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CYCLE DECOMPOSITIONS WITH A SHARPLY VERTEX TRANSITIVE AUTOMORPHISM GROUP

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In some recent papers the method of partial differences introduced by the author in [4] was very helpful in the construction of cyclic cycle systems. Here we use and describe in all details this method for the purpose of constructing, more generally, cycle decompositions with a sharply vertex transitive automorphism group not necessarily cyclic.

1. Introduction.

Throughout the paper we will use some standard notation of graph theory. So, K_v will denote the *complete graph* on v vertices. The *closed trail* of length k whose edges are $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{k-1}, x_0]$ will be denoted by $(x_0, x_1, ..., x_{k-1})$ and it is a *k-cycle* when the vertices $x_0, x_1, ..., x_{k-1}$ are pairwise distinct. It is clear that the same k-trail can be also denoted by $(x_i, x_{i+1}, ..., x_{i+k-1})$ where i is any element in $\{1, ..., k-1\}$ and where the subscripts have to be understood modulo k.

Given an additive group G and a subset Ω of $G - \{0\}$ such that $-\Omega = \Omega$, the *Cayley graph of G on* Ω, denoted by $Cay[G : \Omega]$, is the graph having G as vertex set and edge set E defined by the rule $[x, y] \in E$ if and only if $x - y \in \Omega$. The class of simple graphs with an automorphism group acting sharply transitively on the set of vertices is precisely the class of Cayley graphs (see [21]).

A $(k_1, ..., k_n)$-*cycle decomposition* of a graph Γ is a set $\mathcal{D} = \{C_1, ..., C_n\}$ of cycles with vertices in $V(\Gamma)$ and respective lengths k_1, ..., k_n, with the property that any edge of Γ is edge of exactly one C_i. For its existence it is necessary that

(i) each k_i does not exceed the maximum length of the cycles of Γ;

(ii) $|E(\Gamma)| = k_1 + ... + k_n$;

(iii) the degree of every vertex of Γ is even.

Conditions (i), (ii) are obvious and (iii) immediately follows from the observation that $deg_\Gamma(x) = \sum_{i=1}^{n} deg_{C_i}(x)$ and that $deg_{C_i}(x) = 0$ or 2 according to whether x is or is not a vertex of C_i (by $deg_\Gamma(x)$ and $deg_{C_i}(x)$ we have denoted the degree of x in Γ and C_i, respectively).

If $k_1 = ... = k_n = k$, one simply says that $\mathcal{D}$ is a *k-cycle decomposition* of Γ. In particular, a *k-cycle decomposition* of K_v is also called a *k-cycle system of order* v.

The problem of determining the spectrum of values of v for which there exists a k-cycle system of order v attracted a large number of combinatorialists since the 60's and it has been completely solved very recently.

Theorem 1.1. *There exists a k-cycle system of order v if and only if $k \leq v$, v is odd, and $v(v-1) \equiv 0$ (mod $2k$).*

Of course the *only if part* of the theorem is a trivial consequence of the necessary conditions (i), (ii), (iii) given above for arbitrary cycle decompositions. A crucial ingredient leading to the *if part* was a result (Hoffman, Lindner and Rodger [13], Rodger [17]) that reduce the problem to that of finding a k-cycle system of order v for each admissible v between k and $3k$. This part of the problem was solved by Alspach and Gavlas [1] for k odd (for another proof see also [6] by the present author) and by Šajna [22] for k even.

In the mentioned papers by Alspach and Gavlas and by Šajna the existence problem for k-cycle decompositions of $K_{2v} - I$ (where I denotes a 1-*factor* of K_{2v}) is also solved.

Theorem 1.2. *There exists a k-cycle decomposition of $K_{2v} - I$ if and only if $k \leq 2v$ and $2v(v-1) \equiv 0$ (mod k).*

Given a graph Γ, the action of its automorphism group $Aut(\Gamma)$ on the set $\mathcal{C}(\Gamma)$ of all cycles of Γ is naturally defined by

$$C^\alpha = (x_0^\alpha, x_1^\alpha, ..., x_{k-1}^\alpha) \quad \text{for } C = (x_0, x_1, ..., x_{k-1}) \in \mathcal{C}(\Gamma) \text{ and } \alpha \in Aut(\Gamma).$$

If $\mathcal{D} = \{C_1, ..., C_n\}$ is a cycle decomposition of Γ, its *full automorphism group* $Aut(\mathcal{D})$ is the setwise stabilizer of $\mathcal{D}$ under the action of $Aut(\Gamma)$. In other words, $Aut(\mathcal{D})$ consists of all automorphisms α of Γ such that $(C_1^\alpha, ..., C_n^\alpha)$ is a permutation of $(C_1, ..., C_n)$. Any subgroup of $Aut(\mathcal{D})$ is said to be *an automorphism group* of $\mathcal{D}$.

