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COPURE AND 2-ABSORBING COPURE SUBMODULES

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Let R be a commutative ring with identity and M be an R-module. In this paper, we will introduce the concept of 2-absorbing copure submodules of M as a generalization of copure submodules and obtain some related results. Also, we investigate some results concerning copure submodules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

A submodule N of an R-module M is said to be *pure* if $IN = N \cap IM$ for every ideal I of R [3].

In [7], H. Ansari-Toroghy and F. Farshadifar introduced the dual notion of pure submodules (that is copure submodules) and investigated the first properties of this class of modules. A submodule N of an R-module M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [7].

Following the concept of 2-absorbing ideals of commutative rings as in [10], the concept of 2-absorbing pure submodules of an R-module M as a generalization of pure submodules was introduced in [11]. A submodule N of an R-module M is said to be a *2-absorbing pure submodule of M* if $IJN = IN \cap JN \cap IJM$ for

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every ideals I,J of R. Also, an ideal I of R is said to be a *2-absorbing pure ideal of R* if I is a 2-absorbing pure submodule of R.

The main purpose of this paper is to introduce the concepts of 2-absorbing copure submodules of an R-module M as a generalization of copure submodules and investigate some results concerning this notion and copure submodules.

2. Copure submodules

A proper submodule N of an R-module M is said to be *completely irreducible* if $N=\bigcap_{i\in I}N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N=N_i$ for some $i\in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [12].

Remark 2.1. Let N and K be two submodules of an R-module M. To prove $N\subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K\subseteq L$, then $N\subseteq L$.

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N=(0:_M I)$ [4]. It is easy to see that M is a comultiplication module if and only if $N=(0:_M Ann_R(N))$ for each submodule N of M.

An R-module M satisfies *the double annihilator conditions* (*DAC* for short) if for each ideal I of R, we have $I=Ann_R((0:_M I))$. M is said to be a *strong comultiplication module* if M is a comultiplication R-module which satisfies the double annihilator conditions [7].

Lemma 2.2. *let M be a strong comultiplication R-module. Then $(0:_M I\cap J)=(0:_M I)+(0:_M J)$ for all ideals I and J of R.*

Proof. This follows from the fact that $I=Ann_R((0:_M I))$ far each ideal I of R and [5, 3.3]. □

A family $\{N_i\}_{i\in I}$ of submodules of an R-module M is said to be an *inverse family of submodules of M* if the intersection of two of its submodules again contains a module in $\{N_i\}_{i\in I}$. Also *M satisfies the property $AB5^*$* if for every submodule K of M and every inverse family $\{N_i\}_{i\in I}$ of submodules of M, $K+\cap_{i\in I}N_i=\cap_{i\in I}(K+N_i)$ [14]. For example, every strong comultiplication R-module satisfies the property $AB5^*$ by using Lemma 2.2 and [2, 2.9].

Theorem 2.3. *Let M be an R-module which satisfies the property $AB5^*$. Then we have the following. If $\{N_\lambda\}_{\lambda\in\Lambda}$ is a chain of copure submodules of M, then $\cap_{\lambda\in\Lambda}N_\lambda$ is copure.*

Proof. Let I be an ideal of R. Clearly,

$$\cap_{\lambda\in\Lambda}N_\lambda+(0:_M I)\subseteq(\cap_{\lambda\in\Lambda}N_\lambda :_M I).$$

To see the reverse inclusion, let L be a completely irreducible submodule of M such that $\cap_{\lambda\in\Lambda}N_\lambda+(0:_M I)\subseteq L$. Then $\cap_{\lambda\in\Lambda}N_\lambda+(0:_M I)+L=L$. Since M satisfies the property $AB5^*$, we have

$$\cap_{\lambda\in\Lambda}(N_\lambda+(0:_M I)+L)=L.$$

Now as L is a completely irreducible submodule of M, there exists $\alpha\in\Lambda$ such that $N_\alpha+(0:_M I)+L=L$. It follows that $(N_\alpha :_M I)+L=L$ since N_α is a copure submodule of M. Thus $(N_\alpha :_M I)\subseteq L$. Hence, $(\cap_{\lambda\in\Lambda}N_\lambda :_M I)\subseteq L$. This implies that

