COPURE AND 2-ABSORBING COPURE SUBMODULES

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Let $R$ be a commutative ring with identity and $M$ be an $R$-module. In this paper, we will introduce the concept of 2-absorbing copure submodules of $M$ as a generalization of copure submodules and obtain some related results. Also, we investigate some results concerning copure submodules.

1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

A submodule $N$ of an $R$-module $M$ is said to be pure if $IN = N \cap IM$ for every ideal $I$ of $R$ [3].

In [7], H. Ansari-Toroghy and F. Farshadifar introduced the dual notion of pure submodules (that is copure submodules) and investigated the first properties of this class of modules. A submodule $N$ of an $R$-module $M$ is said to be copure if $(N :_M I) = N + (0 :_M I)$ for every ideal $I$ of $R$ [7].

Following the concept of 2-absorbing ideals of commutative rings as in [10], the concept of 2-absorbing pure submodules of an $R$-module $M$ as a generalization of pure submodules was introduced in [11]. A submodule $N$ of an $R$-module $M$ is said to be a 2-absorbing pure submodule of $M$ if $IJN = IN \cap JN \cap IJM$ for
every ideals $I, J$ of $R$. Also, an ideal $I$ of $R$ is said to be a 2-absorbing pure ideal of $R$ if $I$ is a 2-absorbing pure submodule of $R$.

The main purpose of this paper is to introduce the concepts of 2-absorbing copure submodules of an $R$-module $M$ as a generalization of copure submodules and investigate some results concerning this notion and copure submodules.

2. Copure submodules

A proper submodule $N$ of an $R$-module $M$ is said to be completely irreducible if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$ [12].

**Remark 2.1.** Let $N$ and $K$ be two submodules of an $R$-module $M$. To prove $N \subseteq K$, it is enough to show that if $L$ is a completely irreducible submodule of $M$ such that $K \subseteq L$, then $N \subseteq L$.

An $R$-module $M$ is said to be a comultiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$ [4]. It is easy to see that $M$ is a comultiplication module if and only if $N = (0 :_M \text{Ann}_R(N))$ for each submodule $N$ of $M$.

An $R$-module $M$ satisfies the double annihilator conditions (DAC for short) if for each ideal $I$ of $R$, we have $I = \text{Ann}_R((0 :_M I))$. $M$ is said to be a strong comultiplication module if $M$ is a comultiplication $R$-module which satisfies the double annihilator conditions [7].

**Lemma 2.2.** let $M$ be a strong comultiplication $R$-module. Then $(0 :_M I \cap J) = (0 :_M I) + (0 :_M J)$ for all ideals $I$ and $J$ of $R$.

**Proof.** This follows from the fact that $I = \text{Ann}_R((0 :_M I))$ for each ideal $I$ of $R$ and [5, 3.3].

A family $\{N_i\}_{i \in I}$ of submodules of an $R$-module $M$ is said to be an inverse family of submodules of $M$ if the intersection of two of its submodules again contains a module in $\{N_i\}_{i \in I}$. Also $M$ satisfies the property $AB5^*$ if for every submodule $K$ of $M$ and every inverse family $\{N_i\}_{i \in I}$ of submodules of $M$, $K + \bigcap_{i \in I} N_i = \bigcap_{i \in I} (K + N_i)$ [14]. For example, every strong comultiplication $R$-module satisfies the property $AB5^*$ by using Lemma 2.2 and [2, 2.9].

**Theorem 2.3.** Let $M$ be an $R$-module which satisfies the property $AB5^*$. Then we have the following. If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a chain of copure submodules of $M$, then $\bigcap_{\lambda \in \Lambda} N_\lambda$ is copure.
Proof. Let $I$ be an ideal of $R$. Clearly,

$$\bigcap_{\lambda \in \Lambda} N_{\lambda} + (0 :_{M} I) \subseteq \bigcap_{\lambda \in \Lambda} N_{\lambda} :_{M} I.$$ 

