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PARABOLIC PROBLEMS IN NON-STANDARD SOBOLEV SPACES OF INFINITE ORDER

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This paper is devoted to the study of the existence of solutions for the strongly nonlinear parabolic equation

$$\frac{\partial u}{\partial t} + Au + g(x,t,u) = f(x,t),$$

where A is a Leray-Lions operator acted from $V^{\infty,p(.)}(a_\alpha, Q_T)$ into its dual. The nonlinear term g satisfies growth and sign conditions and the datum f is assumed to be in the dual space $V^{-\infty,p'(\cdot)}(a_\alpha, Q_T)$.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary $\partial\Omega$. Fixing the final time $T > 0$, we denote by Q_T the cylinder $\Omega \times (0,T)$, and by S_T the lateral surface $\partial\Omega \times (0,T)$.

Our aim is to study, in the framework of variable exponent Sobolev Spaces of infinite order, the following strongly nonlinear parabolic problem of Dirichlet

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type

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + g(x,t,u) = f(x,t) & \text{in} \quad Q_T, \\ u(x,0) = 0 & \text{on} \quad S_T, \\ D^w u|_{S_T} = 0 & \text{for any} \quad |w| = 0,1,.... \end{cases} \tag{1}$$

Here A is a nonlinear parabolic operator of infinite order defined by

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha}(A_{\alpha}(x,t,\nabla^{\gamma}u)), \quad |\gamma| \leq |\alpha|$$

where $A_\alpha : \Omega \times [0,T] \times \mathbb{R}^{\lambda_\alpha} \to \mathbb{R}$ is a real function and λ_α is the number of multi-indices γ such that $|\gamma| \leq |\alpha|$. In addition, $A_\alpha(x,t,\xi_\gamma)$ are Carathéodory functions that have polynomial growth in ξ_α for any multi-indice α, and g is a nonlinear term satisfying some growth and sign conditions.

In the stationary case of such problems of infinite order, Dubinskii (see [19]) has studied the Cauchy-Dirichlet problem

$$\begin{cases} L(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, \nabla^{\gamma} u) = f & \text{in } \Omega, \\ D^{\omega} u|_{\partial\Omega} = 0 \qquad |\omega| = 0,1,\dots, \end{cases} \tag{2}$$

in the infinite order Sobolev space

$$W_0^{\infty}(a_\alpha, p_\alpha) = \left\{ u(x) \in C_0^{\infty}(\Omega) \ : \ \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{p_\alpha}^{p_\alpha} < \infty \right\},$$

where $a_\alpha \geq 0$ and $p_\alpha \geq 1$ are numerical sequences (with α is a multi-index). He has proved the existence of solutions for the Dirichlet problem associated with the equation $L(u) = f$ in the Sobolev spaces of infinite order $W_0^\infty(a_\alpha, p_\alpha)$, under some growth and monotonicity conditions with constant exponents $(p_\alpha)_\alpha$.

Note that the study of some elliptic and parabolic equations involving p-Laplace operators, is based on the theory of standard Sobolev spaces. In the case of $p(\cdot)-$Laplace equations, the natural setting for this approach is the use of the variable exponent Lebesgue and Sobolev spaces. Several studies have been devoted to the investigation of related problems in the framework of standard variable exponent Sobolev spaces and a lot of papers have appeared in this direction. Since we are interested in the parabolic case, here we mention the work of Bendahmane, Wittbold and Zimmermann [8], where the authors have studied, the following nonlinear parabolic equation with $g(x,t,s) = 0$,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}\,(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad S_T, \\ u(\cdot,0) = u_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{3}$$

with $f \in L^1(Q_T), u_0 \in L^1(\Omega)$ and $p : \Omega \mapsto (1,+\infty)$ is a continuous function. They proved the existence and uniqueness of renormalized solutions.

Let us mention that the elliptic case for infinite order equations with variable exponents, has been studied by Abdou et al. [1], and the case of constant exponents of the problem (1) has been also investigated by Abdou et al., [2]. As for the case of nonlinear anisotropic parabolic problems of constant exponents (finite order), we refer the reader to the work of Abdou et al. [3].

Variable Sobolev spaces have been used in the last decades to model various phenomena. A major application which uses non-homogeneous operators is related to the modelling of electrorheological fluids, due in the first to Willis Winslow in 1949. For a general account of the underlying physics consult Halsey [24] and for some technical applications Pfeiffer et al. [29]. Electrorheological fluids have been used in robotics and space technology, mainly in the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids, we refer to Acerbi and Mingione [4], Alves and Souto [6], Chabrowski and Fu [13], and Diening [16].

The paper is organized as follows. In section 2 we recall some basic notations and properties of Sobolev spaces with variable exponents in both finite and infinite order. We introduce in section 3 some assumptions on $A_\alpha(x,t,\xi_\gamma)$ and $g(x,t,s)$ essential to assure the existence of weak solutions. The second part will contain some important lemmas which will be useful to prove our main results. The last part of section 3 is devoted to prove the existence of weak solutions for our nonlinear parabolic problem in the framework of Sobolev space of variable exponents with infinite order.

2. Preliminaries

2.1. Variables exponent Lebesgue and Sobolev spaces.

We say that a real-valued continuous function $p(.)$ is log-Hölder continuous in Ω if

$$|p(x) - p(y)| \leq \frac{C}{|log|x-y||} \quad \forall x,y \in \overline{\Omega} \quad \text{such that} \quad |x-y| < \frac{1}{2},$$

with possible different constant C. We denote

$$C_+(\overline{\Omega}) = \{\text{log-Hölder continuous functions } p(\cdot) : \overline{\Omega} \to \mathbb{R} \\ \text{such that } 1 < p_- \leq p_+ < N\},$$

where

$$p_- = \min\{p(x)\,/\, x \in \overline{\Omega}\} \quad \text{and} \quad p_+ = \max\{p(x)\,/\, x \in \overline{\Omega}\}.$$

We define the variable exponent Lebesgue space for $p(\cdot) \in C_+(\overline{\Omega})$ by

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \longrightarrow I\!R \quad \text{measurable } / \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

The space $L^{p(\cdot)}(\Omega)$ under the norm

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

is a uniformly convex Banach space, then reflexive. We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [22], [30]).

Proposition 2.1. *(cf. [22], [30]) (Generalized Hölder inequality)*
(i) *For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega u\,v\,dx\right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(\cdot)}\,\|v\|_{p'(\cdot)}\,.$$

(ii) *For all $p_1(\cdot), p_2(\cdot) \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ in Ω, we have*

$$L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$$

and the embedding is continuous.

Proposition 2.2. *(cf. [22], [30])*
If we denote

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx \quad \forall u \in L^{p(\cdot)}(\Omega),$$

then, the following assertions holds
(i) $\|u\|_{p(\cdot)} < 1 \quad (resp, = 1, > 1) \quad \Leftrightarrow \quad \rho(u) < 1 \quad (resp, = 1, > 1)$,
(ii) $\|u\|_{p(\cdot)} > 1 \;\Rightarrow\; \|u\|^{p_-}_{p(\cdot)} \leq \rho(u) \leq \|u\|^{p_+}_{p(\cdot)}$ *and* $\|u\|_{p(\cdot)} < 1 \;\Rightarrow\; \|u\|^{p_+}_{p(\cdot)} \leq \rho(u) \leq \|u\|^{p_-}_{p(\cdot)}$,
(iii) $\|u\|_{p(\cdot)} \to 0 \;\Leftrightarrow\; \rho(u) \to 0$ *and* $\|u\|_{p(\cdot)} \to \infty \;\Leftrightarrow\; \rho(u) \to \infty$.

Definition 2.3. (cf. [17]) The function $u \in L^{p(\cdot)}(\Omega)$ belongs to the space $W^{k,p(\cdot)}(\Omega)$ where $k \in \mathbb{N}^*$, if its weak partial derivatives $D^\alpha u$ exist and belong to $L^{p(\cdot)}(\Omega)$ for all $|\alpha| \leq k$. We define a modular function on $W^{k,p(\cdot)}(\Omega)$ by

$$\rho_{k,p(\cdot)}(u) = \sum_{|\alpha|=0}^{k} \int_\Omega |D^\alpha u|^{p(x)}\, dx \quad \text{with} \quad D^\alpha u = \frac{\partial^{|\alpha|} u}{(\partial x_1)^{\alpha_1} \dots (\partial x_N)^{\alpha_N}},$$

which induces a norm by

$$\|u\|_{k,p(\cdot)} := \inf\{\lambda > 0 \ : \ \rho_{k,p(\cdot)}(\frac{u}{\lambda}) \leq 1\}.$$

For $k \in \mathbb{N}$, the space $W^{k,p(\cdot)}(\Omega)$ is called Sobolev space and its elements are called Sobolev functions. Clearly $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$.