$$(\cap_{\lambda\in\Lambda}N_\lambda :_M I)\subseteq\cap_{\lambda\in\Lambda}N_\lambda+(0:_M I),$$

by Remark 2.1. □

Theorem 2.4. *Let M be an R-module which satisfies the property $AB5^*$ and N be a submodule of M. Then there is a submodule K of M minimal with respect to $N\subseteq K$ and K is a copure submodule of M.*

Proof. Let

$$\Sigma=\{N\leq H|H\ is\ a\ copure\ submodule\ of\ M\}.$$

Then $M\in\Sigma\neq\emptyset$. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a totally ordered subset of Σ. Then $N\leq\cap_{\lambda\in\Lambda}N_\lambda$ and by Theorem 2.3 (a), $\cap_{\lambda\in\Lambda}N_\lambda$ is a copure submodule of M. Thus by using Zorn's Lemma, one can see that Σ has a minimal element, K say as needed. □

Proposition 2.5. *Let R be a PID, N a submodule of an R-module M, and p_i ($i\in\mathbb{N}$) be a prime element in R. Then $(N:_M p_1^{s_1}...p_t^{s_t})=\sum_{i=1}^{t}(N:_M p_i^{s_i})$.*

Proof. Let p and q be two prime elements in R and $k,s\in\mathbb{N}$. Clearly, $(N:_M p^t)+(N:_M q^s)\subseteq(N:_M p^kq^s)$. Now let $xp^kq^s\in N$. Since R is a *PID*, $p^kR+q^sR=R$. Thus there exist $a,b\in R$ such that $1=ap^k+bq^s$. Hence $x=ap^kx+bq^sx$. This implies that $q^s(x-bq^sx)\in N$. Since $bq^sx\in(N:_M p^k)$, we have $x\in(N:_M p^t)+(N:_M q^s)$. Therefore, $(N:_M p^t)+(N:_M q^s)=(N:_M p^kq^s)$. Now the result follows by induction on t. □

Corollary 2.6. *Let M be a $\mathbb{Z}$-module, m,n be square-free integers such that $(m,n)=1$. Then for each submodule N of M we have*

$$(N:_M(n\mathbb{Z})(m\mathbb{Z}))=(N:_M(n\mathbb{Z}))+(N:_M(m\mathbb{Z})).$$

Proof. This follows from Proposition 2.5. □

Definition 2.7. We say that a copure submodule N of an R-module M is a *minimal copure submodule* of a submodule K of M, if $K \subseteq N$ and there does not exist a copure submodule H of M such that $K \subset H \subset N$.

An R-module M is called *fully cocancellation module* if for each non-zero ideal I of R and for each submodules N_1, N_2 of M such that $(N_1 :_M I) = (N_2 :_M I)$ implies $N_1 = N_2$ [9].

Theorem 2.8. *Every Noetherian fully cocancellation R-module M has only a finite number of minimal copure submodules.*

Proof. Suppose that the result is false. Let Σ denote the collection of proper submodules N of M such that M/N has an infinite number of minimal copure submodules. The collection Σ is non-empty because $0 \in \Sigma$ and hence has a maximal member, S say. Then S is not copure submodule. Thus there exists an ideal I of R such that $(S :_M I) \neq S + (0 :_M I)$. Let V be a minimal copure submodule of M that contains S. If $(S :_M I) \cap V = S$, then $((S :_M I) \cap V) + (0 :_M I) = S + (0 :_M I)$. Hence by the modular law, $(S :_M I) \cap (V + (0 :_M I)) = S + (0 :_M I)$. Now as V is a copure submodule of M, $(S :_M I) \cap (V :_M I) = S + (0 :_M I)$. It follows that $S + (0 :_M I) = (S :_M I)$, a contradiction. If $(S :_M I) \cap V = V$, then $V \subseteq (S :_M I)$ and so

$$(V :_M I) = V + (0 :_M I) \subseteq (S :_M I) + (0 :_M I) = (S :_M I).$$

Thus $(V :_M I) = (S :_M I)$. Since M is a fully cocancellation R-module, $V = S$, a contradiction. Therefore, $S \subset (S :_M I) \cap V \subset V$. Now by the choice of S, the module $(S :_M I) \cap V$ has only finitely many minimal copure submodules. Therefore, there is only a finite number of possibilities for the module S which is a contradiction. □