To see the reverse inclusion, let $L$ be a completely irreducible submodule of $M$ such that $\bigcap_{\lambda \in \Lambda} N_{\lambda} + (0 :_{M} I) \subseteq L$. Then $\bigcap_{\lambda \in \Lambda} N_{\lambda} + (0 :_{M} I) + L = L$. Since $M$ satisfies the property $AB5^*$, we have

$$\bigcap_{\lambda \in \Lambda} (N_{\lambda} :_{M} I) = L.$$ 

Now as $L$ is a completely irreducible submodule of $M$, there exists $\alpha \in \Lambda$ such that $N_{\alpha} + (0 :_{M} I) + L = L$. It follows that $(N_{\alpha} :_{M} I) + L = L$ since $N_{\alpha}$ is a copure submodule of $M$. Thus $(N_{\alpha} :_{M} I) \subseteq L$. Hence, $(\bigcap_{\lambda \in \Lambda} N_{\lambda} :_{M} I) \subseteq L$. This implies that

$$(\bigcap_{\lambda \in \Lambda} N_{\lambda} :_{M} I) \subseteq \bigcap_{\lambda \in \Lambda} N_{\lambda} + (0 :_{M} I),$$

by Remark 2.1. \hfill \Box

**Theorem 2.4.** Let $M$ be an $R$-module which satisfies the property $AB5^*$ and $N$ be a submodule of $M$. Then there is a submodule $K$ of $M$ minimal with respect to $N \subseteq K$ and $K$ is a copure submodule of $M$.

**Proof.** Let

$$\Sigma = \{N \leq H | H \text{ is a copure submodule of } M\}.$$ 

Then $M \in \Sigma \neq \emptyset$. Let $\{N_{\lambda}\}_{\lambda \in \Lambda}$ be a totally ordered subset of $\Sigma$. Then $N \leq \bigcap_{\lambda \in \Lambda} N_{\lambda}$ and by Theorem 2.3 (a), $\bigcap_{\lambda \in \Lambda} N_{\lambda}$ is a copure submodule of $M$. Thus by using Zorn’s Lemma, one can see that $\Sigma$ has a minimal element, $K$ say as needed. \hfill \Box

**Proposition 2.5.** Let $R$ be a PID, $N$ a submodule of an $R$-module $M$, and $p_{i}$ ($i \in \mathbb{N}$) be a prime element in $R$. Then $(N :_{M} p_{1}^{i_{1}} \ldots p_{t}^{i_{t}}) = \Sigma_{i=1}^{t} (N :_{M} p_{i}^{i_{i}})$.

**Proof.** Let $p$ and $q$ be two prime elements in $R$ and $k,s \in \mathbb{N}$. Clearly, $(N :_{M} p^{s}) + (N :_{M} q^{s}) \subseteq (N :_{M} p^{k} q^{k})$. Now let $x p^{k} q^{k} \in N$. Since $R$ is a PID, $p^{k} q^{k} = R$. Thus there exist $a,b \in R$ such that $1 = a p^{k} + b q^{k}$. Hence $x = a p^{k} x + b q^{k} x$. This implies that $q^{k} (x - b q^{k} x) \in N$. Since $b q^{k} x \in (N :_{M} p^{k})$, we have $x \in (N :_{M} p^{k}) + (N :_{M} q^{k})$. Therefore, $(N :_{M} p^{k}) + (N :_{M} q^{k}) = (N :_{M} p^{k} q^{k})$. Now the result follows by induction on $t$. \hfill \Box

**Corollary 2.6.** Let $M$ be a $\mathbb{Z}$-module, $m,n$ be square-free integers such that $(m,n) = 1$. Then for each submodule $N$ of $M$ we have

$$(N :_{M} (n\mathbb{Z}))(m\mathbb{Z})) = (N :_{M} (n\mathbb{Z})) + (N :_{M} (m\mathbb{Z})).$$
Proof. This follows from Proposition 2.5. □

**Definition 2.7.** We say that a copure submodule \( N \) of an \( R \)-module \( M \) is a minimal copure submodule of a submodule \( K \) of \( M \), if \( K \subseteq N \) and there does not exist a copure submodule \( H \) of \( M \) such that \( K \subset H \subset N \).

An \( R \)-module \( M \) is called fully cocancellation module if for each non-zero ideal \( I \) of \( R \) and for each submodules \( N_1, N_2 \) of \( M \) such that \((N_1 :_M I) = (N_2 :_M I)\) implies \( N_1 = N_2 \) [9].

