=2

Let $P$ = $A_1A_2A_m$ a cyclic polygon with m sides, and m ≥4, let S be its associated system.

For any integer n such that 0≤n≤ $\lfloor \frac{m-2}{2} \rfloor$ , the system $S_n$ of the n-anticenters $H''_{ni}$ determine a polygon with m sides $P_n$ = $H''_{n1}$ $H''_{n2}$...$H''_{nm}$ that we call *polygon of the n-anticenters of P*.

Theorem 5.1, in terms of cyclic polygons, can be expressed as it follows:

**Theorem 5.2.** *For any integer n such that 0≤n≤* $\lfloor \frac{m-2}{2} \rfloor$*, the homothety with ratio -(m-1-2n) and center* $H_n$ *maps* $P_n$ *in P.*

Theorem 5.2 contains, as particular case (for $m = 4$ and $n = 1$), the theorem of the quadrilateral of the orthocenters [2, 7]. This theorem states: *the quadri-*

*lateral of the orthocenters of a cyclic quadrilateral $Q$ is symmetric to $Q$ with respect to the anticenter of $Q$.*

Moreover, *the polygon $P_n$ is cyclic; its circumcenter $O_n$ lies on the Eulers line of $P$ and $H_n$ is the n-anticenter of $P_n$.*

Therefore, fixed $P$ the following sequence of points, that lie on the Eulers line of $P$, is defined:

$$O_0, O_1, , O_n, , O_{\overline{m}},$$

with $\overline{m} = \lfloor \frac{m-2}{2} \rfloor$ that satisfy the property:

$$OH_n = (m-1-2n)O_nH_n.$$

## 6. n-altitudes of a cyclic polygon

The medians of an m-system $S$ are the segments that join the centroids, $H_0'$ and $H_0''$, of two complementary subsystems, $S'$ and $S''$, $S$ (paragraph 2). Theorem 2.1 states that each median of $S$ passes through the centroid $H_0$ of $S$. Moreover, $H_0$ divides the median $H_0'H_0''$ relative to a k-subsystem $S'$ of $S$ in two parts such that:

$$H_0'H_0 = \frac{m-k}{k} H_0H_0''$$

Observe that when $m = 2k$, $H_0$ is the middle point of the segment $H_0'H_0''$. By analogy with the definition of median we introduce the one height of a cyclic m-system $S$.

Consider a cyclic m-system $S = \{A_1, A_2, , A_m\}$, with $m \geq 6$; let $O$ and $H_0$ be the circumcenter and the centroid of $S$, respectively.

Fix any k-subsystem $S'$ of $S$, with $3 \leq k \leq \lfloor \frac{m}{2} \rfloor$, and let $S''$ be its complementary $(m-k)-$subsystem. For any integer n such that $0 \leq n \leq \lfloor \frac{k-1}{2} \rfloor$ let $H_n'$ and $H_n''$ be the n-anticenters of $S'$ and $S''$, respectively. We call $n-altitude$ of $S$ relative to $S'$ the segment $H_n'H_n''$ . Note that the 0-heights of $S$ are the medians of $S$.

For the n-altitudes the following theorems hold.

**Theorem 6.1.** *If $S$ is a cyclic m-system with $m \geq 6$ and $m = 2k$, for any integer n such that $0 \leq n \leq \lfloor \frac{k-1}{2} \rfloor$ , the n-altitude $H_n'H_n''$ is parallel to the median $H_0'H_0''$ and it is:*

$$H_n'H_n'' = \frac{k}{k-2n} H_0'H_0'' \tag{8}$$

*Moreover, the 2n-anticenter $H_{2n}$ of $S$ is the middle point of $H_n'H''n$.*

*Proof.* For $n = 0$ the theorem is true because of corollary 2.4.

Suppose the $n \geq 1$ (figure 6 refers to $m = 6$, with $S' = \{A_1, A_2, A_3\}$).

