LE MATEMATICHE
Vol. LIX (2004) – Fasc. I–II, pp. 73–89

FLOCKS, OVOIDS AND GENERALIZED QUADRANGLES

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Let (∞, π_∞) be an incident point-plane pair of PG(3, q). A *tetrad* with respect to (∞, π_∞) is a set $\{X, Y, Z, W\}$ of points of PG$(3, q) \setminus \pi_\infty$ such that $\{\infty, X, Y, Z, W\}$ is a cap of PG(3, q), $\infty \in \langle X, Y, Z\rangle$ and $W \notin \langle X, Y, Z\rangle$. A set Θ of ovoids of PG(3, q) is *tetradic* with respect to (∞, π_∞) if each ovoid contains ∞, has tangent plane π_∞ and is such that each tetrad with respect to (∞, π_∞) is contained in a *unique* ovoid of Θ. From this definition we are able to prove that such a set Θ has a rich and constrained structure, related to flocks and Laguerre planes.

In fact, we shall see that a tetradic set of ovoids always gives rise to a generalized quadrangle (GQ) of order (q, q^2) satisfying Property (G) at a flag, and conversely, suggested by a construction of J. A. Thas from 1999.

In the case where each element of the tetradic set Θ is an elliptic quadric, the existence of the set Θ is equivalent to the existence of a flock of a quadratic cone and the corresponding flock GQ. Using these connections Barwick, Brown and Penttila showed that a GQ satisfying Property (G) at a pair of points and whose associated ovoids are all elliptic quadric must be a dual flock GQ.

In the more general ovoid case, when q is even, Brown showed that a tetradic set with the property that the set of dual ovoids arising from the tangent planes to ovoids of the set, is also a tetradic set, comprises elliptic quadrics. Hence a GQ satisfying Property (G) at a line must be the dual of flock GQ (as conjectured by J. A. Thas at Combinatorics '98).

We shall discuss all of the above results.

1. Introduction.

In this paper we will discuss some of the connections between flocks of quadratic cones, ovoids of $PG(3, q)$ and generalized quadrangles.

In the mid 1980's Kantor and Payne independently gave coset geometry constructions methods for new generalized quadrangles of order (q^2, q) (Payne in [16] for q even and Kantor [14] for q odd). These construction methods have subsequently led to many new constructions of GQs. In the q even case Payne also showed that there were ovals of $PG(2, q)$ associated with his construction method. This has led to constructions of new families of ovals ([16], [7], [6]).

In 1987 J. A. Thas [21] showed that these constructions were equivalent to the existence (in both the q odd and even cases) of a flock of a quadratic cone of $PG(3, q)$. It is now known that flocks form the connection for a nexus of geometrical objects and constructions (see, [13], [19]).

In [15] Knarr gave a geometrical construction of flock quadrangles in the q odd case. Later J. A. Thas [25] gave a geometrical reconstruction of the dual flock quadrangle using ovoids of $PG(3, q)$ (and also another geometrical model for the GQ in [26]). In this paper, motivated by the Thas construction, we take the opposite tack and start with a simple condition defining a set of ovoids in $PG(3, q)$ from which we can construct a GQ of order (q, q^2). Examples of this construction method are the dual flock quadrangles and the Tits GQ $T_3(\Omega)$ for Ω an ovoid of $PG(3, q)$. This construction also allows us to prove new characterisation results for GQs, including a proof that a dual flock GQ is characterised by having Property (G) at a line (see Section 3 for details on Property (G)), as conjectured by J. A. Thas in [24].

We now give the definitions and results required to fill in the details of the above discussion.

An *oval* of $PG(2, q)$ is a set of $q + 1$ points of $PG(2, q)$ no three of which are collinear. Let ℓ be a line of $PG(2, q)$, then ℓ is incident with zero, one or two points of an oval and is accordingly called an *external* line, a *tangent* or a *secant* to the oval.

A *cap* of $PG(3, q)$ is a set of points of $PG(3, q)$ no three of which are collinear. A line of $PG(3, q)$ will be called external, tangent or secant to a cap according to whether it contains zero, one or two points of the cap. An *ovoid* of $PG(3, q)$ is a cap of size $q^2 + 1$ such that the tangents at a point form a plane, called the *tangent plane* at the point. Every plane not tangent to an ovoid meets the ovoid in an oval. If $q > 2$, then a cap of $PG(3, q)$ of maximal size is an ovoid. Every ovoid of $PG(3, q)$, q odd, is a non-degenerate elliptic quadric of $PG(3, q)$. For q even, $q = 2^h$, the two known isomorphism classes of ovoids are the non-degenerate elliptic quadrics, which exist for all $h \geq 1$, and the

Tits ovoids which exist for h odd, $h \geq 3$. (See [10], [11], [12] for details and references for the above.)

Let (∞, π_∞) be an incident point-plane pair of PG$(3, q)$. If X, Y, Z, W are four distinct points of PG$(3, q) \setminus \pi_\infty$, then we say that $\{X, Y, Z, W\}$ is a *tetrad* with respect to (∞, π_∞) if $\{\infty, X, Y, Z, W\}$ is a cap of PG$(3, q)$ such that there exists a plane of PG$(3, q)$ containing ∞ and exactly three of X, Y, Z, W. If ∞ and π_∞ are understood, then we will refer to $\{X, Y, Z, W\}$ as a *tetrad*.

A *tetradic* set of ovoids with respect to (∞, π_∞) is a set of ovoids of PG$(3, q)$ each element of which contains ∞, has tangent plane π_∞ at ∞ and such that every tetrad with respect to (∞, π_∞) is contained in a unique ovoid of the set.

Tetradic sets of ovoids are the fundamental objects in this paper and we shall them and their connection to generalized quadrangles of order (q, q^2). In particular, we will show that a tetradic set of ovoids of PG$(3, q)$ gives rise to a generalized quadrangle of order (q, q^2).

2. Laguerre geometries and flocks.

A *Laguerre plane* is an incidence structure of points, lines and circles with the following properties.

1. Every point lies on a unique line.
2. Any three pairwise non-collinear points lie on a unique circle.
3. For any two non-collinear points P and Q with P on C and Q not on C, there is a unique circle D on Q which meets C in exactly P.