**Theorem 2.8.** Every Noetherian fully cocancellation \( R \)-module \( M \) has only a finite number of minimal copure submodules.

Proof. Suppose that the result is false. Let \( \Sigma \) denote the collection of proper submodules \( N \) of \( M \) such that \( M/N \) has an infinite number of minimal copure submodules. The collection \( \Sigma \) is non-empty because 0 \( \in \Sigma \) and hence has a maximal member, \( S \) say. Then \( S \) is not copure submodule. Thus there exists an ideal \( I \) of \( R \) such that \((S :_M I) \neq S + (0 :_M I)\). Let \( V \) be a minimal copure submodule of \( M \) that contains \( S \). If \((S :_M I) \cap V = S\), then \((S :_M I) \cap V + (0 :_M I) = S + (0 :_M I)\). Hence by the modular law, \((S :_M I) \cap (V + (0 :_M I)) = S + (0 :_M I)\). Now as \( V \) is a copure submodule of \( M \), \((S :_M I) \cap (V :_M I) = S + (0 :_M I)\). It follows that \( S + (0 :_M I) = (S :_M I) \), a contradiction. If \((S :_M I) \cap V = V\), then \( V \subseteq (S :_M I) \) and so

\[
(V :_M I) = V + (0 :_M I) \subseteq (S :_M I) + (0 :_M I) = (S :_M I).
\]

Thus \((V :_M I) = (S :_M I)\). Since \( M \) is a fully cocancellation \( R \)-module, \( V = S\), a contradiction. Therefore, \( S \subset (S :_M I) \cap V \subset V \). Now by the choice of \( S \), the module \((S :_M I) \cap V \) has only finitely many minimal copure submodules. Therefore, there is only a finite number of possibilities for the module \( S \) which is a contradiction. □

3. 2-absorbing copure submodules

**Definition 3.1.** We say that a submodule \( N \) of an \( R \)-module \( M \) is a 2-absorbing copure submodule of \( M \) if \((N :_M IJ) = (N :_M I) + (N :_M J) + (0 :_M IJ)\) for every ideals \( I, J \) of \( R \). This can be regarded as a dual notion of the 2-absorbing pure submodule of \( M \).

**Remark 3.2.** Let \( M \) be an \( R \)-module. Clearly every copure submodule of \( M \) is a 2-absorbing copure submodule of \( M \). But we see in the Example 3.3 that the converse is not true in general.
Example 3.3. The submodule $2\mathbb{Z}_4$ of the $\mathbb{Z}_4$-module $\mathbb{Z}_4$ is a 2-absorbing copure submodule of $\mathbb{Z}_4$ but it is not a copure submodule of $\mathbb{Z}_4$.

Example 3.4. Since $1/4 \in (\mathbb{Z}_4 : \mathbb{Z}(2\mathbb{Z}))(2\mathbb{Z}))$ but

$$1/4 \not\in (\mathbb{Z}_4 : \mathbb{Z}2\mathbb{Z}) + (\mathbb{Z}_4 : \mathbb{Z}2\mathbb{Z}) + (0 : \mathbb{Z}(2\mathbb{Z}))(2\mathbb{Z}))$$.

The submodule $\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Q}$ is not 2-absorbing copure.

Proposition 3.5. Let $M$ be an $R$-module. Then we have the following.

(a) If $N$ is a submodule of $M$ such that $(N :_M IJ) = (N :_M I) + (N :_M J)$ for every ideals $I, J$ of $R$, then $N$ is a 2-absorbing copure submodule of $M$.

(b) If $N$ is a submodule of $M$ such that for each ideal $I$ of $R$, $(N :_M I) =$ a copure submodule of $M$, then $N$ is a 2-absorbing copure submodule of $M$.

(c) If $R$ is a Noetherian ring and $N$ is a 2-absorbing copure submodule of $M$, then for each prime ideal $P$ of $R$, $N_P$ is a 2-absorbing copure submodule of $M_P$ as an $R_P$-module.

(d) If $R$ is a Noetherian ring and $N_P$ is a 2-absorbing copure submodule of an $R_P$-module $M_P$ for each maximal ideal $P$ of $R$, then $N$ is a 2-absorbing copure submodule of $M$.