One says that $\mathcal{D}$ is *sharply-vertex-transitive under a group* G if it admits G as an automorphism group acting sharply transitively on $V(\Gamma)$. From now on, all groups will be written additively. In this case we can identify $V(\Gamma)$ with G and the action of G on $V(\Gamma)$ with the addition on the right by putting $x^g = x + g$ for any $x \in V(\Gamma) = G$ and for any $g \in G$. So, we can say that a cycle decomposition $\mathcal{D}$ of Γ is sharply-vertex-transitive under G if we have:

- $V(\Gamma) = G$;
- $(x_0, x_1, ..., x_{k-1}) \in \mathcal{D} \Rightarrow (x_0 + g, x_1 + g, ..., x_{k-1} + g) \in \mathcal{D}, \ \forall g \in G$.

It is obvious that $Aut(\mathcal{D})$ is a subgroup of $Aut(\Gamma)$. So, if $\mathcal{D}$ is sharply vertex transitive under G then Γ is a sharply vertex transitive graph and hence, for what recalled above, Γ is a Cayley graph.

In this paper the method of partial differences introduced by the present author in [4] for describing sharply vertex transitive graph decompositions in general, is explicitely described for sharply vertex transitive cycle decompositions. Some applications and some new results are also presented.

2. Analysing the structure of a cycle in G with a prescribed G-stabilizer.

Let $A = (a_0, a_1, ..., a_{k-1})$ be a k-cycle. It is well known that $Aut(A)$, the full automorphism group of A, is isomorphic to D_{2k}, the *dihedral group* of order $2k$. Also, we have $Aut(A) =< \alpha, \beta >$ where α and β are the bijections on the set $V(A) = \{a_0, a_1, ..., a_{k-1}\}$ defined by

$$\alpha(a_i) = a_{i+1} \ \text{ and } \ \beta(a_i) = a_{k-i-1} \quad \text{for } i = 0, 1, ..., k-1$$

where the subscripts have to be understood modulo k. More explicitely, we have

$$Aut(A) = \{id_{V(A)}, \alpha, \alpha^2 ..., \alpha^{k-1}, \beta, \beta\alpha, \beta\alpha^2, ..., \beta\alpha^{k-1}\}.$$

The bijections α, α^2, ..., α^{k-1} are *rotations* of A while the bijections β, $\beta\alpha^2$, ..., $\beta\alpha^{k-1}$ are *reflections* of A.

If k is odd, all reflections fix exactly one vertex of A while, in the case of k even, all reflections of the form $\beta\alpha^i$ fix no vertex or exactly two vertices according to whether i is even or odd, respectively.

Now assume that the vertices of A are elements of a group G and let G_A be the stabilizer of A under G. Note that G_A, besides being a subgroup of G, is also, up to isomorphisms, a subgroup of $Aut(A)$ since it acts on A as an automorphism group of it. So, for what seen above, up to isomorphisms G_A is a subgroup of D_{2k}.

If G_A has order t, we say that A is of type γ_t or δ_t according to whether no element or, respectively, at least one element of G_A acts on A as a reflection. Since any reflection is an involution, it is obvious that if a cycle is of type δ_t, then t is even. It is also easy to see that

$$G_A \simeq \begin{cases} Z_t & \text{if } A \text{ is of type } \gamma_t; \\ D_t & \text{if } A \text{ is of type } \delta_t \text{ with } t > 4; \\ Z_2 \times Z_2 & \text{if } A \text{ is of type } \delta_4; \\ Z_2 & \text{if } A \text{ is of type } \delta_2. \end{cases}$$

So, if A and B are cycles of type γ_t and δ_t respectively, we have $G_A \simeq G_B$ if and only if $t = 2$. In this case, however, the nonzero element of G_A acts on A as a rotation while the nonzero element of G_B acts on B as a reflection. Consider, for instance the following 8-cycles with vertices in $G = Z_{16}$:

$$A = (0, 1, 3, 10, 8, 9, 11, 2), \qquad B = (0, 1, 3, 10, 2, 11, 9, 8).$$

They have the same vertices and the same G-stabilizer since we have $G_A = G_B = \{0, 8\}$. By the way, the involution 8 of G acts as a rotation on A while it acts as a reflection on B.