3. 2-absorbing copure submodules

Definition 3.1. We say that a submodule N of an R-module M is a *2-absorbing copure submodule of* M if $(N :_M IJ) = (N :_M I) + (N :_M J) + (0 :_M IJ)$ for every ideals I, J of R. This can be regarded as a dual notion of the 2-absorbing pure submodule of M.

Remark 3.2. Let M be an R-module. Clearly every copure submodule of M is a 2-absorbing copure submodule of M. But we see in the Example 3.3 that the converse is not true in general.

Example 3.3. The submodule $\bar{2}\mathbb{Z}_4$ of the $\mathbb{Z}_4$-module $\mathbb{Z}_4$ is a 2-absorbing copure submodule of $\mathbb{Z}_4$ but it is not a copure submodule of $\mathbb{Z}_4$.

Example 3.4. Since $1/4 \in (\mathbb{Z} :_{\mathbb{Q}} (2\mathbb{Z})(2\mathbb{Z}))$ but

$$1/4 \notin (\mathbb{Z} :_{\mathbb{Q}} 2\mathbb{Z}) + (\mathbb{Z} :_{\mathbb{Q}} 2\mathbb{Z}) + (0 :_{\mathbb{Q}} (2\mathbb{Z})(2\mathbb{Z})).$$

The submodule $\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Q}$ is not 2-absorbing copure.

Proposition 3.5. *Let M be an R-module. Then we have the following.*

(a) If N is a submodule of M such that $(N :_M IJ) = (N :_M I) + (N :_M J)$ for every ideals I, J of R, then N is a 2-absorbing copure submodule of M.

(a) If N is a submodule of M such that for each ideal I of R, $(N :_M I)$ is a copure submodule of M, then N is a 2-absorbing copure submodule of M.

(c) If R is a Noetherian ring and N is a 2-absorbing copure submodule of M, then for each prime ideal P of R, N_P is a 2-absorbing copure submodule of M_P as an R_P-module.

(d) If R is a Noetherian ring and N_P is a 2-absorbing copure submodule of an R_P-module M_P for each maximal ideal P of R, then N is a 2-absorbing copure submodule of M.

Proof. (a) Let I and J be ideals of R. Then $(N :_M IJ) = (N :_M I) + (N :_M J)$ by assumption. Thus $(0 :_M IJ) \subseteq (N :_M IJ) = (N :_M I) + (N :_M J)$. This implies that

$$(0 :_M IJ) + (N :_M I) + (N :_M J) = (N :_M I) + (N :_M J).$$

Therefore, $(0 :_M IJ) + (N :_M I) + (N :_M J) = (N :_M IJ)$ as required.

(b) Let I and J be two ideals of R. Then by assumption,

$$\begin{aligned}(N :_M IJ) &= ((N :_M I) :_M J) \\ &= (N :_M I) + (0 :_M J) \\ &\subseteq (N :_M I) + (N :_M J) + (0 :_M IJ).\end{aligned}$$

It follows that N is a 2-absorbing copure submodule of M since the reverse inclusion is clear.

(c) This follows from the fact that by [13, 9.13], if I is a finitely generated ideal of R, then $((N :_M I))_P = (N_P :_{M_P} I_P)$.