Proof. (a) Let $I$ and $J$ be ideals of $R$. Then $(N :_M IJ) = (N :_M I) + (N :_M J)$ by assumption. Thus $(0 :_M IJ) \subseteq (N :_M IJ) = (N :_M I) + (N :_M J)$. This implies that

$$(0 :_M IJ) + (N :_M I) + (N :_M J) = (N :_M I) + (N :_M J).$$

Therefore, $(0 :_M IJ) + (N :_M I) + (N :_M J) = (N :_M IJ)$ as required.

(b) Let $I$ and $J$ be two ideals of $R$. Then by assumption,

$$(N :_M IJ) = ((N :_M I) :_M J)$$

$$= (N :_M I) + (0 :_M J)$$

$$\subseteq (N :_M I) + (N :_M J) + (0 :_M IJ).$$

It follows that $N$ is a 2-absorbing copure submodule of $M$ since the reverse inclusion is clear.

(c) This follows from the fact that by [13, 9.13], if $I$ is a finitely generated ideal of $R$, then $((N :_M I))_P = (N_P :_{M_P} I_P)$.

(d) Suppose that $I$ and $J$ are two ideals of $R$. Since $R$ is Noetherian, $I$ and $J$ are finitely generated. Hence by [13, 9.13], for each maximal ideal $P$ of $R$, $(N :_M IJ)_P = (N_P :_{M_P} I_P J_P)$. Thus by assumption,

$$(N :_M IJ)_P = (N :_M I)_P + (N :_M J)_P + (0 :_M IJ)_P$$

$$= ((N :_M I) + (N :_M J) + (0 :_M IJ))_P.$$
Therefore
\[(N :_{M} IJ) = (N :_{M} I) + (N :_{M} J) + (0 :_{M} IJ),\]
as desired.

Recall that an \(R\)-module \(M\) is said to be fully copure if every submodule of \(M\) is copure [8].

**Definition 3.6.** We say that an \(R\)-module \(M\) is fully 2-absorbing copure if every submodule of \(M\) is 2-absorbing copure.

**Remark 3.7.** Clearly, every fully copure \(R\)-module is a fully 2-absorbing copure \(R\)-module. But the converse is not true in general. For example, the \(\mathbb{Z}_4\)-module \(\mathbb{Z}_4\) is a fully 2-absorbing copure module but it is not a fully copure \(\mathbb{Z}_4\)-module.

Let \(N\) and \(K\) be two submodules of \(M\). The coproduct of \(N\) and \(K\) is defined by \((0 :_{M} \text{Ann}_R(N)\text{Ann}_R(K))\) and denoted by \(C(NK)\) [6].

**Theorem 3.8.** Let \(M\) be a comultiplication \(R\)-module. Then the following statements are equivalent.

(a) For each submodules \(N, K, H\) of \(M\), we have
\[C(NHK) = C(NK) + C(NH) + C(KH).\]

(b) \(M\) is a fully 2-absorbing copure \(R\)-module.

**Proof.** (a) ⇒ (b) Let \(N\) be a submodule of \(M\) and \(I, J\) be two ideals of \(R\). Then as \(M\) is a comultiplication \(R\)-module,
\[C(N(0 :_{M} I)) = (0 :_{M} \text{Ann}_R(N)\text{Ann}_R((0 :_{M} I)))\]
\[= ((0 :_{M} \text{Ann}_R((0 :_{M} I))) :_{M} \text{Ann}_R(N))\]
\[= ((0 :_{M} I) :_{M} \text{Ann}_R(N)) = (N :_{M} I).\]