Given a finite Laguerre plane, there is an integer $n > 1$ called the *order* of the plane such that there are $n^2 + n$ points, $n + 1$ lines and n^3 circles, every line is incident with n points, every circle is incident with $n + 1$ points, every point is incident with n^2 circles, and every pair of non-collinear points lies on n circles.

Given a Laguerre plane L and a point P of the plane, the *derived affine plane* L_P is the incidence structure with points the points of L not collinear with P, lines the circles of L incident with P and the lines of L not on P and the natural incidence relation. The structure L_P is an affine plane. If L has order n, then L_P has order n.

The known models for Laguerre planes of order q arise from oval cones in PG$(3, q)$. Let $\mathcal{K}$ be a cone in PG$(3, q)$ with vertex V over an oval $\mathcal{O}$ of PG$(2, q)$. The incidence structure with points the points of $\mathcal{K}$ other than V, lines the generators of $\mathcal{K}$, circles the plane sections of $\mathcal{K}$ not containing V and the natural incidence relation, is a Laguerre plane of order q. In the case where $\mathcal{O}$ is a conic, these Laguerre planes are characterised amongst all Laguerre planes by satisfying the configuration of Miquel [27], [8], pp. 245–246, and hence are called *Miquelian*. General references on Laguerre planes are [3], [9], [8], [20].

A flock $\mathcal{F}$ of a Laguerre plane L is a set of circles of L partitioning the points of L. If L has order n, then $\mathcal{F}$ contains n circles. Of particular interest will be the flocks of the Miqeulian Laguerre plane arising from a quadratic cone $\mathcal{K}$ in PG$(3, q)$. Such a flock will also be called a *flock* of the quadratic cone $\mathcal{K}$. For more details on flocks of Laguerre planes see [13].

In the case where the Laguerre plane arises from a quadratic cone in PG$(3, q)$ we have a useful model for flocks in PG$(2, q)$. Let $\mathcal{K}$ be an oval cone in PG$(3, q)$ with vertex V. Let $P \in \mathcal{K} \setminus \{V\}$, $\ell = \langle P, V\rangle$ and let π_ℓ be the plane meeting $\mathcal{K}$ in ℓ. Suppose that π is any plane containing neither V nor P. If we project the points of $\mathcal{K} \setminus \{V\}$ from P onto π, then we have a one-to-one correspondence between the points of $\mathcal{K} \setminus \ell$ and the points of $\pi \setminus (\pi_\ell \cap \pi)$, while the points $\ell \setminus \{P, V\}$ project onto $P' = \ell \cap \pi$. The $q^3 - q^2$ plane sections of $\mathcal{K}$ containing neither P nor V project onto the $q^3 - q^2$ conics of π containing P' and with tangent $\pi \cap \pi_\ell$. A flock $\{\mathcal{C}_1, \ldots, \mathcal{C}_q\}$ of $\mathcal{K}$ projects to a set $\{\mathcal{C}'_1, \ldots, \mathcal{C}'_{q-1}, m\}$ where $\mathcal{C}'_1, \ldots, \mathcal{C}'_{q-1}$ are conics of π with common point P', common tangent $\pi \cap \pi_\ell$, m is a line of π not incident with P' and $\mathcal{C}'_1, \ldots, \mathcal{C}'_{q-1}, m$ partition the points of $\pi \setminus (\pi \cap \pi_\ell)$. Further, no $\mathcal{C}'_i$ is the image of a $\mathcal{C}'_j$, $i, j \in \{1, \ldots, q-1\}$, $i \neq j$, under an elation of π with centre P'. Conversely, any such set $\{\mathcal{C}'_1, \ldots, \mathcal{C}'_{q-1}, m\}$ with these properties corresponds to a flock of $\mathcal{K}$.

If in the definition of a Laguerre plane above we weaken the second axiom to

2.′ Any three pairwise non-collinear points lie on a constant number of circles.

then we have an incidence structure that we will call a *Laguerre geometry*.

We will see that there is a strong connection between these so-called Laguerre geometries and generalized quadrangles of order (s, s^2).

3. Generalized Quadrangles with Property (G).

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

1. Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line.
2. Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point.
3. If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \mathrm{I}\, m \,\mathrm{I}\, Y \,\mathrm{I}\, \ell$.

For a comprehensive introduction to GQs see [18]. The integers s and t are the *parameters* of the GQ and is said to have *order* (s, t). If $s = t$, then is said to have order s. If has order (s, t), then it follows that $|\mathcal{P}| = (s+1)(st+1)$ and $|\mathcal{B}| = (t + 1)(st + 1)$ [18], 1.2.1 . If $= (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order (s, t), then the incidence structure $\mathcal{S}^* = (\mathcal{B}, \mathcal{P}, \mathrm{I})$ is a GQ of order (t, s) called the *dual* of $\mathcal{S}$.

Given two (not necessarily distinct) points X, X' of $\mathcal{S}$, we write $X \sim X'$ and say that X and X' are *collinear*, provided there is some line ℓ for which $X \,\mathrm{I}\, \ell \,\mathrm{I}\, X'$. For $X \in \mathcal{P}$ put $X^\perp = \{X' \in \mathcal{P} : X \sim X'\}$. If $A \subset \mathcal{P}$, then we define $A^\perp = \cap\{X^\perp : X \in A\}$ and $A^{\perp\perp} = (A^\perp)^\perp$.

If $s^2 = t > 1$, then by a result of Bose and Shrikhande ([4]) we have $|\{X, Y, Z\}^\perp| = s + 1$ for any triple $\{X, Y, Z\}$ of pairwise non-collinear points (called a *triad*). We say that $\{X, Y, Z\}$ is 3*-regular* provided $|\{X, Y, Z\}^{\perp\perp}| = s + 1$. The point X is 3*-regular* if and only if each triad $\{X, Y, Z\}$ is 3-regular.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2), $s \neq 1$. Let X_1, Y_1 be distinct collinear points. We say that the pair $\{X_1, Y_1\}$ has *Property (G)*, or that $\mathcal{S}$ has *Property (G) at* $\{X_1, Y_1\}$, if every triad $\{X_1, X_2, X_3\}$ of points, with $Y_1 \in \{X_1, X_2, X_3\}^\perp$, is 3-regular. The GQ $\mathcal{S}$ has *Property (G) at the line* ℓ, or the line ℓ has *Property (G)*, if each pair of points $\{X, Y\}$, $X \neq Y$ and $X \,\mathrm{I}\, \ell \,\mathrm{I}\, Y$, has Property (G). If (X, ℓ) is a flag, then we say that $\mathcal{S}$ has *Property (G) at* (X, ℓ) or that (X, ℓ) has Property (G), if every pair (X, Y), $X \neq Y$ and $Y \,\mathrm{I}\, \ell$ has Property (G).

Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order (q, q^2) satisfying Property (G) at the pair of points $\{X, Y\}$. We now review a construction of AG$(3, q)$ from $\mathcal{S}$, X and Y due to Payne and Thas (see [22]).

We consider the following incidence structure $\mathcal{S}_{XY} = (\mathcal{P}_{XY}, \mathcal{B}_{XY}, \mathrm{I}_{XY})$:

(i) $\mathcal{P}_{XY} = X^\perp \setminus \{X, Y\}^{\perp\perp}$.

(ii) Elements of $\mathcal{B}_{XY}$ are of two types: (a) the sets $\{Y, Z, U\}^{\perp\perp} \setminus \{Y\}$, with $\{Y, Z, U\}$ a triad with $X \in \{Y, Z, U\}^\perp$, and (b) the sets $\{X, W\}^\perp \setminus \{X\}$, with $X \sim W \not\sim Y$.

(iii) I_{XY} is containment.

Then we have the following result.

Theorem 3.1. (Payne and Thas, see [22]) *The incidence structure* $\mathcal{S}_{XY}$ *is the design of points and lines of the affine space* $\mathrm{AG}(3, q)$*. In particular,* q *is a prime power.*

The planes of the affine space $\mathcal{S}_{XY} = \mathrm{AG}(3, q)$ *are of two types:*

(a) The sets $\{X, Z\}^{\perp} \setminus \{Y\}$*, with* $X \not\sim Z$ *and* $Y \in \{X, Z\}^{\perp}$*, and*
(b) each set which is the union of all elements of type (b) of $\mathcal{B}_{XY}$ *containing a point of some line* m *of type (a) of* $\mathcal{B}_{XY}$*.*

This construction leads us to an equivalent formulation of Property (G) at a pair of points (see [2]).

Theorem 3.2. *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a GQ of order* (s, s^2) *and* $X, Y \in \mathcal{P}$ *with* $X \sim Y$*. Then* $\mathcal{S}$ *satisfies Property (G) at* $\{X, Y\}$ *if and only if the incidence structure*

Points:	$X^{\perp} \setminus \langle X, Y\rangle$,
Planes:	$Y^{\perp} \setminus \langle X, Y\rangle$,
Incidence:	*Collinearity in* $\mathcal{S}$

is the point-plane incidence structure of $\mathrm{PG}(3, s)$ *with an incident point-plane pair removed.*

Let $\overline{\mathcal{S}_{XY}}$ be the projective completion of $\mathcal{S}_{XY}$ with plane at infinity π_{∞}. In [25] Thas gives the following interpretation of the GQ $\mathcal{S}$ in $\overline{\mathcal{S}_{XY}}$. The q^2 lines of type (b) of $\mathcal{S}_{XY}$ are parallel, so they define a point ∞ of $\overline{\mathcal{S}_{XY}}$. If we now consider any $Z \in \mathcal{P}$ with $X \not\sim Z \not\sim Y$ and U the point of $\ell = \langle X, Y\rangle$ such that $Z \sim U$, then $\mathcal{V} = \{X, Z\}^{\perp} \setminus \{U\}$ is a set of q^2 points. Clearly each line of $\overline{\mathcal{S}_{XY}}$ on ∞ meets $\mathcal{V}$ in exactly one point. Further, if U_1, U_2, U_3 are points of $\mathcal{V}$ collinear in $\overline{\mathcal{S}_{XY}}$, then it must be that $Y \in \{U_1, U_2, U_3\}^{\perp\perp}$ and so $Z \sim Y$, a contradiction since X, Y, Z is a triangle. It follows from this that $\mathcal{V} \cup \{\infty\}$ is an ovoid of $\overline{\mathcal{S}_{XY}}$ with tangent plane π_{∞} at ∞. We will denote this ovoid by Ω_Z.

Thas also determined the intersections of these ovoids. Consider two distinct points $Z_1, Z_2 \in \mathcal{P}$ with Z_1, Z_2 collinear with points $U_1, U_2 \,\mathrm{I}\, \ell$, respectively, with $U_1, U_2 \neq X, Y$. If $Z_1 \sim Z_2$ and $U_1 = U_2$, then $\Omega_{Z_1} \cap \Omega_{Z_2} = \{\infty\}$, since any larger intersection yields a triangle in $\mathcal{S}$.

If $Z_1 \sim Z_2$ and $U_1 \neq U_2$, then $\Omega_{Z_1} \cap \Omega_{Z_2} = \{\infty, R\}$ where R is the point of the line $\langle Z_1, Z_2\rangle$ in $X^{\perp}$. Further the point of $\langle Z_1, Z_2\rangle$ in $Y^{\perp}$ corresponds, in $\overline{\mathcal{S}_{XY}}$ to a plane which is tangent at R to both Ω_{Z_1} and Ω_{Z_2}.

If $Z_1 \not\sim Z_2$ and $U_1 = U_2$, then $\Omega_{Z_1} \cap \Omega_{Z_2} = (\{X, Z_1, Z_2\}^{\perp} \setminus \{U_1\}) \cup \{\infty\}$, an intersection of size $q + 1$.

For the last case, if $Z_1 \not\sim Z_2$ and $U_1 \neq U_2$, then $\Omega_{Z_1} \cap \Omega_{Z_2} = \{X, Z_1, Z_2\}^{\perp} \cup \{\infty\}$, an intersection of size $q + 2$.