Our aim is to describe the structure of a k-cycle of a prescribed type in a given group G. Before, we need some notation.

Given elements $a_0, a_1, ..., a_{s-1}, x$ in a group G, by $[a_0, a_1, ..., a_{s-1}]_x$ we denote the closed trail represented by the sequence obtainable by concatenation of the sequences

$$\begin{gathered} (a_0, a_1, ..., a_{s-1}) \\ (a_0 + x, a_1 + x, ..., a_{s-1} + x) \\ (a_0 + 2x, a_1 + 2x, ..., a_{s-1} + 2x) \\ \dots \\ (a_0 + (t-1)x, a_1 + (t-1)x, ..., a_{s-1} + (t-1)x) \end{gathered}$$

where t is the order of x in G.

As an example, in $G = Z_{20}$ we have

$$[0, 1, 3, 11]_8 = (0, 1, 3, 11, 8, 9, 11, 19, 16, 17, 19, 7, 4, 5, 7, 15, 12, 13, 15, 3).$$

Now, let x, y be elements of a group G such that both y and $x + y$ are involutions of G. Then, by $[a_0, a_1, ..., a_{s-1}]_{y,x}$ we denote the sequence $[a_0, a_1, ..., a_{s-1}, a_{s-1} + y, ..., a_1 + y, a_0 + y]_x$.

Proposition 2.1. *Let $A = (a_0, a_1, ..., a_{k-1})$ be a closed k-trail in G and let t be a divisor of k, say $k = st$. Then A is a k-cycle of type γ_t if and only if the following conditions hold:*

- *$A = [a_0, a_1, ..., a_{s-1}]_x$ for a suitable x of order t;*
- *$a_0, a_1, ..., a_{s-1}$ lie in pairwise distinct left cosets of $< x >$ in G;*
- *there is no j and no involution $y \in G$ for which we have $a_{k-i} = a_{i+j} + y$ for $0 \leq i \leq k-1$ (where the subscripts have to be understood mod k);*
- *if $A = [a_0, a_1, ..., a_{s'-1}]_{x'}$ for some pair (s', x'), then $o(x') \leq o(x)$.*

The first condition assures that $x \in G_A$ and the second one that the elements $a_0, a_1, ..., a_{k-1}$ are pairwise distinct so that A is actually a k-cycle. The third condition guarantees that no element of G acts on A as a reflection and hence that G_A is of type γ_u for some multiple u of t. Finally, the fourth condition assures that $u = t$ and hence that $G_A =< x >$.

Proposition 2.2. *Let $A = (a_0, a_1, ..., a_{k-1})$ be a closed k-trail in G and let $2t$ be a divisor of k, say $k = 2st$. Then A is a k-cycle of type δ_{2t} if and only if the following conditions hold:*

- *$A = [a_0, a_1, ..., a_{s-1}]_{y,x}$ for suitable elements x, y with $o(y) = o(x+y) = 2$ and $o(x) = t$;*
- *$a_0, a_1, ..., a_{s-1}$ lie in pairwise distinct left cosets of $< x, y >$ in G;*
- *if $A = [a_0, a_1, ..., a_{s'-1}]_{x'}$ for some pair (s', x'), then $o(x') \leq o(x)$.*

The first condition assures that $< x, y > \subseteq G_A$ and the second condition assures that the elements a_0, a_1, ..., a_{k-1} are pairwise distinct so that A is actually a k-cycle. From the first condition we also have that y acts on A as a reflection and that x acts on A as a rotation of order t. Thus G_A is of type δ_{2u} for some multiple u of t. Finally, the third condition assures that $u = t$ and hence that $G_A =< x, y >$.

3. The method of partial differences.

In this section, speaking of a cycle decomposition of $Cay[G : \Omega]$ we understand it admits G as a sharply vertex transitive automorphism group. It is clear that for giving such a decomposition $\mathcal{D}$ it is enough to give a complete system $\mathcal{F}$ of representatives for the G-orbits of its cycles. Such a system will be also called a set of *base cycles* of $\mathcal{D}$. The notion of *list of partial differences* of a cycle is very helpful for recognizing whether a given set $\mathcal{F}$ of cycles in G is a set of base cycles of a cycle decomposition of $Cay[G : \Omega]$.