Similarly, \(C(N(0 :_{M} J)) = (N :_{M} J)\). Now by part (a) and the fact that \(M\) is a comultiplication \(R\)-module,
\[(N :_{M} I) + (N :_{M} J) + (0 :_{M} IJ)\]
\[= C(N(0 :_{M} I)) + C(N(0 :_{M} J)) + C((0 :_{M} I)(0 :_{M} J))\]
\[= C(N(0 :_{M} I)(0 :_{M} J)))\]
\[= (N :_{R} IJ).\]
(b) $\Rightarrow$ (a). As $M$ is a comultiplication $R$-module, we have $C(NH) = (H :_M \text{Ann}_R(N))$ and $C(KH) = (H :_M \text{Ann}_R(K))$. Now since by part (b), $H$ is a 2-absorbing copure submodule of $M$,

$$C(NK) + C(NH) + C(KH)$$

$$= (0 :_M \text{Ann}_R(N)\text{Ann}_R(K)) + (H :_M \text{Ann}_R(N)) + (H :_M \text{Ann}_R(K))$$

$$= (H :_M \text{Ann}_R(N)\text{Ann}_R(K)).$$

But since $M$ is a comultiplication $R$-module,

$$C(NHK) = C(NK) + C(NH) + C(KH).$$

Therefore, $C(NHK) = C(NK) + C(NH) + C(KH)$. $\square$

Let $R$ be a be a principal ideal domain and $M$ be an $R$-module. By [7, 2.12], every submodule of $M$ is pure if and only if it is copure. But the following examples shows that it is not true for 2-absorbing pure and 2-absorbing copure submodules.

**Example 3.9.** Consider the submodule $G_1 := \langle 1/p + \mathbb{Z} \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_p^{\infty}$. Let $m, n$ be two positive integers. If $p$ not divided $m, n$, then

$$G_1 = mnG_1 = mG_1 \cap nG_1 \cap mn\mathbb{Z}_p^{\infty} = G_1.$$

If $p$ divided $m$ or $n$, then

$$0 = mnG_1 = mG_1 \cap nG_1 \cap mn\mathbb{Z}_p^{\infty} = 0.$$

Moreover,

$$G_3 = (G_1 :_{\mathbb{Z}_p^{\infty}} p^2) \neq (G_1 :_{\mathbb{Z}_p^{\infty}} p) + (G_1 :_{\mathbb{Z}_p^{\infty}} p) + (0 :_{\mathbb{Z}_p^{\infty}} p^2) = G_2.$$

Hence, the submodule $G_1 := \langle 1/p + \mathbb{Z} \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_p^{\infty}$ is a 2-absorbing pure submodule but it is not 2-absorbing copure.

**Example 3.10.** Let $m, n$ be two positive integers. If $m, n$ are odd, then we have

$$2\mathbb{Z} = (2\mathbb{Z} :_{\mathbb{Z}} mn) = (2\mathbb{Z} :_{\mathbb{Z}} n) + (2\mathbb{Z} :_{\mathbb{Z}} m) + (0 :_{\mathbb{Z}} mn) = 2\mathbb{Z}.$$

If $m, n$ are even, then we have

$$\mathbb{Z} = (2\mathbb{Z} :_{\mathbb{Z}} mn) = (2\mathbb{Z} :_{\mathbb{Z}} n) + (2\mathbb{Z} :_{\mathbb{Z}} m) + (0 :_{\mathbb{Z}} mn) = \mathbb{Z}.$$

Moreover, $8\mathbb{Z} = (2)(2)(2\mathbb{Z}) \neq (2)(2\mathbb{Z}) \cap (2)(2\mathbb{Z}) \cap (2)(2)(\mathbb{Z}) = 4\mathbb{Z}$. Thus the submodule $2\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is a 2-absorbing copure submodule but it is not 2-absorbing pure.
Theorem 3.11. Let $M$ be an $R$-module which satisfies the property $AB5^*$.

(a) If $\{N_{\lambda}\}_{\lambda \in \Lambda}$ is a chain of 2-absorbing copure submodules of $M$, then
\[
\bigcap_{\lambda \in \Lambda}N_{\lambda} \text{ is a 2-absorbing copure submodule of } M.
\]

(b) If $\{N_{\lambda}\}_{\lambda \in \Lambda}$ is a chain of submodules of $M$ and $K$ is a 2-absorbing copure submodule of $N_{\lambda}$ for each $\lambda \in \Lambda$, then $K$ is a 2-absorbing copure submodule of $\bigcap_{\lambda \in \Lambda}N_{\lambda}$.