Definition 2.4. (cf. [17]) Let $p(\cdot) \in C_+(\overline{\Omega})$ and $k \in \mathbb{N}^*$. The Sobolev space $W_0^{k,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W^{k,p(\cdot)}(\Omega)$, where $C_0^\infty(\Omega)$ is the space of all continuous functions with compact support in Ω, that have continuous derivatives for any order.

Theorem 2.5. *(cf. [17]) Let $p(\cdot) \in C_+(\overline{\Omega})$ and $k \in \mathbb{N}$. The spaces $W^{k,p(\cdot)}(\Omega)$ and $W_0^{k,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.6. *(cf. [22]) In the case of $k = 1$, if $q(\cdot) \in C_+(\bar{\Omega})$ and $q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ for any $x \in \Omega$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.*

Remark 2.7. We denote the dual of the Sobolev space $W_0^{k,p(\cdot)}(\Omega)$ by

$$W^{-k,p'(\cdot)}(\Omega).$$

It is well known (see [17]) that for each $F \in W^{-k,p'(\cdot)}(\Omega)$ there exists $(f_\alpha)_\alpha \subset L^{p'(\cdot)}(\Omega)$ with $|\alpha| = 0, \dots, k$, such that $F = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha f_\alpha$. Moreover, for any $u \in W_0^{k,p(x)}(\Omega)$ we have

$$\langle F, u \rangle = \sum_{|\alpha|=0}^{k} \int_\Omega f_\alpha D^\alpha u\, dx,$$

and we define a norm on the dual space by

$$\|F\|_{-k,p'(\cdot)} = \inf\Big\{ \sum_{|\alpha|=0}^{k} \|f_\alpha\|_{p'(\cdot)} \ / \ F = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha f_\alpha \quad \text{with}\, f_\alpha \subset L^{p'(\cdot)}(\Omega) \ \text{ for } \ |\alpha| = 0, \dots, k \Big\}.$$

Now, let $a_\alpha \geq 0$ be real numbers for multi-indices α. The variable exponent Sobolev space of infinite order is the functional space defined by

$$W^\infty(a_\alpha, p(x))(\Omega) = \left\{ u \in C^\infty(\Omega) : \sigma_{p(\cdot)}(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha u||_{p(x)}^{p^+} < \infty \right\}.$$

Since we shall deal with the Dirichlet problem in this paper, we will use the functional space $W_0^\infty(a_\alpha, p(\cdot))(\Omega)$ defined by

$$W_0^\infty(a_\alpha, p(\cdot))(\Omega) = \left\{ u \in C_0^\infty(\Omega) : \sigma_{p(\cdot)}(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha u||_{p(x)}^{p^+} < \infty \right\}.$$

In contrast with the finite order Sobolev space, the very first question, which arises in the study of the spaces $W_0^\infty(a_\alpha, p(\cdot))(\Omega)$, is the question of their non-triviality (or nonemptiness), i.e. the question of the existence of a function u such that $\sigma_{p(\cdot)}(u) < \infty$.

Definition 2.8. (Dubinskii [18]) The space $W_0^\infty(a_\alpha, p(x))(\Omega)$ is called a non-trivial space if it contains at least one function which not identically equal to zero, i.e. there is a function $u \in C_0^\infty(\Omega)$ such that $\sigma_{p(\cdot)}(u) < \infty$.

It turns out that the answer of this question depends not only on the given parameters a_α, p_α of the spaces $W^\infty(a_\alpha, p(x))(\Omega)$, but also on the domain Ω. The dual space of $W_0^\infty(a_\alpha, p(x))(\Omega)$ is defined as follows

$$W^{-\infty}(a_\alpha, p'(\cdot))(\Omega) = \Big\{ h : h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha h_\alpha,$$

$$\sigma'_{p'(\cdot)}(h) = \sum_{|\alpha|=0}^{\infty} a_\alpha ||h_\alpha||_{p'(\cdot)}^{p'^+} < \infty \Big\},$$

where $h_\alpha \in L^{p'(\cdot)}(\Omega)$ and $p'(\cdot)$ is the conjugate of $p(\cdot)$, i.e., $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. Note that the duality of the space $W^{-\infty}(a_\alpha, p'(\cdot))(\Omega)$ and $W_0^\infty(a_\alpha, p(x))(\Omega)$ is given by the relation

$$\langle h, v \rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega h_\alpha(x) D^\alpha v(x)\, dx,$$

In the particular case when $p(x) = p$ for any multi-indices α, the Sobolev space of infinite order is defined as

$$W_0^\infty(a_\alpha, p)(\Omega) = \left\{ u \in C_0^\infty(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha u||_p^p < \infty \right\},$$

where $a_\alpha \geq 0$, $p > 1$ are real numbers for all multi-indices α and $\|\cdot\|_p$ is the usual norm in the Lebesgue space $L^p(\Omega)$, (see [18], [19]).

2.2. Non-Standard Sobolev spaces of infinite order involving space and time.

Let $Q_T = \Omega \times (0,T)$ with $0 < T < \infty$. Extending the definition of variable exponent $p(\cdot): \overline{\Omega} \longmapsto]1,\infty)$ to $\overline{Q_T}$ by setting $p(x,t) := p(x)$ for all $(x,t) \in Q_T$, we define

$$C_+(\overline{Q_T}) = \{p(\cdot) : \overline{Q_T} \longmapsto I\!R \quad \text{such that} \quad p(x,t) = p(x) \in C_+(\overline{\Omega})\},$$

and we consider for $p(\cdot) \in C_+(\overline{Q_T})$ the generalized Lebesgue space

$$L^{p(\cdot)}(Q_T) := \left\{ u : Q_T \longmapsto I\!R, \text{ measurable, such that } \int_{Q_T} |u(x,t)|^{p(x)}\, dx\, dt < \infty \right\},$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q_T)} := \inf_{\lambda > 0} \left\{ \int_{Q_T} \left| \frac{u(x,t)}{\lambda} \right|^{p(x)} dx\, dt \leq 1 \right\},$$

which, of course shares the same properties as $L^{p(\cdot)}(\Omega)$.

Lemma 2.9. *(cf. [31]) We have the following continuous dense embedding*

$$L^{p_+}(0,T; L^{p(\cdot)}(\Omega)) \hookrightarrow L^{p(\cdot)}(Q_T) \hookrightarrow L^{p_-}(0,T; L^{p(\cdot)}(\Omega)).$$

We introduce, for any $k \in I\!N$, the Lebesgue space with variable exponent involving time $L^{p_-}(0,T; W^{k,p(\cdot)}(\Omega))$ by

$$L^{p_-}(0,T; W^{k,p(\cdot)}(\Omega)) = \left\{ u \text{ measurable function } / \right.$$
$$\left. \sum_{|\alpha|=0}^{k} \int_0^T \|D^\alpha u\|_{p(\cdot)}^{p_-}\, dt < \infty \right\},$$

and we define

$$V^{k,p(\cdot)}(Q_T) = \left\{ u \in L^{p_-}(0,T; W^{k,p(\cdot)}(\Omega)) \,/ \int_{Q_T} |D^\alpha u|^{p(x)}\, dx\, dt < \infty \right.$$
$$\left. \text{for } |\alpha| = 0, \dots, k \right\},$$

equipped with the norm

$$\|u\|_{V^{k,p(\cdot)}(Q_T)} := \sum_{|\alpha|=0}^{k} \|D^\alpha u\|_{L^{p(\cdot)}(Q_T)}.$$

It is clear that for any $u \in V^{k,p(\cdot)}(Q_T)$, we have

$$\begin{aligned}
\|u\|^{p_-}_{V^{k,p(\cdot)}(Q_T)} &= \Big(\sum_{|\alpha|=0}^{k} \|D^\alpha u\|_{L^{p(\cdot)}(Q_T)}\Big)^{p_-} \\
&\leq C\sum_{|\alpha|=0}^{k} \|D^\alpha u\|^{p_-}_{L^{p(\cdot)}(Q_T)} \\
&\leq C\sum_{|\alpha|=0}^{k} \big(\int_{Q_T} |D^\alpha u|^{p(x)}\,dx\,dt + 1\big).
\end{aligned} \tag{4}$$

We introduce the functional space $V_0^{k,p(\cdot)}(Q_T)$, defined by

$$V_0^{k,p(\cdot)}(Q_T) = \Big\{ u \in L^{p_-}(0,T;W_0^{k,p(\cdot)}(\Omega))/ \\
\int_{Q_T} |D^\alpha u|^{p(x)}\,dx\,dt < \infty \text{ for } |\alpha| = 0,\dots,k \Big\},$$