If m is a line of $\mathcal{S}$ such that $m \,\mathrm{I}\, U \,\mathrm{I}\, \ell$ and $U \neq X, Y$, then let the set ovoids of $\mathrm{PG}(3, q) = \overline{\mathcal{S}_{XY}}$ corresponding to points of $m \setminus \{U\}$ be denoted $\mathcal{R}$. The set $\mathcal{R}$ is a set of q ovoids of $\mathrm{PG}(3, q)$ meeting pairwise in a fixed point and with the same tangent plane at that point. We will call such a set $\mathcal{R}$ a *rosette* of ovoids, the fixed point of intersection is called the *base point* of the rosette and the common tangent plane at the base point is called the *base plane* of the rosette. The elements of a rosette partition the points of $\mathrm{PG}(3, q)$ not on the base plane.

If m is a line of $\mathcal{S}$ such that m and ℓ are non-concurrent, then let the set of ovoids of $\mathrm{PG}(3, q) = \overline{\mathcal{S}_{XY}}$ corresponding to points of $m \setminus (X^\perp \cup Y^\perp)$ be denoted $\mathcal{T}$. The set $\mathcal{T}$ is a set of $q - 1$ ovoids of $\mathrm{PG}(3, q)$ meeting pairwise in exactly two fixed points and sharing the tangent planes at those two fixed points. We will call such a set $\mathcal{T}$ a *transversal* of ovoids. These two common points are called the *base points* of the transversal and the two common tangent planes are called the *base planes* of the transversal.

Given this geometric interpretation we also have the following connection between a GQ of order (s, s^2) satisfying Property (G) at a pair of points and Laguerre geometries.

Theorem 3.3. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2) satisfying Property (G) at the pair of points $\{X, Y\}$, $X \sim Y$. Then the incidence structure $\mathcal{L}(X)$*

Points:	$X^\perp \setminus Y^\perp$,
Lines:	$\{\ell : X \,\mathrm{I}\, \ell\} \setminus \{\langle X, Y\rangle\}$,
Circles:	$\{Z^\perp \setminus \{\langle X, Y\rangle\} : Z \in \mathcal{P} \setminus X^\perp\}$,
Incidence:	*Natural,*

is a Laguerre geometry.

Suppose now that the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (q, q^2), $q \neq 1$, satisfies Property (G) at the flag (X, ℓ). Let R be $\{U, U_1, U_2\}^{\perp\perp} = \{U, U_1, U_2, \ldots, U_q\}$, with $\{U, U_1, U_2\}$ a triple of pairwise non-collinear points with $X \in \{U, U_1, U_2\}^\perp$ and $U \,\mathrm{I}\, \ell$, and let ℓ_i be the line incident with X and the point U_i, $i = 1, 2, \ldots, q$. Further, let $\mathcal{P}_{XR}$ be the set of all points, different from X, collinear with X and a point of R, let $\mathcal{B}_{XR} = \{\ell, \ell_1, \ell_2, \ldots, \ell_q\}$, and let $\mathcal{C}_{XR}$ be the set having as elements the sets $\{W, W_1, W_2\}^{\perp\perp}$ with $W \,\mathrm{I}\, \ell$, $W_1 \,\mathrm{I}\, \ell_1$, $W_2 \,\mathrm{I}\, \ell_2$ and $X \notin \{W, W_1, W_2\}$. Also, let incidence I_{XR} between elements of $\mathcal{P}_{XR}$ and elements of $\mathcal{B}_{XR}$ be induced by the incidence in $\mathcal{S}$, and let incidence I_{XR} between elements of $\mathcal{P}_{XR}$ and elements of $\mathcal{C}_{XR}$ be containment. Then Thas shows that these incidence structures are Laguerre planes.

Theorem 3.4. ([22]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (q, q^2) satisfying*

Property (G) at the flag (X, ℓ)*. The incidence structure*

$$\mathcal{L} = (\mathcal{P}_{XR}, \mathcal{B}_{XR}, \mathcal{C}_{XR}, \mathrm{I}_{XR}),$$

with pointset $\mathcal{P}_{SR}$*, lineset* $\mathcal{B}_{XR}$*, circleset* $\mathcal{C}_{XR}$ *and incidence* I_{XR}*, is a Laguerre plane of order* q*. Also, for each points* $Y \,\mathrm{I}\, \ell$*,* $Y \neq X$*, the derived or internal affine plane* $\mathcal{L}_Y$ *of* $\mathcal{L}$ *at* Y *is the affine plane* $\mathrm{AG}(2, q)$*; hence, for* q *odd* $\mathcal{L}$ *is the classical Laguerre plane, that is, arises from the quadratic cone in* $\mathrm{PG}(3, q)$.

If we view $\mathcal{P}_{XR}$ as a subset of the pointset of $\mathcal{L}(X)$, then $\mathcal{L}$ is induced on $\mathcal{P}_{XR}$ by $\mathcal{L}(X)$.

Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a dual flock GQ of order (q, q^2), arising from the flock $\mathcal{F}$, satisfying Property (G) at the line $[\infty]$ and $X, Y \,\mathrm{I}\, [\infty]$, $X \neq Y$. In [25] Thas constructed a set of elliptic quadric ovoids of $\mathrm{PG}(3, q)$ from $\mathcal{F}$ which was then verified to be the set of ovoids $\{\mathcal{O}_Z \ : \ Z \in \mathcal{P} \setminus (X^{\perp} \cup Y^{\perp})\}$ of $\overline{\mathcal{S}_{XY}} = \mathrm{PG}(3, q)$. As a result Thas gave a geometric description of the dual flock GQs valid for both q odd and even (previously Knarr in [15] had given a description valid for only q odd).

The main theorem of [25] is the following result.

Theorem 3.5. ([25], Main Theorem) *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a GQ of order* (q, q^2)*,* $q > 1$*, and assume that* $\mathcal{S}$ *satisfies Property (G) at the flag* (X, ℓ)*. If* q *is odd then* $\mathcal{S}$ *is the dual of a flock GQ. If* q *is even and all ovoids* $\mathcal{O}_Z$ *are elliptic quadrics, then we have the same conclusion.*

In Section 5.1 we will discuss how tetradic sets of ovoids of may be used to weaken the hypotheses of Theorem 3.5 to assume only Property (G) at a pair of collinear points.