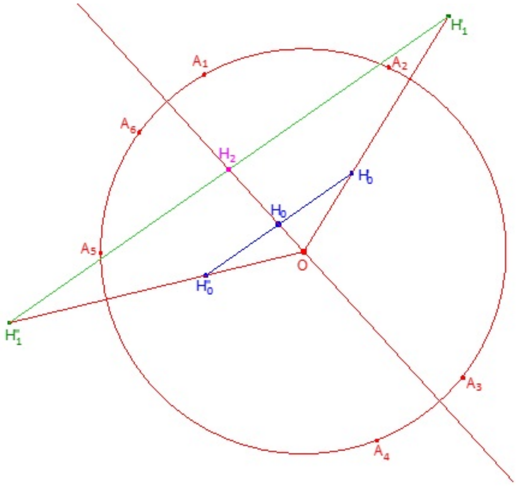


Figure 6: m=6

Consider the homothety $\omega$ with ratio $\frac{k}{k-2n}$ and center $O$. It maps $H'_0$ in $H'_n$, because $O$, $H'_0$, $H'_n$ are collinear on the Eulers line of $S'$ and for (5) it is:

$$OH'_n = \frac{k}{k-2n} OH'_0;$$

analogously $\omega$ maps $H''_0$ in $H''_n$ and it is:

$$OH''_n = \frac{k}{k-2n} OH''_0.$$

Therefore, $\omega$ maps the segment $H'_0H''_0$ in the segment $H'_nH''_n$, then $H'_nH''_n$ is parallel to $H'_0H''_0$ and (8) holds.

Moreover, for (5), it is:

$$OH_{2n} = \frac{k}{k-2n} OH_0.$$

where $H_{2n}$ is the 2n-anticenter of $S$. It follows that $\omega$ maps $H_0$ in $H_{2n}$ and since $H_0$ is the middle point of the segment $H'_0H''_0$, $H_{2n}$ is the middle point of the segment $H'_nH''_n$.

□

**Theorem 6.2.** *If $S$ is any cyclic m-system with $m \geq 6$, for any integer $k$ such that $3 \leq k \leq \lfloor \frac{m}{2} \rfloor$ and for any integer $n$ such that $0 \leq n \leq \lfloor \frac{k-1}{2} \rfloor$, the 2n-anticenter $H_{2n}$ of $S$ lies on the n-altitude $H'_nH''_n$ and it is:*

$$H'_nH_{2n} = \frac{m-k-2nk}{k-2n} H_{2n}H''_n. \tag{9}$$

*Proof.* For $n = 0$ the theorem is true because of theorem 2.1. Suppose then $n \geq 1$.

If $m = 2k$, the theorem is proved because, for theorem 6.1, $H_{2n}$ is the middle point of the segment $H'_nH''_n$. Suppose then that $m > 2k$.

Consider any k-subsystem $S'$ of $S$, with $3 \leq k \leq \lfloor \frac{m}{2} \rfloor$ , and let $S''$ be its complementary (m-k)-subsystem. For any integer n such that $0 \leq n \leq \lfloor \frac{k-1}{2} \rfloor$ let $H'_n$ and $H''_n$ be the anticenters of $S'$ and $S''$, respectively (figure 7 refers to the case $m = 7$ and $k = 3$, with $S' = \{A_1, A_3, A_5\}$).

The points $O$, $H''_0$ and $H''_n$ lie on the Eulers line $S''$ and, for (5), it is:

$$OH''_0 = \frac{m-k-2n}{m-k} OH''_n.$$

Then, the homothety with ratio $\frac{m-k-2n}{m-k}$ and center $O$, that we denote with $\omega_1$, maps $H''n$ in $H''_0$. For corollary 2.2, the homothety with ratio $-\frac{m-k}{k}$ and center $H_0$, that we donote $\omega_2$, maps $H''_0$ in $H'_0$. The points $O$, $H'_0$ and $H'_n$ lie on the Eulers line of $S'$ and, for (5), it is:

$$OH'_n = \frac{k}{k-2n} OH'_n.$$

Then, the homothety with ratio $\frac{k}{k-2n}$ and center $O$, that we denote $\omega_3$, maps $H'_0$ in $H'_n$.

Let $\omega = \omega_1\omega_2\omega_3$ is an homothety with ratio:

$$\frac{m-k-2n}{m-k} \cdot \left(-\frac{m-k}{k}\right) \cdot \frac{k}{k-2n} = -\frac{m-k-2n}{k-2n}.$$

that maps $H''_n$ in $H'_n$, then, if $A$ is its center, then $A$ lies on the n-altitude $H'_nH''_n$ and moreover:

$$H'_nA = \frac{m-k-2n}{k-2n} AH''_n.$$

To finish the proof, lets prove that $A \equiv H_{2n}$.