The spaces $V^{k,p(\cdot)}(Q_T)$ and $V_0^{k,p(\cdot)}(Q_T)$ are separable and reflexive Banach spaces.
The dual space of $V_0^{k,p(\cdot)}(Q_T)$ is defined as follows

$$V^{-k,p'(\cdot)}(Q_T) = \Big\{ F = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha f_\alpha, \text{ with } f_\alpha \in L^{p'(\cdot)}(Q_T) \\
\text{for } |\alpha| = 0,\dots,k \Big\},$$

endowed with the norm

$$\begin{aligned}
\|F\|_{V^{-k,p'(\cdot)}(Q_T)} &= \inf\Big\{ \sum_{|\alpha|=0}^{k} \|f_\alpha\|_{L^{p'(\cdot)}(Q_T)} \;/\; F = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha f_\alpha \\
&\quad \text{with } f_\alpha \in L^{p'(\cdot)}(Q_T) \text{ for } |\alpha| = 0,\dots,k \Big\}.
\end{aligned}$$

The duality pairing of the space $V_0^{k,p(\cdot)}(Q_T)$ with its dual is given by the relation

$$\langle F, v\rangle = \sum_{|\alpha|=0}^{k} \int_{Q_T} f_\alpha(x) D^\alpha v(x)\,dx$$

for all $v \in V_0^{k,p(\cdot)}(Q_T)$.

Now, we define the functional space related to our problem called the Non-Standard Sobolev space of infinite order involving space and time.

Let $(a_\alpha)_\alpha$ be a sequence of nonnegative bounded real numbers for all multi-index α such that $(a_\alpha)_\alpha > 0$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha < \infty$.

The space $L^{p_-}(0,T;W^\infty(a_\alpha,p(\cdot))(\Omega))$ is defined by

$$L^{p_-}(0,T;W^\infty(a_\alpha,p(\cdot))(\Omega)) = \Big\{u(\cdot,t) \in C^\infty(\Omega)/ \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \|D^\alpha u\|_{p(\cdot)}^{p_-}\, dt < \infty\Big\},$$

and we define

$$V^{\infty,p(\cdot)}(a_\alpha,Q_T) = \Big\{u \in L^{p_-}(0,T;W^\infty(a_\alpha,p(\cdot))(\Omega))/ \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\, dx\, dt < \infty\Big\},$$

endowed with the norm

$$\|u\|_{V^{\infty,p(\cdot)}(a_\alpha,Q_T)} := \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{L^{p(\cdot)}(Q_T)}.$$

We have for any $u \in V^{\infty,p(\cdot)}(a_\alpha,Q_T)$

$$\frac{1}{C}\|u\|^{p_-}_{V^{\infty,p(\cdot)}(a_\alpha,Q_T)} - \sum_{|\alpha|=0}^{\infty} a_\alpha \;\leq \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\, dx\, dt. \tag{5}$$

We introduce the functional space $V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)$, defined by

$$V_0^{\infty,p(\cdot)}(a_\alpha,Q_T) = \Big\{u \in L^{p_-}(0,T;W_0^\infty(a_\alpha,p(\cdot))(\Omega))/ \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\, dx\, dt < \infty\Big\}.$$

The spaces $V^{\infty,p(\cdot)}(a_\alpha,Q_T)$ and $V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)$ are separable and reflexive Banach spaces.

The dual space of $V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)$ is defined as

$$V^{-\infty,p'(\cdot)}(a_\alpha,Q_T) = \Big\{F = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha, \text{ with } f_\alpha \in L^{p'(\cdot)}(Q_T) \text{ for } |\alpha| = 0,\dots,k\Big\},$$

and endowed with the norm

$$\|F\|_{V^{-\infty,p'(\cdot)}(a_\alpha,Q_T)} = \inf\Big\{ \sum_{|\alpha|=0}^{\infty} a_\alpha \|f_\alpha\|_{L^{p'(\cdot)}(Q_T)} \;/\; F = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \subset L^{p'(\cdot)}(Q_T) \text{ for } |\alpha| = 0,1,\dots \Big\}.$$

The duality pairing of the space $V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)$ with its dual is given by the relation

$$\langle F, v\rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{Q_T} f_\alpha(x) D^\alpha v(x)\,dx\,dt,$$

for all $v \in V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)$.

3. Main result.

3.1. Essential Assumptions.

Let $Q_T = \Omega \times (0,T)$ with $0 < T < \infty$. and $p(\cdot) \in C_+(\overline{\Omega})$. The nonlinear operator A acted from $V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)$ into its dual $V^{-\infty,p'(\cdot)}(a_\alpha,Q_T)$ is defined by

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x,t,\nabla^\gamma u), \quad |\gamma| \le |\alpha|$$

where $A_\alpha : \Omega \times [0,T] \times \mathbb{R}^{\lambda_\alpha} \to \mathbb{R}$ is a real function and λ_α is the number of multi-indices γ such that $|\gamma| \le |\alpha|$, and that $A_\alpha(x,t,\xi_\gamma)$ are Carathéodory's functions satisfying

$$\big|A_\alpha(x,t,\xi_\gamma)\eta_\alpha\big| \le c_0\, a_\alpha |\xi_\alpha|^{p(x)-1} |\eta_\alpha| \qquad \text{for all } |\alpha| = 0,1,\dots, \tag{6}$$

$$A_\alpha(x,t,\xi_\gamma)\xi_\alpha \ge c_1\, a_\alpha |\xi_\alpha|^{p(x)} \qquad \text{for all } |\alpha| = 0,1,\dots, \tag{7}$$

$$(A_\alpha(x,t,\xi_\gamma) - A_\alpha(x,t,\xi'_\gamma))(\xi_\alpha - \xi'_\alpha) > 0, \qquad \text{for all } |\alpha| = 0,1,\dots, \tag{8}$$

for a.e. $(x,t) \in Q_T$, and all $\xi_\alpha, \xi'_\alpha, \xi_\gamma, \xi'_\gamma$ with $\xi_\alpha \neq \xi'_\alpha$ and $\xi_\gamma \neq \xi'_\gamma$.

$$\text{The space } V_0^{\infty,p(\cdot)}(a_\alpha,Q_T) \text{ is nontrivial.} \tag{9}$$

Here $(a_\alpha)_\alpha$ is a sequence of nonnegative bounded real numbers for all multi-index α such that $a_\alpha > 0$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha < \infty$ and the constants c_1, $c_0 > 0$.

The nonlinear term $g(x,t,s)$ is a Carathéeodory function satisfying

$$|g(x,t,s)| \le |s|^{p(x)-1} + b(x,t), \tag{10}$$

$$g(x,t,s)s \geq 0, \tag{11}$$

for a.e. $(x,t) \in Q_T$ and any $s \in \mathbb{R}$, where the positive function $b(x,t)$ belongs to $L^{p'(\cdot)}(Q_T)$.

We consider the nonlinear $p(x)-$parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + g(x,t,u) = f & \text{in} \quad Q_T, \\ u(x,t) \ = 0 & \text{on} \quad S_T, \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \end{cases} \tag{12}$$

where $u_0 \in L^2(\Omega)$ and $f \in V^{-\infty,p'(\cdot)}(a_\alpha, Q_T)$.

3.2. Some technical Lemmas.

Lemma 3.1. *(cf. [18]) Let B_0, B and B_1 be Banach spaces, and let*

$$Y = \{u \,:\, u \in L^{p_0}(0,T;B_0),\, u_t \in L^{p_1}(0,T;B_1)\}$$

where p_0, $p_1 > 1$ are real numbers.
If the embedding $B_0 \subset B \subset B_1$ is continuous, and the embedding $B_0 \subset B$ is compact, then

$$Y \subset L^{p_0}(0,T;B)$$

and this embedding is compact.

Lemma 3.2. *Let $g \in L^{r(\cdot)}(Q_T)$ and $g_n \in L^{r(\cdot)}(Q_T)$ with $\|g_n\|_{L^{r(\cdot)}(Q_T)} \leq C$ for $r(\cdot) \in C_+(\overline{\Omega})$.*
If $g_n(x,t) \to g(x,t)$ a.e. on Q_T, then $g_n \rightharpoonup g$ in $L^{r(\cdot)}(Q_T)$.

Proof. The proof is similar to the proof of Lemma 3.3 in [7], by considering Q_T instead of Ω.