Definition 3.1. Let $A = (a_0, a_1, ..., a_{k-1})$ be a k-cycle of type γ_t in G. Then the list of partial differences of A is the multiset

$$\partial A = \{\pm(a_{i+1} - a_i) \mid 0 \le i < k/t\}.$$

Let $A = [a_0, a_1, ..., a_{s-1}]_{y,x}$ be a k-cycle of type δ_t in G. Then the list of partial differences of A is the multiset

$$\partial A = \pm\{a_{i+1} - a_i \mid 0 \le i \le s-2\} \cup \{a_{s-1} + y - a_{s-1}, a_0 + x + y - a_0\}.$$

If $\mathcal{F}$ is a collection of cycles in G then the list of partial differences of $\mathcal{F}$ is the multiset $\partial\mathcal{F} = \bigcup_{A\in\mathcal{F}} \partial A$ where elements have to be counted with their respective multiplicities.

Remark 3.2. Note that if A is k-cycle in G, then ∂A has size $\frac{2k}{|G_A|}$.

Let A be a k-cycle in a group G and let $[x, y]$ be any edge of the complete graph with vertex set G. It is not difficult to see that the number of cycles in the G-orbit of A admitting $[x, y]$ as an edge, is exactly equal to the number of times that $x - y$ appears in ∂A. This, thinking at the definition of a Cayley graph, allows us to state the following basic theorem.

Theorem 3.3. *Let Γ be the Cayley graph $Cay[G : \Omega]$. Then a collection $\mathcal{F}$ of cycles in G is a system of representatives for the G-orbits of a cycle decomposition of Γ if and only if $\partial F = \Omega$.*

Given a set $\mathcal{F}$ of base cycles for a cycle decomposition of $Cay[G : \Omega]$, let $\mathcal{F}_\delta$ be the set of cycles of $\mathcal{F}$ admitting at least one element of G as a reflection, and let $\mathcal{F}_\gamma = \mathcal{F} - \mathcal{F}_\delta$.

Proposition 3.4. *If $\mathcal{F}$ is a set of base cycles for a cycle decomposition of $Cay[G : \Omega]$, then we have $|\mathcal{F}_\delta| = \frac{j}{2}$ where j is the number of involutions in Ω.*

Proof. Let A be a cycle of $\mathcal{F}_\gamma$ and note that ∂A can be partitioned into pairs of the form $\{z, -z\}$ so that any involution appears an even number of times in ∂A. On the other hand, by Theorem 3.3 we have $\partial\mathcal{F} = \Omega$ so that no repetiton in $\partial\mathcal{F}$ is allowed. It follows that no involution of G appears in $\partial\mathcal{F}_\gamma$.

Now, let A be a k-cycle of $\mathcal{F}_\delta$, say $A = [a_0, a_1, ..., a_{s-1}]_{y,x}$. Note that $g = a_{s-1} + y - a_{s-1}$ and $h = a_0 + x + y - a_0$ are involutions of G appearing in ∂A. Also note that $\partial A - \{g, h\}$ can be partitioned into pairs of the form $\{z, -z\}$ and hence, reasoning as above, no involution can appear in $\partial A - \{g, h\}$.

We conlude that the number of involutions appearing in $\partial\mathcal{F}$, i.e., the number of involutions in Ω, is $2|\mathcal{F}_\delta|$. The assertion follows. $\square$

We recall that a cycle decomposition of a graph Γ is *Hamiltonian* when its cycles have length $|V(\Gamma)|$. In order to illustrate the method of partial differences explained above, in the following informative proposition we establish which groups of order 12 are sharply vertex transitive automorphism groups of a *Hamiltonian* cycle decomposition of $K_{12} - I$.

Proposition 3.5. *There exists a Hamiltonian cycle decomposition of $K_{12} - I$ admitting G as a sharply vertex transitive automorphism group for each group G of order* 12 *with the only definite exception of the alternating group A_4.*

Proof. If G is a group of order 12, then $K_{12} - I$ may be realized as Cayley graph $Cay[G : G - \{0, \sigma\}]$ where σ is an arbitrary involution of G.

So, for $G = Z_{12}, Z_2 \times Z_6, D_{12}, Q_{12}$ (the dicyclic group) we have to exibhit a 12-cycle decomposition of $Cay[G : G - \{0, \sigma\}]$ for a suitable involution $\sigma \in G$. Also, we have to show that for each involution $\sigma \in A_4$ no cycle decomposition of $Cay[A_4 : A_4 - \{0, \sigma\}]$ exists.