Since $A$ is the center of $\omega$, it is $\omega_3(\omega_2(\omega_1(A))) = A$. Denote $\omega_1(A) = B$ and $\omega_2(B) = C$, then $\omega_3(C) = A$. It is easy to verify that the points $O$, $H_0$, $A$, $B$ and $C$ are collinear, then $A$, $B$, $C$ lie on the Eulers line of $S$.

From $\omega_1(A) = B$, $\omega_2(B) = C$ and $\omega_3(C) = A$ it follows:

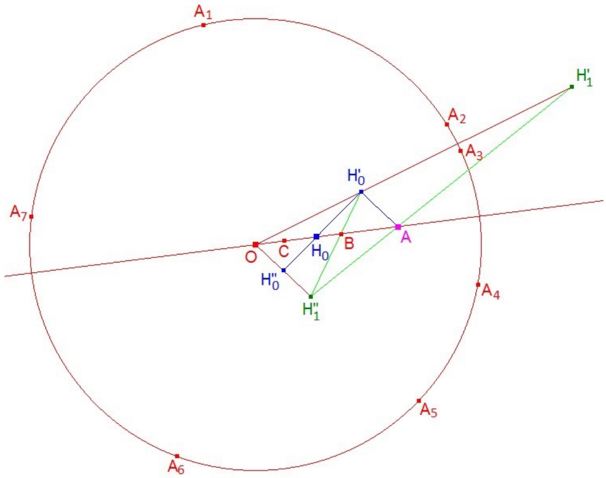


Figure 7: case m=7, k=3

$$OB = \frac{m-k-2n}{m-k} OA \tag{10}$$

$$CH_0 = \frac{m-k}{k} BH_0 \tag{11}$$

$$OA = \frac{k}{k-2n} OC \tag{12}$$

Note that $\frac{m-k-2n}{m-k} < 1$ and $\frac{k}{k-2n} > 1$, then, from (10) and (12) it follows that $OB < OA$ and $OC < OA$. Moreover, for (10) and (12) it is:

$$OB = \frac{m-k-2n}{m-k} \cdot \frac{k}{k-2n} OC = \frac{mk-k^2-2nk}{mk-k^2-2n(m-k)} OC,$$

then, from $m-2k > 0$ it follows $\frac{mk-k^2-2nk}{mk-k^2-2n(m-k)} > 1$, and $OB > OC$.

Therefore, on the semi-line $OH_0$ of the Eulers line of $S$ the points $C$, $B$ and $A$ are collinear in that order. Moreover, $H_0$ lies between $C$ and $B$ (figure 7).

Then, for (11) it is:

$$OC = OH_0 - CH_0 = OH_0 - \frac{m-k}{k} BH_0$$

and, for (12), it is:

$OA = \frac{k}{k-2n} OC = \frac{k}{k-2n} OH_0 - \frac{m-k}{k-2n} BH_0 =$

$= \frac{k}{k-2n}OH_0 + \frac{m-k}{k-2n}OH_0 - \frac{m-k}{k-2n}OH_0 - \frac{m-k}{k-2n}BH_0 = \frac{m}{k-2n}OH_0 - \frac{m-k}{k-2n}OB.$
Therefore, for (10), it is:
$OA = \frac{m}{k-2n}OH_0 - \frac{m-k}{k-2n} \cdot \frac{m-k-2n}{m-k}OA = \frac{m}{k-2n}OH_0 - \frac{m-k-2n}{k-2n}OA$
and then:

$$OA = \frac{m}{m-4n}OH_0.$$

From this last equality and from (5), it follows that $A \equiv H_{2n}$.