Lemma 3.3. *Assuming that $(6)-(8)$ hold and let $(u_n)_n$ be a sequence in $V_0^{k+1,p(\cdot)}(Q_T)$ such that $u_n \rightharpoonup u$ in $V_0^{k+1,p(\cdot)}(Q_T)$ and*

$$\begin{aligned} &\sum_{|\alpha|=k+1} a_\alpha \int_0^T \int_\Omega (|D^\alpha u_n|^{p(x)-2} D^\alpha u_n - |D^\alpha u|^{p(x)-2} D^\alpha u)(D^\alpha u_n - D^\alpha u)\, dx\, dt \\ &\quad + \sum_{|\alpha|=0}^{k} \int_0^T \int_\Omega (A_\alpha(x,t,\nabla^\gamma u_n) - A_\alpha(x,t,\nabla^\gamma u))(D^\alpha u_n - D^\alpha u)\, dx\, dt \longrightarrow 0, \end{aligned} \tag{13}$$

then $u_n \longrightarrow u$ in $V_0^{k+1,p(\cdot)}(Q_T)$ for a subsequence.

Proof. Let

$$\begin{aligned} S_n(x,t) &= \sum_{|\alpha|=k+1} a_\alpha(|D^\alpha u_n|^{p(x)-2}D^\alpha u_n - |D^\alpha u|^{p(x)-2}D^\alpha u)(D^\alpha u_n - D^\alpha u) \\ &\quad + \sum_{|\alpha|=0}^{k} (A_\alpha(x,t,\nabla^\gamma u_n) - A_\alpha(x,t,\nabla^\gamma u))(D^\alpha u_n - D^\alpha u). \end{aligned}$$

Thanks to (8), we have $S_n(x,t)$ is a positive function, and by (13) we have $S_n \to 0$ in $L^1(Q_T)$ *as* $n \to \infty$.
Since $u_n \rightharpoonup u$ in $V_0^{k+1,p(\cdot)}(Q_T)$ and since $S_n \to 0$ a.e in Q_T, then, there exists a subset $B \subset \Omega$ with measure zero such that

$$|D^\alpha u(x)| < \infty \quad \text{for all} \quad |\alpha| = 0,\dots,k+1 \qquad \textit{and} \qquad S_n \to 0,$$

for all $(x,t) \in Q_T \backslash B$. We have

$$\begin{aligned} S_n(x,t) &= \sum_{|\alpha|=k+1} a_\alpha(|D^\alpha u_n|^{p(x)-2}D^\alpha u_n - |D^\alpha u|^{p(x)-2}D^\alpha u)(D^\alpha u_n - D^\alpha u) \\ &+ \sum_{|\alpha|=0}^{k} (A_\alpha(x,t,\nabla^\gamma u_n) - A_\alpha(x,t,\nabla^\gamma u))(D^\alpha u_n - D^\alpha u) \\ &= \sum_{|\alpha|=k+1} a_\alpha(|D^\alpha u_n|^{p(x)} + |D^\alpha u|^{p(x)} - |D^\alpha u_n|^{p(x)-2}D^\alpha u_n\, D^\alpha u - |D^\alpha u|^{p(x)-2}D^\alpha u\, D^\alpha u_n) \\ &+ \sum_{|\alpha|=0}^{k} (A_\alpha(x,t,\nabla^\gamma u_n)D^\alpha u_n + A_\alpha(x,t,\nabla^\gamma u)D^\alpha u - A_\alpha(x,t,\nabla^\gamma u_n)D^\alpha u - A_\alpha(x,t,\nabla^\gamma u)D^\alpha u_n) \\ &\geq \sum_{|\alpha|=k+1} a_\alpha(|D^\alpha u_n|^{p(x)} + |D^\alpha u|^{p(x)} - |D^\alpha u_n|^{p(x)-2}D^\alpha u_n\, D^\alpha u - |D^\alpha u|^{p(x)-2}D^\alpha u\, D^\alpha u_n) \\ &+ \sum_{|\alpha|=0}^{k} a_\alpha(c_1|D^\alpha u_n|^{p(x)} + c_1|D^\alpha u|^{p(x)} - c_0|D^\alpha u_n|^{p(x)-1}|D^\alpha u| - c_0|D^\alpha u|^{p(x)-1}|D^\alpha u_n|) \\ &\geq \underline{c_1} \sum_{|\alpha|=0}^{k+1} a_\alpha|D^\alpha u_n|^{p(x)} - C_{x,t}(1 + \overline{c_0}\sum_{|\alpha|=0}^{k+1} a_\alpha|D^\alpha u_n|^{p(x)-1} + \overline{c_0}\sum_{|\alpha|=0}^{k+1} a_\alpha|D^\alpha u_n|), \end{aligned}$$

with $\underline{c_1} = \min(c_1,1)$, $\overline{c_0} = \max(c_0,1)$ and the constant $C_{x,t}$ depends only on x and t and does not depend on n. By taking

$$R_{n,p(x)} = \sum_{|\alpha|=0}^{k+1} a_\alpha|D^\alpha u_n|^{p(x)},\ R_{n,p(x)-1} = \sum_{|\alpha|=0}^{k+1} a_\alpha|D^\alpha u_n|^{p(x)-1} \text{ and}$$

$$R_{n,1} = \sum_{|\alpha|=0}^{k+1} a_\alpha|D^\alpha u_n|,$$

we obtain

$$S_n(x,t) \geq R_{n,p(x)}\left(\underline{c_1} - \frac{C_{x,t}}{R_{n,p(x)}} - \frac{C_{x,t}\overline{c_0}R_{n,p(x)-1}}{R_{n,p(x)}} - \frac{C_{x,t}\overline{c_0}R_{n,1}}{R_{n,p(x)}}\right).$$

Hence, $(D^\alpha u_n)_n$ is bounded almost everywhere in Q_T for all $|\alpha| = 0,\dots,k+1$. (Indeed, assuming for some $0 \le |\alpha_0| \le k+1$, that $|D^{\alpha_0}u_n| \to \infty$ in a measurable subset $E \subset Q_T$, then

$$\begin{aligned}
&\lim_{n\to\infty}\int_{Q_T} S_n(x,t)\,dx\,dt \\
&\ge \limsup_{n\to\infty}\int_E R_{n,p(x)}\left(\underline{c_1} - \frac{C_{x,t}}{R_{n,p(x)}} - \frac{C_{x,t}\overline{c_0}R_{n,p(x)-1}}{R_{n,p(x)}} - \frac{C_{x,t}\overline{c_0}R_{n,1}}{R_{n,p(x)}}\right)dx\,dt \\
&\ge \limsup_{n\to\infty}\int_E |D^{\alpha_0}u_n|^{p(x)}\left(\underline{c_1} - \frac{C_{x,t}}{R_{n,p(x)}} - \frac{C_{x,t}\overline{c_0}R_{n,p(x)-1}}{R_{n,p(x)}} - \frac{C_{x,t}\overline{c_0}R_{n,1}}{R_{n,p(x)}}\right)dx\,dt \\
&= \infty,
\end{aligned}$$

which is absurd since $S_n \to 0$ in $L^1(Q_T)$).

Let ω_α be an accumulation point of $(D^\alpha u_n)_n$ for $|\alpha| = 0,\dots,k+1$, we have $|\omega_\alpha| < \infty$. Thanks to the continuity of A_α, we have

$$\begin{aligned}
\sum_{|\alpha|=k+1} a_\alpha(|\omega_\alpha|^{p(x)-2}\omega_\alpha - |D^\alpha u|^{p(x)-2}D^\alpha u)(\omega_\alpha - D^\alpha u)& \\
+\sum_{|\alpha|=0}^{k}(A_\alpha(x,t,\omega_\gamma) - A_\alpha(x,t,\nabla^\gamma u))(\omega_\alpha - D^\alpha u) &= 0.
\end{aligned}$$

Thus by (8), we deduce that $\omega_\alpha = D^\alpha u$, and the uniqueness of the accumulation point implies that $D^\alpha u_n \to D^\alpha u$ a.e in Q_T.
Now, remark that the operator $(A_\alpha(x,t,\nabla^\gamma u_n))_n$ is bounded in $L^{p'(\cdot)}(Q_T)$ and $A_\alpha(x,t,\nabla^\gamma u_n) \to A_\alpha(x,t,\nabla^\gamma u)$ a.e in Q_T. Using Lemma 3.2, we can establish that

$$A_\alpha(x,t,\nabla^\gamma u_n) \rightharpoonup A_\alpha(x,t,\nabla^\gamma u) \quad \text{in} \quad L^{p'(\cdot)}(Q_T) \qquad \text{for all} \quad |\alpha| = 0,\dots,k,$$

and, in view of (8) and (13), we obtain

$$\int_{Q_T}|D^\alpha u_n|^{p(x)}\,dx\,dt \longrightarrow \int_{Q_T}|D^\alpha u|^{p(x)}\,dx\,dt \ \text{ for all } |\alpha| = k+1, \tag{14}$$

and

$$\int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_n)D^\alpha u_n\,dx\,dt \longrightarrow \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u)D^\alpha u\,dx\,dt \tag{15}$$

for all $|\alpha| = 0,\dots,k$. Thanks to the coercivity condition, we have

$$c_1\,a_\alpha|D^\alpha u_n|^{p(x)} \le A_\alpha(x,t,\nabla^\gamma u_n)D^\alpha u_n.$$