1st case: $G = Z_{12}$. Take the 12-cycles

$$A=(0, 1, 5, 3, 4, 8, 6, 7, 11, 9, 10, 2) \text{ and } B=(0, 9, 2, 11, 4, 1, 6, 3, 8, 5, 10, 7).$$

We have $A = [0, 1, 5]_3$ and $B = [0, 9]_2$. Applying Proposition 2.1, it is easily seen that A is of type γ_4 and B is of type γ_6. So we have:

$$\partial A = \{\pm 1, \pm 4, \pm 2\}, \quad \partial B = \{\pm 3, \pm 5\}$$

and hence $\partial\{A, B\} = G - \{0, 6\}$. So, $\{A, B\}$ is a set of base cycles of a 12-cycle decomposition of $Cay[G : G - \{0, 6\}]$ by Theorem 3.3.

2nd case: $G = Z_2 \times Z_6$. Take the 12-cycles

$$A = (00, 12, 01, 13, 02, 14, 03, 15, 04, 10, 05, 11),$$

$$B = (00, 01, 05, 02, 04, 03, 13, 14, 12, 15, 11, 10).$$

Note that we have $A = [00, 12]_{01}$ and $B = [00, 01, 05]_{03,13}$. Applying Propositions 2.1 and 2.2, it is easily seen that A is of type γ_6 and B is of type δ_4. So we have:

$$\partial A = \{12, 14, 11, 15\}, \quad \partial B = \{01, 05, 02, 04, 10, 03\}$$

and hence $\partial\{A, B\} = G - \{00, 13\}$. So, $\{A, B\}$ is a set of base cycles of a 12-cycle decomposition of $Cay[G : G - \{00, 13\}]$ by Theorem 3.3.

3rd case: $G = D_{12}$. Here, as usual, we adopt multiplicative notation representing G with defining relations $G =< x, y \mid x^6 = y^2 = 1; yx = x^5y >$. Take the 12-cycles

$$A = (1, x, x^5, x^5y, xy, y, x^3, x^4, x^2, x^2y, x^4y, x^3y),$$

$$B = (1, xy, x, y, x^2, x^5y, x^3, x^4y, x^4, x^3y, x^5, x^2y),$$

$$C = (1, x^5y, x, x^4y, x^2, x^3y, x^3, x^2y, x^4, xy, x^5, y).$$

Note that we have $A = [1, x, x^5]_{y,x^3}$, $B = [1]_{xy,x}$ and $C = [1]_{x^5y,x}$. Applying Proposition 2.1, it is easily seen that A is of type δ_4 while both B and C are of type δ_{12}. So we have:

$$\partial A = \{x, x^5, x^2, x^4, x^3y, x^4y\}, \quad \partial B = \{xy, x^2y\}, \quad \partial C = \{x^5y, y\}$$

and hence $\partial\{A, B\} = G - \{1, x^3\}$. So, $\{A, B, C\}$ is a set of base cycles of a 12-cycle decomposition of $Cay[G : G - \{1, x^3\}]$ by Theorem 3.3.

4th case: $G = Q_{12}$. Also here we adopt multiplicative notation representing G with defining relations $G =< x, y \mid x^6 = 1; y^2 = x^3; yx = x^5y >$. Take the 12-cycles

$$A = (1, x, x^5, y, xy, x^5y, x^3, x^4, x^2, x^3y, x^4y, x^2y),$$

$$B = (1, y, x, x^5y, x^2, x^4y, x^3, x^3y, x^4, x^2y, x^5, xy).$$

Note that we have $A = [1, x, x^5]_y$ and $B = [1, y]_x$. Applying Proposition 2.1, it is easily seen that A is of type γ_4 while B is of type γ_6. So we have:

$$\partial A = \{x, x^5, x^2, x^4, x^5y, x^2y\}, \quad \partial B = \{y, x^3y, xy, x^4y\}$$

and hence $\partial\{A, B\} = G - \{1, x^3\}$. So, $\{A, B\}$ is a set of base cycles of a 12-cycle decomposition of $Cay[G, G - \{1, x^3\}]$ by Theorem 3.3.