4. Geometric constructions of flock quadrangles.

In this section we consider geometrical constructions of flock GQs. First we consider the construction of Knarr [15] (see also [23]). Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_q\}$ be a flock of the quadratic cone K with vertex X_0 of $\mathrm{PG}(3, q)$, with q odd. The plane of $\mathcal{C}_i$ is denoted by π_i, $i = 1, 2, \dots, q$. Let K be embedded in the non-singular quadric $\mathcal{Q}$ of $\mathrm{PG}(4, q)$. The polar line of π with respect to the polarity of $\mathcal{Q}$ is denoted by ℓ_i and let $\ell_i \cap \mathcal{Q} = \{X_0, X_i\}$, for $i = 1, 2, \dots, q$. Then no point of $\mathcal{Q}$ is collinear with all three of X_0, X_i, X_j, $1 \leq i < j \leq q$. Such a set $U = \{X_0, X_1, \dots, X_q\}$ is called a BLT-set and has the property that no point of $\mathcal{Q}$ is collinear with three points of U (see Bader, Lunardon and Thas [1]). Applying the duality from the GQ $Q(4, q)$ to the GQ

$W(q)$ we have a BLT-set of lines of $W(q)$, that is, a set V of $q+1$ lines of $W(q)$ such that no line of $W(q)$ is concurrent with three distinct lines of V. Knarr [15] shows that the GQ of order (q^2, q) corresponding to the flock $\mathcal{F}$ is isomorphic to the following incidence structure.

Let ϕ be a symplectic polarity of PG$(5, q)$. Let $P \in$ PG$(5, q)$ and let PG$(3, q)$ be a 3-dimensional subspace of PG$(5, q)$ for which $P \notin$ PG$(3, q) \subset P^{\phi}$. In PG$(3, q)$ the polarity ϕ induces a symplectic polarity ϕ', and hence a GQ $W(q)$. Let V be a BLT-set of $W(q)$ and let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be defined as follows.

Points: (i) P; (ii) lines of PG$(5, q)$ not containing P but contained in one of the planes $\pi_t = \langle P, \ell_t \rangle$, with ℓ_t an element of V; (iii) points of PG$(5, q)$ not in P^{ϕ}.

Lines: (a) $\pi_t = \langle P, \ell_t \rangle$, with $\ell_t \in V$; totally isotropic planes of ϕ not contained in P^{ϕ} and meeting some π_t in a line.

The incidence relation I is just the natural incidence inherited from PG$(5, q)$.

Next we give the construction of Thas which is valid for both q odd and even ([25], see also [24]). Let $\widetilde{F} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q-1}, n\}$ be the projection of a flock of a quadratic cone into the plane $\zeta \cong$ PG$(2, q)$ (see Section 2), with $\mathcal{C}_i$, $i = 1, 2, \ldots, q-1$, conics meeting pairwise in the point ∞, common tangent t and with n a line of ζ not incident with ∞. Let ζ be embedded in PG$(3, q)$ and consider planes $\pi_\infty \neq \zeta$ and $\mu \neq \zeta$ of PG$(3, q)$, respectively containing t and n. Let R be a point of $\mu \setminus (\zeta \cup \pi_\infty)$. Let Ω_i be the non-singular quadric which contains $\mathcal{C}_i$, which is tangent to π_∞ at ∞ and which is tangent to μ at R, with $i = 1, 2, \ldots, q-1$. As $\mathcal{C}_i \cap n = \emptyset$, the quadric is elliptic for $i = 1, 2, \ldots, q-1$.

Let $\mathcal{S}$ be the following incidence structure.

Points:
- (a) The non-singular elliptic quadrics Ω containing $\Omega_i \cap \pi_\infty = \ell_\infty^{(i)} \cup m_\infty^{(i)}$ (over GF(q^2)) such that the intersection multiplicity of Ω_i and Ω at ∞ is at least three (that are Ω_i, the non-singular elliptic quadrics $\Omega \neq \Omega_i$ containing $\ell_\infty^{(i)} \cup m_\infty^{(i)}$ (over GF(q^2)) and intersecting Ω_i over GF(q) in a non-singular conic containing ∞, and the non-singular elliptic quadrics $\Omega \neq \Omega_i$ for which $\Omega \cap \Omega_i$ over GF(q^2) is $\ell_\infty^{(i)} \cup m_\infty^{(i)}$ counted twice), with $i = 1, 2, \ldots, q-1$.
- (b) The points of PG$(3, q) \setminus \pi_\infty$.
- (c) The planes of PG$(3, q)$ not containing ∞.
- (d) The set ω_i consisting of the q^3 quadrics corresponding with Ω_i, $i = 1, 2, \ldots, q-1$.
- (e) The plane π_∞.
- (f) The point ∞.

Lines: (i) Let (w, ζ_w) be a point-plane flag of $\mathrm{PG}(3, q)$, with $w \notin \pi_\infty$ and $\infty \notin \zeta_w$. Then all quadrics Ω of type (a) which are tangent to ζ_w at w, together with w and ζ_w, form a line of type (i). Any two distinct quadrics of such a line have exactly two points (R and ∞) in common.

(ii) Let Ω be a point of type (a) which corresponds to the quadric Ω_i, $i \in \{1, 2, \ldots, q-1\}$. If $\Omega \cap \pi_\infty = \Omega_i \cap \pi_\infty = \ell_\infty^{(i)} \cup m_\infty^{(i)}$ (over $\mathrm{GF}(q^2)$), then all points Ω' of type (a) for which $\Omega' \cap \Omega$ over $\mathrm{GF}(q^2)$ is $\ell_\infty^{(i)} \cup m_\infty^{(i)}$ counted twice, together with Ω and Ω_i, form a line of type (ii).

(iii) A set of q parallel planes of $\mathrm{AG}(3, q) = \mathrm{PG}(3, q) \setminus \pi_\infty$, together with π_∞, is a line of type (iii)

(iv) Lines of type (iv) are the lines of $\mathrm{PG}(3, q)$ containing ∞.

(v) $\{\infty, \pi_\infty, \Omega_1, \Omega_2, \ldots, \Omega_{q-1}\}$ is the unique line of type (v).

Incidence: Incidence is containment.