Hence, in view of Fatou's Lemma, we get for all $0 \leq |\alpha| \leq k$,

$$\begin{aligned} 2\int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_n)D^\alpha u_n\,dx\,dt \leq \liminf_{n\to\infty}\bigg(&\int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_n)D^\alpha u_n\,dx\,dt \\ &+\int_{Q_T} A_\alpha(x,t,\nabla^\gamma u)D^\alpha u\,dx\,dt \\ &-\frac{c_1 a_\alpha}{2^{p_+-1}}\int_{Q_T}|D^\alpha u_n - D^\alpha u|^{p(x)}\,dx\,dt\bigg), \end{aligned}$$

then

$$0 \leq -\limsup_{n\to\infty}\int_{Q_T}|D^\alpha u_n - D^\alpha u|^{p(x)}\,dx\,dt.$$

It follows that

$$0 \leq \liminf_{n\to\infty}\int_{Q_T}|D^\alpha u_n - D^\alpha u|^{p(x)}\,dx\,dt \leq \limsup_{n\to\infty}\int_{Q_T}|D^\alpha u_n - D^\alpha u|^{p(x)}\,dx\,dt \leq 0,$$

which implies that

$$\int_{Q_T}|D^\alpha u_n - D^\alpha u|^{p(x)}\,dx\,dt \longrightarrow 0$$

as $n \to \infty$. Thus, we obtain

$$D^\alpha u_n \longrightarrow D^\alpha u \quad in \quad L^{p(\cdot)}(Q_T) \qquad \text{for all} \quad |\alpha| = 0,\ldots,k.$$

Finally, thanks to (14), we have $D^\alpha u_n \to D^\alpha u$ in $L^{p(\cdot)}(Q_T)$ for $|\alpha| = k+1$. Consequently, we deduce that

$$u_n \longrightarrow u \quad \text{in} \quad V_0^{k+1,p(\cdot)}(Q_T).$$

This completes our proof.

3.3. Main result.

Theorem 3.4. *Assuming that* (6) - (11) *hold, then for any* $f \in V^{-\infty,p'(\cdot)}(a_\alpha, Q_T)$, *there exists* $u \in V_0^{\infty,p(\cdot)}(a_\alpha, Q_T)$ *such that*

$$\begin{cases} g(x,t,u) \in L^1(Q_T), \quad g(x,t,u)u \in L^1(Q_T), \text{ and} \\ \displaystyle\int_0^T \langle \frac{\partial u}{\partial t}, v\rangle dt + \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T\int_\Omega A_\alpha(x,t,\nabla^\gamma u)D^\alpha v\,dx\,dt \\ \qquad \displaystyle + \int_{Q_T} g(x,t,u)v\,dx\,dt = \int_0^T \langle f, v\rangle\,dt, \text{for all } v \in V_0^{\infty,p(\cdot)}(a_\alpha, Q_T). \end{cases}$$

Proof of the Theorem 3.4

Step 1: Approximate problems

We consider for all $k \geq 1,$ the approximate problems

$$\begin{cases} \dfrac{\partial u_k}{\partial t} + A_{2k+2}u_k + g(x,t,u_k) = f_k(x,t) & \text{in} \quad Q_T, \\ u_k = 0 & \text{on} \quad S_T, \\ u_k(x,0) = u_0 & \text{in} \quad \Omega, \end{cases}$$

where

$$f_k(x,t) = \sum_{|\alpha|=0}^{k+1} (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha(x,t) \text{ for } f_\alpha \in L^{p'(\cdot)}(Q),$$

and A_{2k+2} is the operator acting from $V_0^{k+1,p(\cdot)}(Q_T)$ into $V^{-k-1,p'(\cdot)}(Q_T)$ defined by

$$\begin{aligned}\int_0^T \langle A_{2k+2}u, v\rangle\, dt = & \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)-2} D^\alpha u\, D^\alpha v\, dx\, dt \\ & + \sum_{|\alpha|=0}^{k} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u)\, D^\alpha v\, dx\, dt.\end{aligned}$$

Using generalized Hölder's type inequality, we have for any $u,v \in V_0^{k+1,p(\cdot)}(Q_T)$

$$\begin{aligned}\left| \int_0^T \langle A_{2k+2}u, v\rangle\, dt \right| \leq & \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)-1}\, |D^\alpha v|\, dx\, dt \\ & + \sum_{|\alpha|=0}^{k} \int_{Q_T} |A_\alpha(x,t,\nabla^\gamma u)|\, |D^\alpha v|\, dx\, dt \\ \leq & \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)-1}\, |D^\alpha v|\, dx\, dt \\ & + c_0 \sum_{|\alpha|=0}^{k} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)-1}\, |D^\alpha v|\, dx\, dt \\ \leq & \, \overline{c_0} \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)-1}\, |D^\alpha v|\, dx\, dt \\ \leq & \, \overline{c_0}\, c_p \sum_{|\alpha|=0}^{k+1} a_\alpha \|\, |D^\alpha u|^{p(x)-1} \|_{L^{p'(\cdot)}(Q_T)}\, \|D^\alpha v\|_{L^{p(\cdot)}(Q_T)},\end{aligned}$$

with $\overline{c_0} = \max(1, c_0)$ and $c_p = \left(\dfrac{1}{p_-} + \dfrac{1}{(p'(\cdot))_-} \right)$. Since

$$\|\, |D^\alpha u|^{p(x)-1} \|_{L^{p'(\cdot)}(Q_T)} \leq \left(\int_{Q_T} |D^\alpha u|^{p(x)} dx\, dt \right)^{\frac{1}{p'_-}} + 1 \leq \|D^\alpha u\|_{L^{p(\cdot)}(Q_T)}^{\frac{p_+}{p'_-}} + 2,$$

then

$$\begin{aligned}\left|\int_0^T \langle A_{2k+2}u,v\rangle\,dt\right| &\leq \overline{c_0}c_p\sum_{|\alpha|=0}^{k+1} a_\alpha(\|D^\alpha u\|_{L^{p(\cdot)}(Q_T)}^{\frac{p_+}{p'_-}}+2)\|D^\alpha v\|_{L^{p(\cdot)}(Q_T)}\\ &\leq C(\|v\|_{V_0^{k+1,p(\cdot)}(Q_T)}^{\frac{p_+}{p'_-}}+2)\|v\|_{V_0^{k+1,p(\cdot)}(Q_T)}. \end{aligned}\tag{16}$$

We define the operator $G : V_0^{k+1,p(\cdot)}(Q_T) \longmapsto V^{-k-1,p'(\cdot)}(Q_T)$, by

$$\int_0^T \langle Gu,v\rangle\,dt = \int_{Q_T} g(x,t,u)v\,dx\,dt \qquad \forall v\in V_0^{k+1,p(\cdot)}(Q_T).$$

By using Young's inequality, we have for any u, $v \in V_0^{k+1,p(\cdot)}(Q_T)$

$$\begin{aligned}\left|\int_{Q_T} g(x,t,u)v\,dx\,dt\right| &\leq \int_{Q_T}(|u|^{p(x)-1}+b(x,t))|v|\,dx\,dt\\ &\leq c_p\,(\|\,|u|^{p(x)-1}\|_{L^{p'(\cdot)}(Q_T)}+\|b(x,t)\|_{L^{p'(\cdot)}(Q_T)})\big\|v\big\|_{L^{p(\cdot)}(Q_T)}\\ &\leq c'\|v\|_{V_0^{k+1,p(\cdot)}(Q_T)}\,. \end{aligned}\tag{17}$$

Lemma 3.5. *$B_k = A_{2k+2} + G$ is pseudo-monotone operator from $V_0^{k+1,p(\cdot)}(Q_T)$ into $V^{-k-1,p'(\cdot)}(Q_T)$. Moreover, B_n is coercive in the following sense*

$$\frac{\int_0^T\langle B_k v,v\rangle\,dt}{\|v\|_{V_0^{k+1,p(\cdot)}(Q_T)}} \longrightarrow +\infty \qquad as \quad \|v\|_{V_0^{k+1,p(\cdot)}(Q_T)} \longrightarrow +\infty,$$

for all $v\in V_0^{k+1,p(\cdot)}(Q_T)$.