Proof. (a) Let $I$ and $J$ be two ideals of $R$. Clearly,
\[
(\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} I) + (\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} J) + (0 :_{M} IJ) \subseteq (\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} IJ).
\]

Let $L$ be a completely irreducible submodule of $M$ such that
\[
(\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} I) + (\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} J) + (0 :_{M} IJ) \subseteq L.
\]

Then
\[
\bigcap_{\lambda \in \Lambda}(N_{\lambda} :_{M} I) + (\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} J) + (0 :_{M} IJ) + L = L.
\]

Since $M$ satisfies the property $AB5^*$, we have
\[
\bigcap_{\lambda \in \Lambda}((N_{\lambda} :_{M} I) + (\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} J) + (0 :_{M} IJ) + L) = L.
\]

Now as $L$ is a completely irreducible submodule of $M$, there exists $\alpha \in \Lambda$ such that $(N_{\alpha} :_{M} I) + (\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} J) + (0 :_{M} IJ) + L = L$. Since $M$ satisfies the property $AB5^*$, $\bigcap_{\lambda \in \Lambda}((N_{\alpha} :_{M} I) + (N_{\lambda} :_{M} J) + (0 :_{M} IJ) + L) = L$. Now again as $L$ is a completely irreducible submodule of $M$, there exists $\beta \in \Lambda$ such that $(N_{\alpha} :_{M} I) + (N_{\beta} :_{M} J) + (0 :_{M} IJ) + L = L$. We can assume that $N_{\alpha} \subseteq N_{\beta}$. Therefore, $(N_{\alpha} :_{M} I) + (N_{\alpha} :_{M} J) + (0 :_{M} IJ) \subseteq L$. It follows that $(N_{\alpha} :_{M} IJ) \subseteq L$ since $N_{\alpha}$ is a 2-absorbing copure submodule of $M$. Hence, $(\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} IJ) \subseteq L$. This implies that
\[
(\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} IJ) \subseteq (\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} I) + (\bigcap_{\lambda \in \Lambda}N_{\lambda} :_{M} J) + (0 :_{M} IJ),
\]

by Remark 2.1.

(b) Let $I$ and $J$ be two ideals of $R$. Clearly,
\[
(K :_{\bigcap_{\lambda \in \Lambda}N_{\lambda}} I) + (K :_{\bigcap_{\lambda \in \Lambda}N_{\lambda}} J) + (0 :_{\bigcap_{\lambda \in \Lambda}N_{\lambda}} IJ) \subseteq (K :_{\bigcap_{\lambda \in \Lambda}N_{\lambda}} IJ).
\]

To see the reverse inclusion, let $L$ be a completely irreducible submodule of $M$ such that
\[
(K :_{\bigcap_{\lambda \in \Lambda}N_{\lambda}} I) + (K :_{\bigcap_{\lambda \in \Lambda}N_{\lambda}} J) + (0 :_{\bigcap_{\lambda \in \Lambda}N_{\lambda}} IJ) \subseteq L.
\]
Then
\[ \cap_{\lambda \in \Lambda} (K : N_\lambda I) + (K : \cap_{\lambda \in \Lambda} N_\lambda J) + (0 : \cap_{\lambda \in \Lambda} N_\lambda IJ) + L = L. \]

Since \( M \) satisfies the property \( AB5^* \), we have
\[ \cap_{\lambda \in \Lambda} ((K : N_\lambda I) + (K : \cap_{\lambda \in \Lambda} N_\lambda J) + (0 : \cap_{\lambda \in \Lambda} N_\lambda IJ) + L) = L. \]

Now as \( L \) is a completely irreducible submodule of \( M \), there exists \( \alpha \in \Lambda \) such that
\[ (K : N_\alpha I) + (K : \cap_{\lambda \in \Lambda} N_\lambda J) + (0 : \cap_{\lambda \in \Lambda} N_\lambda IJ) + L = L. \]

By similar argument, since \( M \) satisfies the property \( AB5^* \) and \( L \) is a completely irreducible submodule of \( M \), there exist \( \beta \in \Lambda \) and \( \gamma \in \Lambda \) such that,
\[ (K : N_\alpha I) + (K : N_\beta J) + (0 : N_\gamma IJ) + L = L. \]