In the proof of the construction Thas [25] confirms that this structure is indeed that arising from the corresponding dual flock GQ as in Section 3. In the following section we will see how it is possible to characterise the set of elliptic quadrics with a simple axiom that allows us to prove the properties required to prove the above construction yields a GQ of order (q, q^2), without referring to the dual flock GQ. We will then see how these concepts can be extended to sets of ovoids (rather than just sets of elliptic quadric ovoids as above) to extend the Thas construction above.

These concepts will also allow us to prove improved characterisation results for GQs with Property (G).

5. Tetradic sets of ovoids of PG(3, q) and generalized quadrangles.

First we consider the dual nature of rosettes and transversals of ovoids.

Theorem 5.1. *Let $\mathcal{R} = \{\Omega_1, \ldots, \Omega_q\}$ be a rosette of ovoids of* $\mathrm{PG}(3, q)$ *with base point ∞ and base plane π_∞. If Ω_i^* is the set of tangent planes to the ovoid Ω_i, then $\mathcal{R}^* = \{\Omega_1^*, \Omega_2^*, \ldots, \Omega_q^*\}$ is a rosette in the dual space* $\mathrm{PG}(3, q)^*$ *with base point π_∞^* and base plane ∞^*.*

Proof. Consider a plane π of $\mathrm{PG}(3, q)$, not π_∞ and not incident with ∞. The elements of $\mathcal{R}$ partition the q^2 points of $\pi \setminus \pi_\infty$ into q sets of size 1 or $q+1$. Consequently, π is tangent to exactly one element of $\mathcal{R}$. So in the dual space $\mathcal{R}^* = \{\Omega_1^*, \Omega_2^*, \ldots, \Omega_q^*\}$ is a set of q ovoids containing the point π_∞^* with common tangent plane ∞^* at π_∞^*. Further each point of $\mathrm{PG}(3, q)^* \setminus \infty^*$

is contained in exactly one element of $\mathcal{R}^*$. Hence the elements of $\mathcal{R}^*$ must intersect pairwise in π^*_∞ and $\mathcal{R}^*$ a rosette in $\mathrm{PG}(3,q)^*$. □

Recall a *transversal* $\mathcal{T}$ of ovoids of $\mathrm{PG}(3,q)$ is a set of $q-1$ ovoids of $\mathrm{PG}(3,q)$ meeting pairwise in exactly two fixed points and sharing the tangent planes at those two fixed points. These two common points are called the *base points* of the transversal and the two common tangent planes are called the *base planes* of the transversal. The elements of a transversal partition the points that are not on the base planes and not on the line spanned by the base points. Similarly to the rosette case a transversal is a self-dual object.

Theorem 5.2. *Let $\mathcal{T} = \{\Omega_1, \Omega_2, \dots, \Omega_{q-1}\}$ be a transversal of ovoids of* $\mathrm{PG}(3,q)$ *with base points ∞ and R and corresponding base planes π_∞ and π_R, respectively. If Ω_i^* is the set of tangent planes to the ovoid Ω_i, then the set $\mathcal{T}^* = \{\Omega_1^*, \Omega_2^*, \dots, \Omega_{q-1}^*\}$ is a transversal in the dual space* $\mathrm{PG}(3,q)^*$ *with base points π^*_∞ and π^*_R, and corresponding base planes ∞^* and R^*, respectively.*

Proof. Let π be a plane of $\mathrm{PG}(3,q)$, $\pi \neq \pi_\infty, \pi_R$ and not incident with ∞ nor R. The q^2-q-1 points of $\pi \setminus (\pi_\infty \cup \pi_R \cup \langle \infty, R\rangle)$ are partitioned by the elements of $\mathcal{T}$ into $q-1$ sets of size 1 or $q+1$. Hence it follows that π is tangent to exactly one element of $\mathcal{T}$ and from this that $\mathcal{T}^*$ is a transversal in the dual space $\mathrm{PG}(3,q)^*$. □

5.1. Tetradic sets of elliptic quadrics of PG(3, q).

This section is based on the work of Barwick, Brown and Penttila [2].

Let Θ be a tetradic set of ovoids of $\mathrm{PG}(3,q)$, with respect to the incident point-plane pair (∞, π_∞), consisting entirely of elliptic quadrics.

Let X, Y, Z be three points such that $\{X, Y, Z, \infty\}$ form a quadrangle in the plane $\pi = \langle X, Y, Z\rangle$. If Ω is an element of Θ containing X, Y, Z, ∞, then the conic $\Omega \cap \pi$ also contains X, Y, Z, ∞ and further has tangent $\pi \cap \pi_\infty$ at ∞ and so is completely determined. Any point $W \in \mathrm{PG}(3,q) \setminus (\pi \cup \pi_\infty)$ when added to $\{X, Y, Z\}$ forms a tetrad and since there is unique element of Θ on any tetrad, there must be q elliptic quadrics of Θ on X, Y, Z and partitioning the points of $\mathrm{PG}(3,q) \setminus (\pi \cup \pi_\infty)$. Calculations show that this set of q elliptic quadrics must be acted on regularly by the elations of $\mathrm{PG}(3,q)$ with centre ∞ and axis π and in fact the full group of elations with centre ∞ must act semi-regularly on Θ. This group action defines equivalence classes on Θ which implies that elements of an equivalence class must intersect in exactly $\{\infty\}$ or in a conic containing ∞.

Now let X, Y, Z be three points spanning a plane not containing ∞. In the plane $\pi = \langle X, Y, \infty\rangle$ there are $q-1$ conics in π containing X, Y, ∞ and with tangent $\pi \cap \pi_\infty$ at ∞, each of which lies in a unique element of Θ with Z.

Consider the incidence structure with points $\mathrm{PG}(3, q) \setminus \pi_\infty$, lines the lines of $\mathrm{AG}(3, q)$ in the parallel class defined by ∞, circles the elements of Θ plus the planes of $\mathrm{PG}(3, q)$ not incident with ∞ and natural incidence. By the above, any three non-collinear points are contained in a constant number q of circles. In fact, in [2] it was proved that for non-collinear points P and Q with P on circle C and Q not on C, there is a unique circle D on Q which meets C in exactly P. So we have a Laguerre geometry as discussed in Section 2. More particularly, from [2] we have the following results on the structure of Θ.