Proof of Lemma 3.5

In view of inequalities (16) and (17), the operator B_k is bounded. For the coercivity, thanks to (7) and (10), we have for all $u \in V_0^{k+1,p(\cdot)}(Q_T)$

$$\begin{aligned}\int_0^T \langle B_k u, u\rangle\, dt &= \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\, dx\, dt \\ &\quad + \sum_{|\alpha|=0}^{k} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u) D^\alpha u\, dx\, dt \\ &\quad + \int_{Q_T} g(x,t,u) u\, dx\, dt \\ &\geq \underline{c_1} \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\, dx\, dt \\ &\geq \frac{\underline{c_1}\,\underline{a_{k+1}}}{C} \|u\|^{p_-}_{V_0^{k+1,p(\cdot)}(Q_T)} - \underline{c_1} \sum_{|\alpha|=0}^{k+1} a_\alpha,\end{aligned}$$

with $\underline{c_1} = \min(1, c_1)$ and $0 < \underline{a_{k+1}} = \min\limits_{|\alpha|\leq k+1} a_\alpha$. Then, it follows that

$$\frac{\int_0^T \langle B_k u, u\rangle\, dt}{\|u\|_{V_0^{k+1,p(\cdot)}(Q_T)}} \longrightarrow +\infty \qquad \text{as} \quad \|u\|_{V_0^{k+1,p(\cdot)}(Q_T)} \longrightarrow +\infty.$$

Now it remains to show that B_k is pseudo-monotone. Let $(u_h)_h$ a sequence in $V_0^{k+1,p(\cdot)}(Q_T)$ such that

$$\left\{\begin{array}{l} u_h \rightharpoonup u \quad \text{in } V_0^{k+1,p(\cdot)}(Q_T), \\ B_k u_h \rightharpoonup \chi \text{ in } V^{-k-1,p'(\cdot)}(Q_T), \\ \limsup\limits_{h\to\infty} \int_0^T \langle B_k u_h, u_h\rangle\, dt \leq \int_0^T \langle \chi, u\rangle\, dt. \end{array}\right. \tag{18}$$

We shall prove that

$$\chi = B_k u \quad \text{and} \quad \langle B_k u_h, u_h\rangle \longrightarrow \langle \chi, u\rangle \text{ as } h \to +\infty.$$

First, since $a_\alpha > 0$ for $|\alpha| \leq 1$, then $V_0^{k+1,p(\cdot)}(Q_T) \subset V_0^{1,p(\cdot)}(Q_T)$. Hence the embedding $V_0^{k+1,p(\cdot)}(Q_T) \hookrightarrow\hookrightarrow L^{p(\cdot)}(Q_T)$ is compact and there exists a subsequence still denoted by $(u_h)_h$ such that $u_h \to u$ in $L^{p(\cdot)}(Q_T)$.

Since $(u_h)_h$ is a bounded sequence in $V_0^{k+1,p(\cdot)}(Q_T)$, and in view of (6), it follows that $(A_\alpha(x,t,\nabla^\gamma u_h))_h$ is bounded in $L^{p'(\cdot)}(Q_T)$. Therefore, there exists a function $\varphi_\alpha \in L^{p'(\cdot)}(\Omega)$ such that

$$A_\alpha(x,t,\nabla^\gamma u_h) \rightharpoonup \varphi_\alpha \text{ in } L^{p'(\cdot)}(Q_T) \text{ for } |\alpha| = 0,\ldots,k, \tag{19}$$

and

$$|D^\alpha u_h|^{p(x)-2}D^\alpha u_h \rightharpoonup |D^\alpha u|^{p(x)-2}D^\alpha u \text{ in } L^{p'(\cdot)}(Q_T) \text{ for } |\alpha| = k+1. \quad (20)$$

Since $u_h \to u$ in $L^{p(\cdot)}(Q_T)$ then

$$g(x,t,u_h) \longrightarrow g(x,t,u) \qquad \text{in} \quad L^{p'(\cdot)}(Q_T) \text{ as } h \to \infty. \tag{21}$$

Thus, for all $v \in V_0^{k+1,p(\cdot)}(Q_T)$ we have

$$\begin{aligned}\int_0^T \langle \chi, v\rangle\, dt &= \lim_{h\to\infty}\int_0^T \langle B_k u_h, u\rangle\, dt \\ &= \lim_{h\to\infty}\sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_h|^{p(x)-2}D^\alpha u_h\, D^\alpha v\, dx\, dt \\ &\quad + \lim_{h\to\infty}\sum_{|\alpha|=0}^{k}\int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_h)\, D^\alpha v\, dx\, dt \\ &\quad + \lim_{h\to\infty}\int_{Q_T} g(x,t,u_h)v\, dx\, dt \\ &= \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)-2}D^\alpha u\, D^\alpha v\, dx\, dt \\ &\quad + \sum_{|\alpha|=0}^{k}\int_{Q_T} \varphi_\alpha\, D^\alpha v\, dx\, dt + \int_{Q_T} g(x,t,u)v\, dx\, dt.\end{aligned} \tag{22}$$

Combining (18) and (22), we obtain

$$\begin{aligned}\limsup_{h\to\infty}\int_0^T \langle B_k u_h, u_h\rangle\, dt &= \limsup_{h\to\infty}\Bigg(\sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_h|^{p(x)}\, dx\, dt \\ &\quad + \sum_{|\alpha|=0}^{k}\int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_h)\, D^\alpha u_h\, dx\, dt \\ &\quad + \int_{Q_T} g(x,t,u_h)u_h\, dx\, dt\Bigg) \\ &\le \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\, dx\, dt \\ &\quad + \sum_{|\alpha|=0}^{k}\int_{Q_T} \varphi_\alpha\, D^\alpha u\, dx\, dt + \int_{Q_T} g(x,t,u)u\, dx\, dt,\end{aligned}$$

and since

$$\int_{Q_T} g(x,t,u_h)u_h\, dx\, dt \longrightarrow \int_{Q_T} g(x,t,u)u\, dx\, dt \quad \text{as } h \to \infty, \tag{23}$$

then

$$\limsup_{h\to\infty}\Big(\sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_h|^{p(x)}\,dx\,dt + \sum_{|\alpha|=0}^{k} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_h)\,D^\alpha u_h\,dx\,dt\Big)$$
$$\leq \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\,dx\,dt + \sum_{|\alpha|=0}^{k} \int_{Q_T} \varphi_\alpha\,D^\alpha u\,dx\,dt. \tag{24}$$

On the other hand, in view of (8), we get

$$\sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} (|D^\alpha u_h|^{p(x)-2} D^\alpha u_h - |D^\alpha u|^{p(x)-2} D^\alpha u)(D^\alpha u_h - D^\alpha u)\,dx\,dt$$
$$+ \sum_{|\alpha|=0}^{k} \int_{Q_T} (A_\alpha(x,t,\nabla^\gamma u_h) - A_\alpha(x,t,\nabla^\gamma u))(D^\alpha u_h - D^\alpha u)\,dx\,dt \geq 0.$$

This implies that

$$\sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_h|^{p(x)}\,dx\,dt + \sum_{|\alpha|=0}^{k} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_h)\,D^\alpha u_h\,dx\,dt$$
$$\geq - \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\,dx\,dt + \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_h|^{p(x)-2} D^\alpha u_h\,D^\alpha u\,dx\,dt$$
$$+ \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)-2} D^\alpha u\,D^\alpha u_h\,dx\,dt + \sum_{|\alpha|=k+1} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_h)\,D^\alpha u\,dx\,dt$$
$$+ \sum_{|\alpha|=0}^{k} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u)\,D^\alpha u_h\,dx\,dt - \sum_{|\alpha|=k+1} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u)\,D^\alpha u\,dx\,dt.$$

Using (19) and (20), we deduce that

$$\liminf_{h\to\infty}\Big(\sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_h|^{p(x)}\,dx\,dt + \sum_{|\alpha|=0}^{k} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_h)\,D^\alpha u_h\,dx\,dt\Big)$$
$$\geq \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\,dx\,dt + \sum_{|\alpha|=0}^{k} \int_{Q_T} \varphi_\alpha\,D^\alpha u\,dx\,dt. \tag{25}$$

Hence, in view of (24) and (25), we get

$$\lim_{h\to\infty}\Big(\sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_h|^{p(x)}\,dx\,dt + \sum_{|\alpha|=0}^{k} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_h)\,D^\alpha u_h\,dx\,dt\Big)$$
$$= \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\,dx\,dt + \sum_{|\alpha|=0}^{k} \int_{Q_T} \varphi_\alpha\,D^\alpha u\,dx\,dt. \tag{26}$$

Thanks to (23), we conclude that

$$\langle B_k u_h, u_h\rangle \longrightarrow \langle \chi, u\rangle \text{ as } h \to +\infty.$$