(d) Suppose that I and J are two ideals of R. Since R is Noetherian, I and J are finitely generated. Hence by [13, 9.13], for each maximal ideal P of R, $(N :_M IJ)_P = (N_P :_{M_P} I_P J_P)$. Thus by assumption,

$$\begin{aligned}(N :_M IJ)_P &= (N :_M I)_P + (N :_M J)_P + (0 :_M IJ)_P \\ &= ((N :_M I) + (N :_M J) + (0 :_M IJ))_P.\end{aligned}$$

Therefore

$$(N :_M IJ) = (N :_M I) + (N :_M J) + (0 :_M IJ),$$

as desired. □

Recall that an R-module M is said to be *fully copure* if every submodule of M is copure [8].

Definition 3.6. We say that an R-module M is *fully 2-absorbing copure* if every submodule of M is 2-absorbing copure.

Remark 3.7. Clearly, every fully copure R-module is a fully 2-absorbing copure R-module. But the converse is not true in general. For example, the $\mathbb{Z}_4$-module $\mathbb{Z}_4$ is a fully 2-absorbing copure module but it is not a fully copure $\mathbb{Z}_4$-module.

Let N and K be two submodules of M. The *coproduct* of N and K is defined by $(0 :_M Ann_R(N)Ann_R(K))$ and denoted by $C(NK)$ [6].

Theorem 3.8. *Let M be a comultiplication R-module. Then the following statements are equivalent.*

(a) For each submodules N, K, H of M, we have

$$C(NHK) = C(NK) + C(NH) + C(KH).$$

(b) M is a fully 2-absorbing copure R-module.

Proof. $(a) \Rightarrow (b)$ Let N be a submodule of M and I, J be two ideals of R. Then as M is a comultiplication R-module,

$$\begin{aligned} C(N(0 :_M I)) &= (0 :_M Ann_R(N)Ann_R((0 :_M I))) \\ &= ((0 :_M Ann_R((0 :_M I))) :_M Ann_R(N)) \\ &= ((0 :_M I) : Ann_R(N)) = (N :_M I). \end{aligned}$$

Similarly, $C(N(0 :_M J)) = (N :_M J)$. Now by part (a) and the fact that M is a comultiplication R-module,

$$\begin{aligned} &(N :_M I) + (N :_M J) + (0 :_M IJ) \\ &\quad = C(N(0 :_M I)) + C(N(0 :_M J)) + C((0 :_M I)(0 :_M J)) \\ &\quad = C(N(0 :_M I)(0 :_M J))) \\ &\quad = (N :_R IJ). \end{aligned}$$

$(b) \Rightarrow (a)$. As M is a comultiplication R-module, we have $C(NH) = (H :_M Ann_R(N))$ and $C(KH) = (H :_M Ann_R(K))$. Now since by part (b), H is a 2-absorbing copure submodule of M,

$$\begin{aligned} C(NK)+&C(NH)+C(KH) \\ &= (0 :_M Ann_R(N)Ann_R(K)) + (H :_M Ann_R(N)) + (H :_M Ann_R(K)) \\ &= (H :_M Ann_R(N)Ann_R(K)). \end{aligned}$$

But since M is a comultiplication R-module,

$$(H :_M Ann_R(N)Ann_R(K)) = (0 :_M Ann_R(N)Ann_R(K)Ann_R(H)).$$

Therefore, $C(NHK) = C(NK) + C(NH) + C(KH)$. □

Let R be a be a principal ideal domain and M be an R-module. By [7, 2.12], every submodule of M is pure if and only if it is copure. But the following examples shows that it is not true for 2-absorbing pure and 2-absorbing copure submodules.

Example 3.9. Consider the submodule $G_1 := \langle 1/p + \mathbb{Z} \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$. Let m, n be two positive integers. If p not divided m, n, then

$$G_1 = mnG_1 = mG_1 \cap nG_1 \cap mn\mathbb{Z}_{p^\infty} = G_1.$$

If p divided m or n, then

$$0 = mnG_1 = mG_1 \cap nG_1 \cap mn\mathbb{Z}_{p^\infty} = 0.$$

Moreover,

$$G_3 = (G_1 :_{\mathbb{Z}_{p^\infty}} p^2) \neq (G_1 :_{\mathbb{Z}_{p^\infty}} p) + (G_1 :_{\mathbb{Z}_{p^\infty}} p) + (0 :_{\mathbb{Z}_{p^\infty}} p^2) = G_2.$$

Hence, the submodule $G_1 := \langle 1/p + \mathbb{Z} \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a 2-absorbing pure submodule but it is not 2-absorbing copure.