Theorem 5.3.

1. $|\Theta| = q^3(q-1)$.
2. *Θ is divided into $q-1$ equivalence classes of size q^3. Each equivalence class is acted on regularly by the elations of* $\mathrm{PG}(3, q)$ *with centre ∞.*
3. *For each $\Omega \in \Theta$ there is a unique rosette of equivalent elliptic quadrics in Θ (necessarily with base point ∞ and base plane π_∞) containing Ω. This rosette is acted on regularly by the elations with centre ∞ and axis π_∞.*
4. *For each incident point-plane pair (P, π), $P \notin \pi_\infty$, $\infty \notin \pi$ there is a unique transversal of elliptic quadrics in Θ with base points P, ∞ and base planes π, π_∞.*

With these properties it is possible to prove that the following construction yields a GQ of order (q, q^2). The construction is directly equivalent to that of Thas presented in Section 4, although in [2] there is a direct proof that the following incidence structure $\mathrm{GQ}(\Theta)$ is a generalized quadrangle.

Points:
- (a) The elements of Θ.
- (b) The points of $\mathrm{PG}(3, q) \setminus \pi_\infty$.
- (c) The planes of $\mathrm{PG}(3, q)$ not containing ∞.
- (d) The $q-1$ equivalence classes of Θ.
- (e) The plane π_∞.
- (f) The point ∞.

Lines:
- (i) Triples $(P, \pi, \mathcal{T})$ where (P, π) is an incident point-plane pair of $\mathrm{PG}(3, q)$, with $\infty \notin \pi$, $P \notin \pi_\infty$ and $\mathcal{T}$ the unique transversal in Θ with base points P, ∞ and base planes π, π_∞.
- (ii) A rosette of elliptic quadrics in Θ together with the equivalence class of the elliptic quadrics is a line of type (ii).
- (iii) A set of q parallel planes of $\mathrm{AG}(3, q) = \mathrm{PG}(3, q) \setminus \pi_\infty$, together with π_∞, is a line of type (iii)
- (iv) Lines of type (iv) are the lines of $\mathrm{PG}(3, q)$ containing ∞.
- (v) $[\infty] = \{\infty, \pi_\infty, \Omega_1, \Omega_2, \ldots, \Omega_{q-1}\}$ is the unique line of type (v).

Incidence: Incidence is containment.

Interpreting [25] in the context of tetradic sets we have the following connection between the flock of a quadratic cone and the above GQ construction. Let $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q-1}, m\}$ be the elements of a flock of a quadratic cone projected onto a plane π (as in Section 2) with common point ∞ and common tangent ℓ. Embedding π in $\mathrm{PG}(3, q)$ we choose planes π_∞, π_R distinct from π and such that $\pi_\infty \cap \pi = \ell$ and $\pi_R \cap \pi = m$. Choosing any point $R \in \pi_R \setminus (\pi \cup \pi_\infty)$ there is a unique transversal of elliptic quadrics $\Omega_1, \Omega_2, \ldots, \Omega_{q-1}$ with base points ∞, R, corresponding base planes π_∞, π_R and $\mathcal{C}_i = \Omega_i \cap \pi$ for $i = 1, 2, \ldots, q-1$. Taking images of this set under the elations of $\mathrm{PG}(3, q)$ with centre ∞ yields a tetradic set which constructs a GQ by the above. In [25] Thas identifies this as the dual of the corresponding flock GQ.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (q, q^2) satisfying Property (G) at a pair of points. In Section 3 we saw that there is a set of associated ovoids and that the intersections sizes and other properties of elements of this set have been determined. If we assume that each ovoid of this set is an elliptic quadric, then properties of elliptic quadrics force this set to be tetradic and the corresponding GQ must be that constructed above which is the dual of a flock GQ. Hence we have the following theorem which strengthens the main result of [25].

Theorem 5.4. ([2]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2) satisfying Property (G) at a pair of collinear points (X, Y). If s is odd, then $\mathcal{S}$ is the dual of a flock GQ. If s is even and all ovoids $\mathcal{O}_Z$ of $\mathcal{S}_{XY}$ for $Z \in \mathcal{P} \setminus (X^\perp \cup Y^\perp)$ are elliptic quadrics, then we have the same conclusion.*

5.2. Tetradic sets of ovoids of PG(3, q), q even.

In this section we consider the more general case of Θ being a tetradic sets of ovoids of $\mathrm{PG}(3, q)$. Since in the q odd case we have only elliptic quadric ovoids, which was discussed in the previous section, we can assume that q is even. This section is based on the work in [5].

Let X, Y, Z be three points such that $\{X, Y, Z, \infty\}$ form a quadrangle in the plane $\pi = \langle X, Y, Z\rangle$. For any $W \in \mathrm{PG}(3, q) \setminus (\pi \cup \pi_\infty)$ the set $\{X, Y, Z, W\}$ is a tetrad and is contained in a unique element of Θ and so there are q elements of Θ partitioning the points of $\mathrm{PG}(3, q) \setminus (\pi \cup \pi_\infty)$. These q ovoids intersect pairwise in the plane π, but it is not clear prima facie that they intersect pairwise in a fixed oval. Employing combinatorial arguments it is proved that this is the case, however it does not provide the group action that defines the equivalence classes of Θ as in the previous section. For this we need to consider polarities of $\mathrm{PG}(3, q)$ that interchange ∞ and π_∞.

The singular lines of a symplectic polarity form a linear complex of $\mathrm{PG}(3, q)$ which, via the klein correspondence becomes a non-singular hyperplane section of the klein quadric $Q^+(5, q)$. In particular, if the symplectic polarity has as singular lines the lines incident with ∞ and contained in π_∞, then in $Q^+(5, q)$ the corresponding hyperplane contains a line which we will denote by α. The following lemma defines an equivalence relation on the non-singular hyperplane sections of $Q^+(5, q)$ containing α.