Now, by (26), we obtain

$$\sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} (|D^\alpha u_h|^{p(x)-2} D^\alpha u_h - |D^\alpha u|^{p(x)-2} D^\alpha u)(D^\alpha u_h - D^\alpha u)\, dx\, dt \to 0$$

as $h \to +\infty$, and

$$\sum_{|\alpha|=0}^{k} \int_{Q_T} (A_\alpha(x,t,\nabla^\gamma u_h) - A_\alpha(x,t,\nabla^\gamma u))(D^\alpha u_h - D^\alpha u)\, dx\, dt \to 0 \text{ as } h \to +\infty.$$

In view of Lemma 3.3, we deduce that

$$u_h \longrightarrow u \quad \text{in} \quad V_0^{k+1,p(\cdot)}(Q_T).$$

Then, it follows that

$$|D^\alpha u_h|^{p(x)-2} D^\alpha u_h \longrightarrow |D^\alpha u|^{p(x)-2} D^\alpha u \quad \text{in} \quad L^{p'(\cdot)}(Q_T) \quad \text{for all} \quad |\alpha| = k+1, \tag{27}$$

and $A_\alpha(x,t,\nabla^\gamma u_h) \to A_\alpha(x,t,\nabla^\gamma u)$ a.e in Q_T, since $(A_\alpha(x,t,\nabla^\gamma u_h))$ is bounded in $L^{p'(\cdot)}(Q_T)$. Applying Lemma 3.2, we obtain

$$A_\alpha(x,t,\nabla^\gamma u_h) \rightharpoonup A_\alpha(x,t,\nabla^\gamma u) \text{ in } L^{p'(\cdot)}(Q_T) \quad \text{for all} \quad |\alpha| = 0,\dots,k. \tag{28}$$

Finally, combining (21), (27) and (28) we conclude that $\chi = B_n u$, which completes the proof of Lemma 3.5.

Therefore, by Lemma 3.5, there exists at least one weak solution $u_k \in V_0^{k+1,p(\cdot)}(Q_T)$ of quasilinear parabolic problem (16), we refer the reader to ([27], Theorem 2.7, page 180).

Step 2: A priori estimates.

Taking $u_k(x,t)$ as a test function in (16), we obtain

$$\int_0^T \langle \frac{\partial u_k}{\partial t}, u_k\rangle dt + \int_0^T \langle A_{2k+2} u_k, u_k\rangle dt + \int_{Q_T} g(x,t,u_k) u_k\, dxdt = \int_0^T \langle f_k, u_k\rangle\, dt. \tag{29}$$

We have

$$\begin{aligned}
&\int_0^T \langle A_{2k+2}u_k, u_k\rangle dt \\
&= \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\,dx\,dt + \sum_{|\alpha|=0}^{k} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u_k) D^\alpha u_k\,dx\,dt \\
&\geq \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\,dx\,dt + c_1 \sum_{|\alpha|=0}^{k} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\,dx\,dt \\
&\geq \underline{c_1} \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\,dx\,dt,
\end{aligned} \tag{30}$$

and

$$\int_0^T \langle \frac{\partial u_k}{\partial t}, u_k\rangle dt = \int_\Omega \frac{|u_k(T)|^2}{2}\,dx - \int_\Omega \frac{|u_0|^2}{2}\,dx. \tag{31}$$

For the term on the right-hand side of (29), by using Young's inequality we get

$$\begin{aligned}
\int_0^T \langle f_k, u_k\rangle\,dt &= \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} f_\alpha D^\alpha u_k\,dx\,dt \\
&\leq c_2 \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |f_\alpha|^{p'(\cdot)}\,dx\,dt \\
&\quad + \frac{\underline{c_1}}{2} \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\,dx\,dt.
\end{aligned} \tag{32}$$

By combining $(29)-(32)$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{|u_k(T)|^2}{2}\,dx + \frac{\underline{c_1}}{2} \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\,dx\,dt + \int_{Q_T} g(x,t,u_k)u_k\,dx\,dt \\
&\qquad \leq c_2 \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |f_\alpha|^{p'(\cdot)}\,dx\,dt + \int_\Omega \frac{|u_0|^2}{2}\,dx.
\end{aligned} \tag{33}$$

Since $u_0 \in L^2(Q_T)$ and $\|f_k\|_{V^{-k-1,p'(\cdot)}(Q_T)} \leq C\|f\|_{V^{-\infty,p'(\cdot)}(a_\alpha,Q_T)}$, then, there exists a constant c_3 that doesn't depend on k, such that

$$\int_\Omega \frac{|u_k(T)|^2}{2}\,dx + \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\,dx\,dt + \int_{Q_T} g(x,t,u_k)u_k\,dx\,dt \leq c_3. \tag{34}$$

Step 3: The sequence $\left(\dfrac{\partial u_k}{\partial t}\right)_{k\in\mathbb{N}^*}$ **is bounded in** $V^{-\infty,p'(\cdot)}(a_\alpha,Q_T)$.

Using $v\in V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)$ as a test function in (16), we get

$$\int_0^T \langle\frac{\partial u_k}{\partial t},v\rangle dt+\int_0^T \langle A_{2k+2}u_k,v\rangle dt+\int_{Q_T} g(x,t,u_k)v\,dxdt=\int_0^T \langle f_k,v\rangle\,dt,$$

then, we have

$$\left|\int_0^T \langle\frac{\partial u_k}{\partial t},v\rangle dt\right|\le\left|\int_0^T \langle A_{2k+2}u_k,v\rangle dt\right|+\int_{Q_T}|g(x,t,u_k)|\,|v|\,dx\,dt$$
$$+\left|\int_0^T \langle f_k,v\rangle\,dt\right|. \quad (35)$$

For the first term on the right-hand side of (35), using generalized Hölder's type inequality and (6), we obtain

$$\begin{aligned}
&\left|\int_0^T \langle A_{2k+2}u_k,v\rangle dt\right|\\
&\le\sum_{|\alpha|=k+1}a_\alpha\int_{Q_T}|D^\alpha u_k|^{p(x)-1}|D^\alpha v|\,dx\,dt\\
&+\sum_{|\alpha|=0}^{k}\int_{Q_T}|A_\alpha(x,t\nabla^\gamma u_k)|\,|D^\alpha v|\,dx\,dt\\
&\le\overline{c_0}\sum_{|\alpha|=0}^{k+1}a_\alpha\int_{Q_T}|D^\alpha u_k|^{p(x)-1}|D^\alpha v|\,dx\,dt\\
&\le c_p\,\overline{c_0}\sum_{|\alpha|=0}^{k+1}a_\alpha\|\,|D^\alpha u_k|^{p(x)-1}\|_{L^{p'(\cdot)}(Q_T)}\|D^\alpha v\|_{L^{p(\cdot)}(Q_T)}\\
&\le c_p\,\overline{c_0}\sum_{|\alpha|=0}^{k+1}a_\alpha((\int_{Q_T}|D^\alpha u_k|^{p(x)}\,dx\,dt)^{\frac{1}{p'_-}}+1)\|D^\alpha v\|_{L^{p(\cdot)}(Q_T)}\\
&\le c_4\|v\|_{V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)}. \qquad (36)
\end{aligned}$$

Concerning the second term on the right-hand side of (35), thanks to (10) we get

$$\begin{aligned}
&\int_{Q_T}|g(x,t,u_k)|\,|v|\,dx\,dt\\
&\le\int_{Q_T}|u_k|^{p(x)-1}|v|\,dx\,dt+\int_{Q_T}b(x,t)|v|\,dx\,dt\\
&\le c_p(\|\,|u_k|^{p(x)-1}\,\|_{L^{p'(\cdot)}(Q_T)}+\|b(x,t)\|_{L^{p'(\cdot)}(Q_T)})\|v\|_{L^{p(\cdot)}(Q_T)}\\
&\le c_5\|v\|_{V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)}. \qquad (37)
\end{aligned}$$

For the last term on the right-hand side of (35), we have

$$\left|\int_0^T \langle f_k, v\rangle\, dt\right| \leq \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |f_\alpha|\,|D^\alpha v|\, dx\, dt$$
$$\leq c_p \sum_{|\alpha|=0}^{k+1} a_\alpha \|f_\alpha\|_{L^{p'(\cdot)}(Q_T)} \|D^\alpha v\|_{L^{p(\cdot)}(Q_T)}$$
$$\leq c_p c_6 \|f\|_{V^{-\infty,p'(\cdot)}(a_\alpha,Q_T)} \|v\|_{V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)}. \tag{38}$$

By combining $(35)-(38)$, we deduce that

$$\left|\int_0^T \langle \frac{\partial u_k}{\partial t}, v\rangle dt\right| \leq c_7 \|v\|_{V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)}, \tag{39}$$

with c_7 is a constant that does not depend on k. Then it follows that the sequence $\left(\frac{\partial u_k}{\partial t}\right)_{k\in I\!N^*}$ is bounded in $V^{-\infty,p'(\cdot)}(a_\alpha,Q_T)$.