□

By analogy with what is said about theorem 2.1, we can state that the following corollaries are directly derived from theorem 6.2

**Corollary 6.3.** *If $S$ is any cyclic m-system with $m \geq 6$, for any integer $k$ such that $3 \leq k \leq \lfloor \frac{m}{2} \rfloor$ and for any integer $n$ such that $0 \leq n \leq \lfloor \frac{k-1}{2} \rfloor$, in the homothety with ratio $-\frac{m-k-2n}{k-2n}$ and center in the 2n-anticenter $H_{2n}$ of $S$, the n-anticenter of any k-subsystem $S'$ of $S$ is the image of the n-anticenter of the complementary (m-k)-subsystem of $S'$.*

**Corollary 6.4.** *If $S$ is any cyclic m-system with $m \geq 6$, for any integer $k$ such that $3 \leq k \leq \lfloor \frac{m}{2} \rfloor$ and for any integer $n$ such that $0 \leq n \leq \lfloor \frac{k-1}{2} \rfloor$, the segment $H'_{n1}H'_{n2}$ joining the n-anticenter of two k-subsystems $S'_1$, $S'_2$ of $S$ is parallel to the segment $H''_{n1}H''_{n2}$ joining the n-anticenters of the complementary (m-k)-subsystems of $S'_1$, $S'_2$, and moreover it is:*

$$H'_{n1}H'_{n2} = \frac{m-k-2n}{k-2n}H''_{n1}H''_{n2}.$$

**Corollary 6.5.** *If $S$ is any cyclic m-system with $m = 2k$, for any integer $k$ such that $0 \leq n \leq \lfloor \frac{k-1}{2} \rfloor$, the n-anticenters of any two k-subsystems of $S$ are symmetric with respect to the 2n-anticenter $H_{2n}$ of $S$.*

Let $P$ be a cyclic polygon with m sides, with $m \geq 6$, and let $S$ be its associated system. We call n-altitude of $P$ any n-altitude of $S$. Theorem 6.2 and its corollaries can easily be formulated in terms of polygons. In particular, theorem 6.2 can be formulated as it follows:

**Theorem 6.6.** *If $P$ is any cyclic polygon with m sides, with $m \geq 6$, for any integer $k$ such that $3 \leq k \leq \frac{m}{2}$ and for any integer $n$ such that $0 \leq n \leq \lfloor \frac{k-1}{2} \rfloor$, the 2n-anticenter $H_{2n}$ of $P$ lies on the n-altitude $H'_nH''_n$ and (9) holds.*

## 7. Loci of anticenters

Consider a cyclic m-system $S = \{A_1, A_2, , A_m\}$, with $m \geq 4$, and let $\gamma$ be the circle containing the points of $S$. Let $O$ and $R$ be the circumcenter and the circumradius of $S$, respectively. For any integer n such that $0 \leq n \leq \lfloor \frac{m-1}{2} \rfloor$ let $H_n$ be the n-anticenter of $S$.

Let $S_i = \{A_1, A_2, , A_{i-1}, A_i, , A_m\}$ and let $H_i^{(0)}$ be its centroid. Let $\gamma_i^{(n)}$ be the locus described by $H_n$ when $A_i$ moves on $\gamma$.

In [5] it has been proved that $\gamma_i^{(o)}$ is the circle of radius $\frac{r}{m}$, with center $O_i^{(0)}$ belonging to the segment $OH_i^{(0)}$ and such that:

$$OH_i^{(0)} = mO_i^{(0)}H_i^{(0)}. \tag{13}$$

**Theorem 7.1.** *For any integer n such that $0 \leq n \leq \lfloor \frac{m-1}{2} \rfloor$, $\gamma_i^{(n)}$ is a circle with radius $\frac{r}{m-2n}$. Moreover, the center $O_i^{(n)}$ of $\gamma_i^{(0)}$ is such that $O_i^{(0)}$ lies on the segment $OO_i^{(n)}$ and it is:*

$$OO_i^{(n)} = \frac{m}{m-2n} OO_i^{(0)}. \tag{14}$$

*Proof.* For (5) it is $OH_n = \frac{m}{m-2n} OH_0$, then the homothety $\omega$ with ratio $\frac{m}{m-2n}$ and center $O$ maps $H_0$ in $H_n$ and $\gamma_i^{(o)}$ in $\gamma_i^{(n)}$ (figure 8 refers to case $m = 6$). It follows that $\gamma_i^{(n)}$ is a circle with radius $\frac{r}{m-2n}$.