Lemma 5. *5. Let α be a line on the quadric $\mathcal{Q} = Q^+(5, q)$, q even, and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two non-singular hyperplane sections of $\mathcal{Q}$ containing α. If $\mathcal{L}_1 \bowtie \mathcal{L}_2$ if and only if $\mathcal{L}_1 \cap \mathcal{L}_2$ contains a common three-dimensional quadratic cone section of $\mathcal{Q}$, then $\bowtie$ is an equivalence relation on the set of non-singular hyperplane sections of $\mathcal{Q}$ containing α.*

Proof. Clearly $\bowtie$ is reflexive and symmetric, so it only remains to show transitivity. Now suppose that $\mathcal{L}_1 \bowtie \mathcal{L}_2$ and $\mathcal{L}_2 \bowtie \mathcal{L}_3$ for distinct $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$. Let $\mathcal{K}_{12} = \mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{K}_{23} = \mathcal{L}_2 \cap \mathcal{L}_3$. Then $\mathcal{K}_{12}$ and $\mathcal{K}_{23}$ are distinct quadratic cone sections of $\mathcal{Q}$ both containing α. Now if $\mathcal{K}_{12}$ and $\mathcal{K}_{23}$ have the same vertex (necessarily on α) then since they are both contained in $\mathcal{L}_2$ they must conicide and $\mathcal{L}_1 \bowtie \mathcal{L}_3$. So now we suppose that $\mathcal{K}_{12}$ and $\mathcal{K}_{23}$ have distinct vertices. In this case the cones intersect in a single line, and the two three spaces generated by the cones intersect in a plane π intersecting $\mathcal{Q}$ in just α. Since $\pi \subset \langle \mathcal{L}_1 \rangle \cap \langle \mathcal{L}_3 \rangle$ it must be the case that $\mathcal{L}_1 \cap \mathcal{L}_3$ is a quadratic cone section of $\mathcal{Q}$ and hence $\mathcal{L}_1 \bowtie \mathcal{L}_3$. $\square$

Employing this equivalence relation it is possible to show that it induces an equivalence relation on Θ whereby each polarity of $\mathrm{PG}(3, q)$, interchanging ∞ and π_∞, is induced by exactly q ovoids of Θ which form a rosette. Hence we have our $q - 1$ equivalence classes of Θ each of size q^3. Note that if Θ consists entirely of elliptic quadrics, then this equivalence relation on Θ is the same as described in Section 5.1.

In fact, this equivalence relation allows the proof of the equivalent structural theorem for a general tetradic set Θ as was proved for a tetradic set of elliptic quadrics in Theorem 5.3. (In Theorem 5.3 replace “elliptic quadric” by “ovoid” and remove references to group actions to obtain the more general theorem.) The construction of the incidence structure $\mathrm{GQ}(\Theta)$ from Section 5.1 (with “elliptic quadric” replaced by “ovoid”) also carries through to give a generalized quadrangle.

Theorem 5.7. *Let Θ be a tetradic set of ovoids of $\mathrm{PG}(3, q)$, q even, with respect to (∞, π_∞). Then the incidence structure $\mathrm{GQ}(\Theta)$ is a GQ of order (q, q^2) satisfying Property (G) at the flag $(\infty, [\infty])$. Conversely, any GQ of order (q, q^2) satisfying Property (G) at a flag gives rise to tetradic set of ovoids.*

The known examples of GQs of order (q, q^2) which have Property (G) at a flag are the dual flock GQs (which have Property (G) at a line) and the Tits GQs $T_3(\Omega)$ for Ω an ovoid of PG(3, q). The tetradic set associated with a GQ $T_3(\Omega)$ can be characterised in the following ways.

Theorem 5.7. *Let Θ be a tetradic set of ovoids of* PG(3, q). *Then* GQ(Θ) *being isomorphic to the GQ $T_3(\Omega)$ with 3-regular point ∞ is equivalent to each of the following:*

1. *$\Omega \in \Theta$ and the group of collineations of* PG(3, q) *with centre ∞ acts regularly on Θ.*
2. *$\Omega \in \Theta$ and Θ has the property that if $\Omega_1, \Omega_2 \in \Theta$ are inequivalent, then $\Omega_1 \cap \Omega_2$ is either ∞ or the union of ∞ and an oval.*

If a tetradic set of ovoids Θ has the property that if, under a duality of PG(3, q) that interchanges ∞ and π_∞, the elements of Θ are mapped to the set of tangent planes to a tetradic set of ovoids, then we say that Θ is also *dual tetradic*. In the construction of GQ(Θ) interchanging the role of ∞ and π_∞ induces a duality of PG(3, q). Hence if Θ is both tetradic and dual tetradic, then GQ(Θ) satisfies Property (G) at the flags $(\infty, [\infty])$ and $(\pi_\infty, [\infty])$. Conversely, any GQ of order (q, q^2) that satisfies Property (G) at two distinct flags (X, ℓ) and (Y, ℓ) gives rise to a tetradic set of ovoids that is also dual tetradic.

Suppose that Θ is both tetradic and dual tetradic. Dualising Θ preserves the equivalence classes of Θ, in the sense that if $\Omega_1, \Omega_2 \in \Theta$ are equivalent and $*$ the duality of PG(3, q), then Ω_1^*, Ω_2^* are equivalent ovoids. It follows that if $\Omega_1, \Omega_2 \in \Theta$ are equivalent and intersect in the oval $\mathcal{O}$ containing ∞, then there is also a point P such that Ω_1 and Ω_2 share tangent planes incident with P. This is a strong configurational property possessed by ovoids in an equivalence class, from which it is possible to prove that the ovoids must, in fact, be elliptic quadrics. That is, if Θ is a set of ovoids that is both tetradic and dual tetradic, then it consists entirely of elliptic quadrics. Hence, by employing Theorem 5.4, we have the following theorem.

Theorem 5.8. ([5]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (q, q^2) and ℓ a line of $\mathcal{S}$ such that $\mathcal{S}$ satisfies Property (G) at distinct flags (X, ℓ) and (Y, ℓ). Then $\mathcal{S}$ is the dual of a flock GQ.*

As a special case of this we have the answer to a conjecture made by J. A. Thas in the proceedings of Combinatorics '98 [24].

Theorem 5.9. ([5]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (q, q^2) and assume that $\mathcal{S}$ satisfies Property (G) at some line ℓ. Then $\mathcal{S}$ is the dual of a flock GQ.*

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