Example 3.10. Let m, n be two positive integers. If m, n are odd, then we have

$$2\mathbb{Z} = (2\mathbb{Z} :_{\mathbb{Z}} mn) = (2\mathbb{Z} :_{\mathbb{Z}} n) + (2\mathbb{Z} :_{\mathbb{Z}} m) + (0 :_{\mathbb{Z}} mn) = 2\mathbb{Z}.$$

If m, n are even, then we have

$$\mathbb{Z} = (2\mathbb{Z} :_{\mathbb{Z}} mn) = (2\mathbb{Z} :_{\mathbb{Z}} n) + (2\mathbb{Z} :_{\mathbb{Z}} m) + (0 :_{\mathbb{Z}} mn) = \mathbb{Z}.$$

Moreover, $8\mathbb{Z} = (2)(2)(2\mathbb{Z}) \neq (2)(2\mathbb{Z}) \cap (2)(2\mathbb{Z}) \cap (2)(2)(\mathbb{Z}) = 4\mathbb{Z}$. Thus the submodule $2\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is a 2-absorbing copure submodule but it is not 2-absorbing pure.

Theorem 3.11. *Let M be an R-module which satisfies the property $AB5^*$. Then we have the following.*

(a) If $\{N_\lambda\}_{\lambda\in\Lambda}$ is a chain of 2-absorbing copure submodules of M, then $\cap_{\lambda\in\Lambda}N_\lambda$ is a 2-absorbing copure submodule of M.

(b) If $\{N_\lambda\}_{\lambda\in\Lambda}$ is a chain of submodules of M and K is a 2-absorbing copure submodule of N_λ for each $\lambda\in\Lambda$, then K is a 2-absorbing copure submodule of $\cap_{\lambda\in\Lambda}N_\lambda$.

Proof. (a) Let I and J be two ideals of R. Clearly,

$$(\cap_{\lambda\in\Lambda}N_\lambda :_M I)+(\cap_{\lambda\in\Lambda}N_\lambda :_M J)+(0:_M IJ)\subseteq(\cap_{\lambda\in\Lambda}N_\lambda :_M IJ).$$

Let L be a completely irreducible submodule of M such that

$$(\cap_{\lambda\in\Lambda}N_\lambda :_M I)+(\cap_{\lambda\in\Lambda}N_\lambda :_M J)+(0:_M IJ)\subseteq L.$$

Then

$$\cap_{\lambda\in\Lambda}(N_\lambda :_M I)+(\cap_{\lambda\in\Lambda}N_\lambda :_M J)+(0:_M IJ)+L=L.$$

Since M satisfies the property $AB5^*$, we have

$$\cap_{\lambda\in\Lambda}((N_\lambda :_M I)+(\cap_{\lambda\in\Lambda}N_\lambda :_M J)+(0:_M IJ)+L)=L.$$

Now as L is a completely irreducible submodule of M, there exists $\alpha\in\Lambda$ such that $(N_\alpha :_M I)+(\cap_{\lambda\in\Lambda}N_\lambda :_M J)+(0:_M IJ)+L=L$. Since M satisfies the property $AB5^*$, $\cap_{\lambda\in\Lambda}((N_\alpha :_M I)+(N_\lambda :_M J)+(0:_M IJ)+L)=L$. Now again as L is a completely irreducible submodule of M, there exists $\beta\in\Lambda$ such that $(N_\alpha :_M I)+(N_\beta :_M J)+(0:_M IJ)+L=L$. We can assume that $N_\alpha\subseteq N_\beta$. Therefore, $(N_\alpha :_M I)+(N_\alpha :_M J)+(0:_M IJ)\subseteq L$. It follows that $(N_\alpha :_M IJ)\subseteq L$ since N_α is a 2-absorbing copure submodule of M. Hence, $(\cap_{\lambda\in\Lambda}N_\lambda :_M IJ)\subseteq L$. This implies that

$$(\cap_{\lambda\in\Lambda}N_\lambda :_M IJ)\subseteq(\cap_{\lambda\in\Lambda}N_\lambda :_M I)+(\cap_{\lambda\in\Lambda}N_\lambda :_M J)+(0:_M IJ),$$

by Remark 2.1.