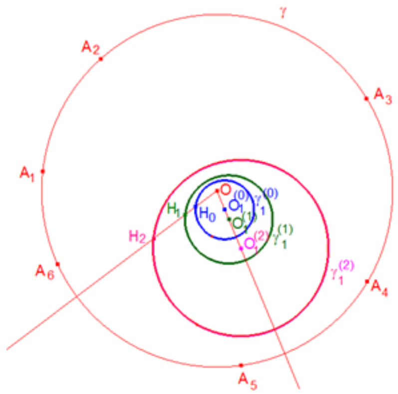

Figure 8: case m=6

The center $O_i^{(n)}$ of $\gamma_i^{(n)}$ is the correspondent point of $O_i^{(o)}$ in the homothety $\gamma$, then $O_i^{(n)}$ is the point of the semi-line $OO_i^{(0)}$ such that (14) holds. Moreover, since $n > 0$, $O_i^{(0)}$ lies on the segment $OO_i^{(n)}$ because $\frac{r}{m} < \frac{r}{m-2n}$. □

For any integer $n$ such that $0 < n \leq \lfloor \frac{m-1}{2} \rfloor$ consider the m-system $C_n = \{O_1^{(n)}, O_2^{(n)}, , O_m^{(n)}\}$; we call it *system of the centers* of $S$.

**Theorem 7.2.** *For any integer n such that* $0 \leq n \leq \lfloor \frac{m-1}{2} \rfloor$*,* $C_n$ *is mapped in S from the homothety with ratio -(m-2n) and center K, where the point K is such that* $H_0$ *lie on the segment OK and:*

$$OK = \frac{m}{m-2n+1} OH_0. \tag{15}$$

*Proof.* For (14), the homothety $\omega'$ with ratio $\frac{m-2n}{m}$ and center $O$ maps $O_i^{(n)}$ in $O_i^{(o)}$. From (13) it follows $OH_i^{(0)} = m(OH_i^{(0)} - OO_i^{(0)})$, and then:

$$OH_i^{(o)} = \frac{m}{m-1} OO_i^{(o)}.$$

Therefore, the homothety $\omega''$ with ratio $\frac{m}{m-1}$ and center $O$ maps $O_i^{(0)}$ in $H_i^{(0)}$. Then, $\omega_1 = \omega' \omega''$is the homothety with ratio $\frac{m-2n}{m-1}$ and center $O$; $\omega_1$ maps $O_i^{(n)}$ in $H_i^{(0)}$. Moreover, for theorem 5.1, the homothety with ratio $-(m-1)$ and center $H_0$, that we denote by $\omega_2$, maps $H_i^{(0)}$ in $A_i$ (figure 9 refers to case $m = 4$, $n = 1$).

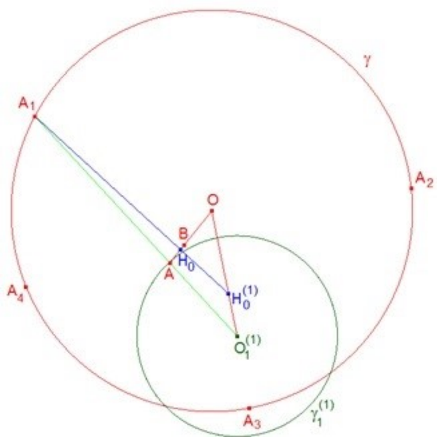


Figure 9: case m=4, n=1

Let $\omega = \omega_1 \omega_2$; $\omega$ is an homothety with ratio $-(m-2n)$, that maps $O_i^{(n)}$ in $A_i$, then maps $C_n$ in $S$. Let $A$ be its center. To finish the proof, lets prove that $A \equiv K$.

Since A is the center of $\omega$, it is $\omega_2(\omega_1(A)) = A$. If $\omega_1(A) = B$, then it is $\omega_2(B) = A$. The points $O$, $H_0$, $A$ and $B$ are collinear, then $A$ and $B$ lie on the Eulers line of $S$.