5th case: $G = A_4$. Assume that $\mathcal{F}$ is a set of base cycles of a 12-cycle decomposition $\mathcal{D}$ of $Cay[G : G - \{0, \sigma\}]$ for some involution σ of G. By Proposition 3.4, $\mathcal{F}_\delta$ is a singleton since A_4 has exactly three involutions. The only cycle A of $\mathcal{F}_\delta$ cannot be of type δ_2 otherwise its orbit would have size 6 that is absurd since $\mathcal{D}$ has size 5. Thus A is necessarily of type δ_4 since G does not have subgroups isomorphic to a dihedral group. It follows, by Remark

3.2, that ∂A has size 6 and hence that $\mathcal{F}_\gamma$ have to produce exactly 4 differences. This, by the same remark, would be possible only if $\mathcal{F}_\gamma$ consists of two cycles of type γ_{12} or it consists of exactly one cycle of type γ_6. On the other hand, this is absurd since the maximum order of the elements of G is 3. We conclude that no 12-cycle decomposition of $Cay[G : G - \{0, \sigma\}]$ exists. $\square$

4. Sharply vertex transitive cycle systems.

From now on we only consider k-cycle decompositions of the complete graph, i.e., k-cycle systems. Since the complete graph K_v may be realized as Cayley graph $Cay[G : G - \{0\}]$ for any group G of order v, then, in view of Theorem 3.3 we have.

Theorem 4.1. *There exists a k-cycle system admitting G as a sharply vertex transitive automorphism group if and only if there exists a set $\mathcal{F}$ of k-cycles with vertices in G such that $\partial\mathcal{F} = G - \{0\}$.*

In the following propositions the method of partial differences allows us to get two non-existence results on sharply vertex transitive cycle systems.

Proposition 4.2. *Assume that $\mathcal{D}$ is a k-cycle system of order v admitting G as a sharply vertex transitive automorphism group and let $gcd(v, k) = p_1^{\alpha_1}...p_n^{\alpha_n}$ be the prime factorization of $gcd(v, k)$. Then G has a cyclic subgroup of order $p_i^{\alpha_i}$ for each i.*

Proof. Assume that G does not have a cyclic subgroup of order $p_i^{\alpha_i}$ for some i. Then, if γ_t is the type of a cycle A of $\mathcal{D}$, we have that t cannot be divisible by $p_i^{\alpha_i}$ and hence, by Remark 3.2, the size $2k/t$ of ∂A is divisible by p_i. So, p_i would be a divisor of $|\partial\mathcal{F}|$ for any set $\mathcal{F}$ of base cycles of $\mathcal{D}$ and hence, by Theorem 4.1, p_i would be a divisor of $v - 1$. This is obviously false since, by hypothesis, p_i is also a divisor of v. $\square$

Corollary 4.3. *There is no Hamiltonian cycle system of order a nonprime primepower admitting a sharply vertex transitive automorphism group.*

Proof. Let $\mathcal{D}$ be a Hamiltonian cycle system of order $v = p^n$ with p a prime and $n \geq 2$. It is obvious from Proposition 4.2 that if $\mathcal{D}$ is sharply vertex transitive under a group G, then G is cyclic. On the other hand we already know from [8] that a cyclic Hamiltonian cycle system of order v does not exist. $\square$

Proposition 4.4. *Let k and v be positive integers such that $k < v < 2k$ and $gcd(v, k)$ is a power of a prime p. Then, if G is a group of order v whose Sylow*

p-subgroup is cyclic and normal in G, there is no k-cycle system of order v admitting G as a sharply vertex transitive automorphism group.

Proof. Assume that $\mathcal{D}$ is a k-cycle system of order v admitting G as a sharply vertex transitive automorphism group and let $\mathcal{F}$ be a set of base cycles of $\mathcal{D}$. Observe, first, that no cycle of $\mathcal{D}$ has trivial G-stabilizer. In fact, in the opposite case we would have cycles whose G-orbit has size v and hence $|\mathcal{D}| \geq v$. On the other hand this is absurd since $v < 2k$ implies that $v - 1 > \frac{v(v-1)}{2k} = |\mathcal{D}|$.

Set $gcd(v, k) = p^n$ and let A be any cycle of $\mathcal{F}$. For what seen above, A is of type γ_{p^α} for some $\alpha \in \{1, ..., n\}$. So, by Proposition 2.1, we have $A = [a_0, a_1, ..., a_{s-1}]_x$ where x is an element of G of order p^α and where the elements $a_0, a_1, ..., a_{s-1}$ are in pairwise distinct cosets of $< x >$. This easily implies that no element of $< x >$ appears in ∂A. Now note that in view of the hypothesis on the Sylow p-subgroup of G, each element of order p belongs to $< x >$ and hence no of these elements belongs to ∂A.