In order to apply Lemma 3.1, let us consider

$$B_0 = W^{k+1,p(\cdot)}(\Omega), \quad B = W^{k,p(\cdot)}(\Omega) \quad \text{and} \quad B_1 = W^{-\infty}(a_\alpha, p'(\cdot))(\Omega)$$

with $p_0 = p_-$ and $p_1 = p'_-$, where $k \in I\!N^*$ is arbitrary. Then, in view of estimates (34) and (39), we deduce that the family $(u_k)_{k\in I\!N^*}$ of solutions of the problems (16) is compact in the space $V_0^{k,p(\cdot)}(Q_T)$.

Consequently, by similar argument as in the elliptic case (using the diagonal process), (see [9] and [18]), one gets that the sequence $(u_k)_{k\in I\!N^*}$ converges strongly together with all derivatives $(D^\omega u_k)_{k\in I\!N^*}$ in the space $L^{p(\cdot)}(Q)$ to some function $u \in V_0^{\infty,p(\cdot)}(a_\alpha,Q_T)$, i.e

$$u_k \longrightarrow u \quad \text{in} \quad V_0^{\infty,p(\cdot)}(a_\alpha,Q_T). \tag{40}$$

Step 4: The equi-integrability of $g(x,t,u_k)$.

We shall prove that

$$g(x,t,u_k) \longrightarrow g(x,t,u) \quad \text{strongly in} \quad L^1(Q_T).$$

Indeed, using Vitali's theorem, it is sufficient to prove that $g(x,t,u_k)$ is uniformly equi-integrable.

Let $m > 1$ and E a be measurable subset of Q_T, we have

$$\begin{aligned}
&\int_E |g(x,t,u_k)|\,dx\,dt \\
&\le \int_{E\cap\{|u_k|\le m\}} |g(x,t,u_k)|\,dx\,dt + \frac{1}{m}\int_{E\cap\{|u_k|>m\}} g(x,t,u_k)u_k\,dx\,dt \\
&\le \int_{E\cap\{|u_k|\le m\}} (|T_m(u_k)|^{p(x)-1} + b(x,t))\,dx\,dt + \frac{1}{m}\int_{Q_T} g(x,t,u_k)u_k\,dx\,dt \\
&\le |m|^{p_+-1}|E| + \int_E b(x,t)\,dx\,dt + \frac{c_3}{m},
\end{aligned}$$

where c_3 is the constant of (34) which is independent of k. Then, for all $\varepsilon > 0$, there exists m large enough such that $\dfrac{c_3}{m} < \dfrac{\varepsilon}{2}$, and $|E|$ sufficiently small to obtain $|m|^{p_+-1}|E| + \displaystyle\int_E b(x,t)\,dx\,dt < \frac{\varepsilon}{2}$, we get

$$\int_E |g(x,t,u_k)|\,dx\,dt \le \varepsilon.$$

Using Vitali's theorem, and since $g(x,t,u_k) \to g(x,t,u)$ a.e. in Q_T, we deduce that

$$g(x,t,u_k) \longrightarrow g(x,t,u) \qquad \text{in} \quad L^1(Q_T). \tag{41}$$

On the other hand, in view of Fatou's lemma and (34), we obtain

$$\int_{Q_T} g(x,t,u)u\,dx\,dt \le \liminf_{k\to+\infty} \int_{Q_T} g(x,t,u_k)u_k\,dx\,dt \le c_3,$$

which implies that $g(x,t,u)u \in L^1(Q_T)$.

Step 5: Passage to the limit.

Now, we will prove that

$$\lim_{k\to+\infty} \int_0^T \langle A_{2k+2}(u_k), v\rangle\,dt = \int_0^T \langle A(u), v\rangle\,dt \qquad \text{for all} \quad v \in V_0^{\infty,p(\cdot)}(a_\alpha, Q_T). \tag{42}$$

Indeed, let $k > 0$ large enough and $v \in V_0^{\infty,p(\cdot)}(a_\alpha, Q_T)$, we have

$$\begin{aligned}
&\int_0^T \langle A(u) - A_{2k+2}(u_k), v\rangle\, dt \\
&= \sum_{|\alpha|=0}^{k} \int_{Q_T} (A_\alpha(x,t,\nabla^\gamma u) - A_\alpha(x,t,\nabla^\gamma u_k))\, D^\alpha v\, dx\, dt \\
&\quad - \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)-2} D^\alpha u_k D^\alpha v\, dx\, dt \\
&\quad + \sum_{|\alpha|=k+1}^{\infty} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u)\, D^\alpha v\, dx\, dt. \qquad (43)
\end{aligned}$$

On one hand, since $(A_\alpha(x,t,\xi_\gamma))_\alpha$ are Carathéodory functions and thanks to (40), we obtain

$$\sum_{|\alpha|=0}^{k} \int_{Q_T} (A_\alpha(x,t,\nabla^\gamma u) - A_\alpha(x,t,\nabla^\gamma u_k))\, D^\alpha v\, dx\, dt \longrightarrow 0 \quad \text{as} \quad k \to \infty. \qquad (44)$$

Now since $u_k \in V_0^{k+1,p(\cdot)}(a_\alpha, Q_T)$, then we get

$$\sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\, dx\, dt \le C \quad \text{for all} \quad k > 0,$$

it follows that

$$\lim_{k\to\infty} \sum_{|\alpha|=0}^{k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\, dx dt \le C \Longrightarrow \lim_{k\to\infty} \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\, dx dt = 0.$$

We deduce, using Young's inequality, that

$$\begin{aligned}
&\left| \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)-2} D^\alpha u_k D^\alpha v\, dx\, dt \right| \\
&\le \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)-1} |D^\alpha v|\, dx\, dt \\
&\le \frac{1}{p'_-} \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)}\, dx\, dt + \frac{1}{p_-} \sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha v|^{p(x)}\, dx\, dt.
\end{aligned}$$

Hence, we conclude that

$$\sum_{|\alpha|=k+1} a_\alpha \int_{Q_T} |D^\alpha u_k|^{p(x)-2} D^\alpha u_k D^\alpha v\, dx\, dt \longrightarrow 0 \quad \text{as} \quad k \to \infty. \qquad (45)$$

On the other hand, using Young inequality, we have

$$\begin{aligned}
&\Big| \sum_{|\alpha|=k+1}^{\infty} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u)\, D^\alpha v\, dx\, dt \Big| \\
&\leq c_0 \sum_{|\alpha|=k+1}^{\infty} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)-1}\, |D^\alpha v|\, dx\, dt \\
&\leq \frac{c_0}{p'_-} \sum_{|\alpha|=k+1}^{\infty} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\, dx\, dt + \frac{c_0}{p_-} \sum_{|\alpha|=k+1}^{\infty} a_\alpha \int_{Q_T} |D^\alpha v|^{p(x)}\, dx\, dt,
\end{aligned}$$

and since $u,\, v \in V_0^{\infty,p(\cdot)}(a_\alpha, Q_T)$ then, we have

$$\sum_{|\alpha|=0}^{\infty} a_\alpha \int_{Q_T} |D^\alpha u|^{p(x)}\, dx\, dt < \infty \quad \text{and} \quad \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{Q_T} |D^\alpha v|^{p(x)}\, dx\, dt < \infty,$$

which implies that

$$\sum_{|\alpha|=k+1}^{\infty} \int_{Q_T} A_\alpha(x,t,\nabla^\gamma u)\, D^\alpha v\, dx\, dt \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{46}$$

Finally, by combining $(43)-(46)$, we conclude that

$$\int_0^T \langle A_{2k+2}(u_k), v\rangle\, dt \longrightarrow \int_0^T \langle A(u), v\rangle\, dt \quad \text{for all} \quad v \in V_0^{\infty,p(\cdot)}(a_\alpha, Q_T). \tag{47}$$

Moreover, it is clear that

$$\int_0^T \langle f_k, v\rangle\, dt \longrightarrow \int_0^T \langle f, v\rangle\, dt \qquad \text{as} \quad k \to \infty. \tag{48}$$

Consequently, by passing to the limit in (16), we obtain

$$\int_0^T \langle \frac{\partial u}{\partial t}, v\rangle dt + \int_0^T \langle A(u), v\rangle\, dt + \int_{Q_T} g(x,t,u) v\, dx\, dt = \int_0^T \langle f, v\rangle\, dt,$$

for all $v \in V_0^{\infty,p(\cdot)}(a_\alpha, Q_T)$, which achieves the proof of Theorem 3.4.

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