(b) Let I and J be two ideals of R. Clearly,

$$(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I)+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} J)+(0:_{\cap_{\lambda\in\Lambda}N_\lambda} IJ)\subseteq(K:_{\cap_{\lambda\in\Lambda}N_\lambda} IJ).$$

To see the reverse inclusion, let L be a completely irreducible submodule of M such that

$$(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I)+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} J)+(0:_{\cap_{\lambda\in\Lambda}N_\lambda} IJ)\subseteq L.$$

Then

$$\cap_{\lambda\in\Lambda}(K :_{N_\lambda} I)+(K :_{\cap_{\lambda\in\Lambda}N_\lambda} J)+(0 :_{\cap_{\lambda\in\Lambda}N_\lambda} IJ)+L=L.$$

Since M satisfies the property $AB5^*$, we have

$$\cap_{\lambda\in\Lambda}((K :_{N_\lambda} I)+(K :_{\cap_{\lambda\in\Lambda}N_\lambda} J)+(0 :_{\cap_{\lambda\in\Lambda}N_\lambda} IJ)+L)=L.$$

Now as L is a completely irreducible submodule of M, there exists $\alpha\in\Lambda$ such that

$$(K :_{N_\alpha} I)+(K :_{\cap_{\lambda\in\Lambda}N_\lambda} J)+(0 :_{\cap_{\lambda\in\Lambda}N_\lambda} IJ)+L=L.$$

By similar argument, since M satisfies the property $AB5^*$ and L is a completely irreducible submodule of M, there exist $\beta\in\Lambda$ and $\gamma\in\Lambda$ such that,

$$(K :_{N_\alpha} I)+(K :_{N_\beta} J)+(0 :_{N_\gamma} IJ)+L=L.$$

Since $\{N_\lambda\}_{\lambda\in\Lambda}$ is a chain, we can assume that $N_\alpha\subseteq N_\beta\subseteq N_\gamma$. Therefore, $(K :_{N_\alpha} I)+(K :_{N_\alpha} J)+(0 :_{N_\alpha} IJ)\subseteq L$. It follows that $(K :_{N_\alpha} IJ)\subseteq L$ since K is a 2-absorbing copure submodule of N_α. Therefore, $(K :_{\cap_{\lambda\in\Lambda}N_\lambda} IJ)\subseteq L$. This implies that

$$(K :_{\cap_{\lambda\in\Lambda}N_\lambda} IJ)\subseteq(K :_{\cap_{\lambda\in\Lambda}N_\lambda} I)+(K :_{\cap_{\lambda\in\Lambda}N_\lambda} J)+(0 :_{\cap_{\lambda\in\Lambda}N_\lambda} IJ),$$

by Remark 2.1. □

Theorem 3.12. *Let M be an R-module which satisfies the property $AB5^*$ and N be a submodule of M. Then there is a submodule K of M minimal with respect to $N\subseteq K$ and K is a 2-absorbing copure submodule of M.*

Proof. Let

$$\Sigma=\{N\leq H|H \text{ is a } 2-\text{absorbing copure submodule of } M\}.$$

Then $M\in\Sigma\neq\emptyset$. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a totally ordered subset of Σ. Then $N\leq\cap_{\lambda\in\Lambda}N_\lambda$ and by Theorem 3.11 (a), $\cap_{\lambda\in\Lambda}N_\lambda$ is a 2-absorbing copure submodule of M. Therefore by using Zorn's Lemma, one can see that Σ has a minimal element, K say as desired. □