Since \( \{N_\lambda\}_{\lambda \in \Lambda} \) is a chain, we can assume that \( N_\alpha \subseteq N_\beta \subseteq N_\gamma \). Therefore, \( (K : N_\alpha I) + (K : N_\alpha J) + (0 : N_\alpha IJ) \subseteq L \). It follows that \( (K : N_\alpha IJ) \subseteq L \) since \( K \) is a 2-absorbing copure submodule of \( N_\alpha \). Therefore, \( (K : \cap_{\lambda \in \Lambda} N_\lambda IJ) \subseteq L \). This implies that
\[ (K : \cap_{\lambda \in \Lambda} N_\lambda IJ) \subseteq (K : \cap_{\lambda \in \Lambda} N_\lambda I) + (K : \cap_{\lambda \in \Lambda} N_\lambda J) + (0 : \cap_{\lambda \in \Lambda} N_\lambda IJ), \]
by Remark 2.1.

**Theorem 3.12.** Let \( M \) be an \( R \)-module which satisfies the property \( AB5^* \) and \( N \) be a submodule of \( M \). Then there is a submodule \( K \) of \( M \) minimal with respect to \( N \subseteq K \) and \( K \) is a 2-absorbing copure submodule of \( M \).

**Proof.** Let
\[ \Sigma = \{N \leq H \mid H \text{ is a } 2\text{-absorbing copure submodule of } M\}. \]

Then \( M \in \Sigma \neq 0 \). Let \( \{N_\lambda\}_{\lambda \in \Lambda} \) be a totally ordered subset of \( \Sigma \). Then \( N \leq \cap_{\lambda \in \Lambda} N_\lambda \) and by Theorem 3.11 (a), \( \cap_{\lambda \in \Lambda} N_\lambda \) is a 2-absorbing copure submodule of \( M \). Therefore by using Zorn’s Lemma, one can see that \( \Sigma \) has a minimal element, \( K \) say as desired.

**Theorem 3.13.** Let \( M \) be a strong comultiplication \( R \)-module and \( N \) be a submodule of \( M \). Then \( N \) is a 2-absorbing copure submodule of \( M \) if and only if \( \text{Ann}_R(N) \) is a 2-absorbing pure ideal of \( R \).
Proof. Since $M$ is a comultiplication $R$-module, $N = (0 :_M \text{Ann}_R(N))$. Let $N$ be a 2-absorbing copure submodule of $M$ and let $I$ and $J$ be any two ideals of $R$. Then

$$
(N :_M IJ) = (N :_M I) + (N :_M J) + (0 :_M IJ) \Rightarrow ((0 :_M \text{Ann}_R(N)) :_M IJ)
= ((0 :_M \text{Ann}_R(N)) :_M I) + ((0 :_M \text{Ann}_R(N)) :_M J) + (0 :_M IJ).
$$

It follows that

$$(0 :_M \text{Ann}_R(N)IJ) = (0 :_M \text{Ann}_R(N)I) + (0 :_M \text{Ann}_R(N)J) + (0 :_M IJ).$$

Thus by Lemma 2.2,

$$(0 :_M \text{Ann}_R(N)IJ) = (0 :_M \text{Ann}_R(N)I \cap \text{Ann}_R(N)J \cap IJ).$$

This implies that $\text{Ann}_R(N)IJ = \text{Ann}_R(N)I \cap \text{Ann}_R(N)J \cap IJ$ since $M$ is a strong comultiplication $R$-module. Hence $\text{Ann}_R(N)$ is a 2-absorbing pure ideal of $R$. Conversely, let $\text{Ann}_R(N)$ be a 2-absorbing pure ideal of $R$ and let $I$ and $J$ be any two ideals of $R$. Then

$$\text{Ann}_R(N)IJ = \text{Ann}_R(N)I \cap \text{Ann}_R(N)J \cap IJ.$$ 

Hence by using Lemma 2.2,

$$(0 :_M \text{Ann}_R(N)IJ) = (0 :_M \text{Ann}_R(N)I) + (0 :_M \text{Ann}_R(N)J) + (0 :_M IJ).$$

Therefore, as $M$ is a comultiplication $R$-module,

$$(N :_M IJ) = (N :_M I) + (N :_M J) + (0 :_M IJ),$$ 

as desired. \qed

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REFERENCES

COPURE AND 2-ABSORBING COPURE SUBMODULES


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