Thus, considering that A has been taken arbitrarily in $\mathcal{F}$, we conclude that no element of order p appears in $\partial\mathcal{F}$. This contradicts Theorem 4.1. □

5. Cyclic cycle systems.

A cycle system is said to be *cyclic* if it admits a cyclic sharply vertex transitive automorphism group. At the moment the major result on cyclic cycle systems is the following.

Theorem 5.1. *If $v \equiv 1$ (mod $2k$) or $v \equiv k$ (mod $2k$) with k odd, then there exists a cyclic k-cycle system of order v with the only definite exceptions of* $(v, k) = (9, 3), (15, 15)$ *and* (q, q) *with q a nonprime prime power.*

The papers contributing to this result are [7], [8], [14], [15], [18], [19], [20], [23]. Additional references are [2], [3], [9], [10].

The problem of establishing the set of v's for which there exists a cyclic k-cycle system of order v for a *fixed* value of k, has been settled for $k = q$ or $2q$ with q a prime power, and for $k \leq 30$. If k is a prime power the answer to this problem is given by Theorem 5.1 since in this case the necessary condition $v(v - 1) \equiv 0$ (mod $2k$) becomes $v \equiv 1$ or k (mod $2k$). In the cases where $k = 2q$ with q a prime power or where $k \leq 30$, the answer can be found in a very recent paper by Fu and Wu [10].

As immediate consequence of Proposition 4.4 we have:

Proposition 5.2. *If k and v are positive integers such that $gcd(v, k)$ is a prime power and $k < v < 2k$, then no cyclic k-cycle system of order v exists.*

In the following table we report the admissible pairs (v, k) with $k \leq 100$ for which the above Proposition guarantees the non-existence of a cyclic k-system of order v:

(9, 6)	(21, 14)	(21, 15)	(25, 15)	(25, 20)	(33, 22)	(33, 24)
(49, 28)	(45, 33)	(55, 33)	(57, 38)	(65, 40)	(49, 42)	(57, 42)
(55, 45)	(81, 45)	(69, 46)	(69, 51)	(85, 51)	(65, 52)	(81, 54)
(81, 60)	(93, 62)	(91, 63)	(99, 63)	(91, 65)	(105, 65)	(121, 66)
(93, 69)	(115, 69)	(85, 70)	(81, 72)	(99, 77)	(133, 77)	(105, 78)
(129, 86)	(117, 87)	(145, 87)	(145, 90)	(105, 91)	(169, 91)	(161, 92)
(141, 94)	(115, 95)	(171, 95)	(129, 96).			

Table1

We conjecture that the exceptional (admissible) pairs (v, k) for which no cyclic k-cycle system of order v exists are exactly those given by Theorem 5.1 and Proposition 5.2. So, explicitly:

Conjecture. There exists a cyclic k-cycle system of order v for all admissible pairs (v, k) with the only exceptions of

- $(v, k) = (9, 3)$;
- $v = k = 15$;
- $v = k = p^n$ with p a prime and $n > 1$;
- $k < v < 2k$ with $gcd(v, k)$ a prime power.

6. Some sharply vertex transitive cycle systems of small order.

In this section we examine the first exceptional pairs (v, k) reported in Table 1 and we show that for some of them it is possible to exibhit a k-cycle system of order v with a sharply vertex transitive automorphism group.

a) A sharply vertex transitive 6-cycle system of order 9.
Let $G = Z_3 \times Z_3$ and consider the following 6-cycles with vertices in G:

$$A = (00, 01, 10, 11, 20, 21), \qquad B = (00, 11, 01, 12, 02, 10).$$

Both A and B are of type γ_3 since we have:

$$A = [00, 01]_{10} \qquad \text{and} \qquad B = [00, 11]_{01}$$

So we have:

$$\partial A = \{01, 02, 12, 21\} \qquad \text{and} \qquad \partial B = \{11, 22, 10, 20\}.$$

We see that $\partial\{A, B\} = G - \{00\}$ and hence $\{A, B\}$ is a set of base cycles of the required cycle system by Theorem 4.1.

b) A sharply vertex transitive 14-cycle system of order 21 does not exist.

There are only two groups of order 21; Z_{21} and the non-abelian group G with defining relations

$$G =< x, y \mid x^7 = y^3 = 1;\ yx = x^2y > .$$

It is obvious that the group of order 7 generated by $< x >$ is normal in G. So, since we have $gcd(14, 21) = 7$, the non-existence follows by Proposition 4.4.

c) A sharply vertex transitive 15-cycle system of order 21.