From $\omega_1(A) = B$ and $\omega_2(B) = A$ it follows:

$$OB = \frac{m-2n}{m-1} OA \tag{16}$$

$$AH_0 = (m-1)BH_0 \tag{17}$$

Since $n > 0$ it is $\frac{m-2n}{m-1} < 1$ ; moreover, $m-1 > 1$. Then $OB < OA$ and $AH_0 > BH_0$. It follows that $B$ lies between $O$ and $A$ and that $B$ lies between $H_0$ and $O$ (figure 9). Then, for (17) it follows:

$$OB = OH_0 - BH_0 = OH_0 - \frac{1}{m-1} AH_0.$$

Then, for (16), it follows:

$$OA = \frac{m-1}{m-2n} OB = \frac{m-1}{m-2n} OH_0 - \frac{1}{m-2n} AH_0 =$$

$$= \frac{m-1}{m-2n} OH_0 + \frac{1}{m-2n} OH_0 - \frac{1}{m-2n} OH_0 - \frac{1}{m-2n} AH_0 = \frac{m}{m-2n} OH_0 - \frac{1}{m-2n} OA.$$

Therefore, it follows:

$$\frac{m-2n+1}{m-2n} OA = \frac{m}{m-2n} OH_0,$$

and then:

$$OA = \frac{m}{m-2n+1} OH_0.$$

It follows that $A \equiv K$ (Figure 10 refers to m=4, n=1).

$\square$

Let $P = A_1 A_2 A_m$ be a cyclic polygon with m sides, with $m \geq 4$, and $S$ its associated system. For any integer n such that $0 < n \leq \lfloor \frac{m-1}{2} \rfloor$, the system $C_n$ of the centers $O_i^{(n)}$ of $S$ determine a polygon with m sides $P_c = O_1^{(n)} O_2^{(n)} O_m^{(n)}$ that we call *polygon of the centers* of P. Theorem 7.2, in terms of polygon, can be formulated as it follows:

**Theorem 7.3.** *For any integer n such that* $0 \leq n \leq \lfloor \frac{m-1}{2} \rfloor$, $P_c$ *is mapped in* $P$ *from the homothety with ratio* $-(m-2n)$ *and center* $K$*, where the point* $K$ *is such that* $H_0$ *lies on the segment* $OK$ *and (15) holds.*

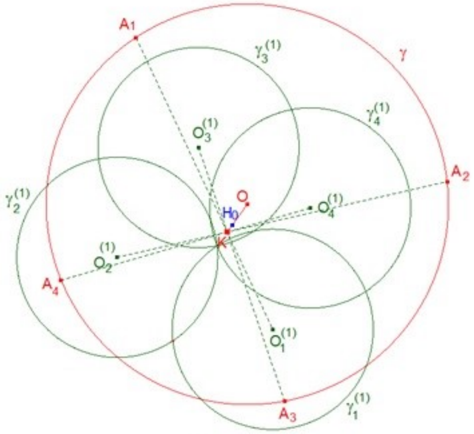


Figure 10: case m=4, n=1

## 8. Maltitudes of a cyclic polyedron

All the definitions and theorems given in the previous paragraphs for the plane systems $S$, i.e. for the systems of points that lie on a plane, still hold, with appropriate changes, for three-dimensional systems $S$ , i.e. for systems of points of the three dimensional space. First of all, observe that definitions and theorems of paragraph 1 still hold [8]. In particular, a three dimensional system $S$ is said to be cyclic if all its points lie on a sphere.

The notion of 1-maltitude, introduced for plane systems in paragraph 2, still hold for a three dimensional system $S$, with the following change.

Let $S = \{A_1, A_2, , A_m\}$ be any three dimensional m-system, with $m \geq 4$. Consider the 2-subsystems $S' = \{A_i, A_j\}$, for i, j = 1, 2, , m and $i \neq j$. Let $H_0^{(ij)}$ be the centroid of the subsystem $S''$ complementary of $S'$. We call 1-maltitude relative to the subsystem $S'$ the plane through $H_0^{(ij)}$ and perpendicular to $A_iA_j$ .

Theorem 3.1 still holds for $S$, and the proof is the analogous to the one of a plane system. Then, we can easily introduce the definitions of 1-anticenter and of Eulers line of a cyclic three dimensional system $S$ and, then, of a cyclic polyhedron $P$ .

Note that the concept of 1-maltitude of a polyhedron generalises the one of Monge plane of a tetrahedra and that the concept of 1-anticenter generalises the one of Monge point of a tetrahedron [1, 9].

Analogously, definitions of n-maltitude and n-altitude and theorems examined for cyclic plane systems $S$ and for cyclic polygons $P$ in paragraphs 3-7 still hold for cyclic three dimensional systems $S$ and for cyclic polyhedra $P$ .

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