Theorem 3.13. *Let M be a strong comultiplication R-module and N be a submodule of M. Then N is a 2-absorbing copure submodule of M if and only if $Ann_R(N)$ is a 2-absorbing pure ideal of R.*

Proof. Since M is a comultiplication R-module, $N = (0 :_M Ann_R(N))$. Let N be a 2-absorbing copure submodule of M and let I and J be any two ideals of R. Then

$$(N :_M IJ) = (N :_M I) + (N :_M J) + (0 :_M IJ) \Rightarrow ((0 :_M Ann_R(N)) :_M IJ)$$
$$= ((0 :_M Ann_R(N)) :_M I) + ((0 :_M Ann_R(N)) :_M J) + (0 :_M IJ).$$

It follows that

$$(0 :_M Ann_R(N)IJ) = (0 :_M Ann_R(N)I) + (0 :_M Ann_R(N)J) + (0 :_M IJ).$$

Thus by Lemma 2.2,

$$(0 :_M Ann_R(N)IJ) = (0 :_M Ann_R(N)I \cap Ann_R(N)J \cap IJ).$$

This implies that $Ann_R(N)IJ = Ann_R(N)I \cap Ann_R(N)J \cap IJ$ since M is a strong comultiplication R-module. Hence $Ann_R(N)$ is a 2-absorbing pure ideal of R. Conversely, let $Ann_R(N)$ be a 2-absorbing pure ideal of R and let I and J be any two ideals of R. Then

$$Ann_R(N)IJ = Ann_R(N)I \cap Ann_R(N)J \cap IJ.$$

Hence by using Lemma 2.2,

$$(0 :_M Ann_R(N)IJ) = (0 :_M Ann_R(N)I) + (0 :_M Ann_R(N)J) + (0 :_M IJ).$$

Therefore, as M is a comultiplication R-module,

$$(N :_M IJ) = (N :_M I) + (N :_M J) + (0 :_M IJ),$$

as desired. □

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REFERENCES

[1] M.S. Abbass, *On fully stable modules*, Ph.D. Thesis, University of Baghdad, 1991.

[2] Y. Al-Shaniafi and P. F. Smith, *Comultiplication modules over commutative rings*, J. Commut. Algebra, **3** (1) (2011), 1-29.

[3] W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York-Heidelberg-Berlin, 1974.

[4] H. Ansari-Toroghy and F. Farshadifar, The dual notion of multiplication modules, *Taiwanese J. Math.* **11** (4) (2007) 1189-1201.

[5] H. Ansari-Toroghy and F. Farshadifar, *Comultiplication modules and related results*, Honam Math. J. **30** (1) (2008), 91–99.

[6] H. Ansari-Toroghy and F. Farshadifar, *Product and dual product of submodules*, Far East J. Math. Sci., **25** (3) (2007), 447–455.

[7] H. Ansari-Toroghy and F. Farshadifar, *Strong comultiplication modules*, CMU. J. Nat. Sci. **8** (1) (2009), 105–113.

[8] H. Ansari-Toroghy and F. Farshadifar, *Fully idempotent and coidempotent modules*, Bull. Iranian Math. Soc. **38** (4) (2012), 987-1005.

[9] H. Ansari-Toroghy and F. Farshadifar, *The dual notion of fully cancellation modules*, submitted.

[10] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. **75** (2007), 417-429.

[11] F. Farshadifar, *2-absorbing pure submodules*, submitted.

[12] L. Fuchs, W. Heinzer, and B. Olberding, *Commutative ideal theory without finiteness conditions: Irreducibility in the quotient filed*, in : Abelian Groups, Rings, Modules, and Homological Algebra, Lect. Notes Pure Appl. Math. **249** (2006), 121–145.

[13] R. Y. Sharp, *Step in commutative algebra*, Cambridge University Press, 1990.

[14] R. Wisbauer, *Foundations of Modules and Rings Theory*, Gordon and Breach, Philadelphia, PA, 1991.

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