Let G be the non-abelian group of order 21 that we have presented in b). Consider the following 15-cycles with vertices in G:

$$A = (1, x, x^6, x^2, yx^5, y^2, y^2x, y^2x^5, y^2x^4, x^3, y, yx^2, yx^3, yx, y^2x^6),$$

$$B = (1, y, y^2x^4, yx, x^6, y^2x^3, x^3, yx^4, x^5, y^2x, yx^2, y^2x^2, x^4, y^2x^6, yx^5).$$

It is straightforward to check that both A and B are of type γ_3 since we have:

$$A = [1, x, x^6, x^2, yx^5]_{y^2} \quad \text{and} \quad B = [1, y, y^2x^4, yx, x^6]_{y^2x^3}.$$

So we have:

$$\partial A = \{x^i \mid i = 1, ..., 6\} \cup \{yx^3, y^2x, yx^4, y^2x^6\},$$

$$\partial B = \{y, y^2, yx, y^2x^5, yx^6, y^2x^2, yx^2, y^2x^3, yx^5, y^2x^4\}.$$

We see that all elements of $G - \{1\}$ are covered exactly once by $\partial\{A, B\}$ so that $\{A, B\}$ is a set of base cycles of the required cycle system by Theorem 4.1.

d) A sharply vertex transitive 15-cycle system of order 25.

Let $G = Z_5 \times Z_5$ and consider the following 15-cycles with vertices in G:

$$A = (00, 01, 12, 10, 11, 22, 20, 21, 32, 30, 31, 42, 40, 41, 02),$$

$$B = (00, 12, 31, 10, 22, 41, 20, 32, 01, 30, 42, 11, 40, 02, 21),$$

$$C = (00, 13, 22, 40, 03, 12, 30, 43, 02, 20, 33, 42, 10, 23, 32),$$

$$D = (00, 10, 32, 02, 12, 34, 04, 14, 31, 01, 11, 33, 03, 13, 30),$$

It is straightforward to check all these cycles are of type γ_5 since we have:

$$A = [00, 01, 12]_{10}, \qquad B = [00, 12, 31]_{10},$$

$$C = [00, 13, 22]_{40}, \qquad D = [00, 10, 32]_{02}.$$

So we have:

$$\partial A = \{01, 04, 11, 44, 02, 03\}, \quad \partial B = \{12, 43, 24, 31, 21, 34\},$$

$$\partial C = \{13, 42, 14, 41, 23, 32\}, \quad \partial D = \{10, 40, 22, 33, 20, 30\}.$$

We see that $\partial\{A, B, C, D\} = G - \{00\}$ and hence $\{A, B, C, D\}$ is a set of base cycles of the required cycle system by Theorem 4.1.

e) A sharply vertex transitive 20-cycle system of order 25.
Let $G = Z_5 \times Z_5$ and consider the following 20-cycles with vertices in G:

$$A = (00, 10, 40, 22, 01, 11, 41, 23, 02, 12, 42, 24, 03, 13, 43, 20, 04, 14, 44, 21),$$

$$B = (00, 11, 30, 42, 01, 12, 31, 43, 02, 13, 32, 44, 03, 14, 33, 40, 04, 10, 34, 41),$$

$$C = (00, 13, 41, 40, 43, 01, 34, 33, 31, 44, 22, 21, 24, 32, 10, 14, 12, 20, 03, 02).$$

It is straightforward to check that A, B and C are of type γ_5 since we have:

$$A = [00, 10, 40, 22]_{01}, \quad B = [00, 11, 30, 42]_{01} \text{ and } C = [00, 13, 41, 40]_{43}.$$

So we have:

$$\partial A = \{10, 40, 20, 30, 23, 32, 21, 34\},$$

$$\partial B = \{11, 44, 24, 31, 12, 43, 14, 41\},$$

$$\partial C = \{13, 42, 22, 33, 01, 04, 02, 03\}.$$

We see that all elements of $G - \{00\}$ are covered exactly once by $\partial\{A, B, C\}$ and hence $\{A, B, C\}$ is a set of base cycles of the required cycle system by Theorem4.1.

e) Sharply vertex transitive 33-cycle systems of order 22 and of order 24 do not exist.

Up to isomorphisms, there is exactly one group of order 33, that is the cyclic one. So, the non-existence follows from Proposition 4.4.

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