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APPLICATIONS OF FORMATIVE PROCESSES TO THE DECISION PROBLEM IN SET THEORY

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As part of a project aimed at the implementation of a proof-checker based on the set-theoretic formalism, the decision problem in set theory has been studied very intensively, starting in the late seventies.

Several results have been produced in the first decade of research, giving rise to the novel field of *computable set theory*. At that point, it already was clear that to face the tremendous amount of technicalities involved in the combination of smaller decidable fragments into larger ones, new techniques were in order.

Such techniques have recently emerged, by a careful analysis of the formation process of disjoint families of sets. This has led to the characterization of suitable decidable conditions for the satisfiability of set-theoretic formulae belonging to specific collections.

In this paper we give an elementary introduction to the formative process technique and discuss some open problems.

1. Introduction.

The systematic investigation of the decision problem in set theory can be traced back to the late seventies, when a full-fledged programming language based on the set-theoretic formalism—named SETL—began to be used quite

Key words Satisfiability decision problem, satisfaction algorithm, Zermelo-Fraenkel set theory.

extensively in prototyping large-scale projects and the urge of program verification became more pressing. [1]

Plainly, any mature program verification system must be based on a powerful, general purpose proof checker, i.e., an interactive programmed system into which one can enter sequences of logical/mathematical formulae, which it will accept as long as it can perform some computation which guarantees that each new formula is a logical consequence of preceding formulae. Such a verifier ensures rigorously against logical error, possibly at the price of requiring its user to enter a burdensome mass of intermediate detail. On the other hand, the weight of detail required is inversely proportional to the size of the inferences which the system is able to make automatically. For this reason, the technology of program verification requires considerably powerful automatic theorems provers, which are ultimately based on systematic proof methods, such as the very general resolution principle or collections of more particularized decision procedures.

After the initial enthusiasm, it appeared clear that resolution was a too general principle to attain efficiency, and that new proof techniques to cover important specialized classes of statements were needed.

Its expressivity and widespread use in the whole body of mathematics, motivated the adoption of the formalism of set theory as primitive language for a proof checker and, in turn, a thorough investigation on automatic deduction methods in set theory.

The study of the decision problem in set theory progressed at a very fast pace, yielding in a few years a large body of results which where collected in the novel field of *Computable Set Theory*; the reader is referred to [3], [5] for a very comprehensive account.

Using specialized techniques, several fragments of set theory were shown to have a solvable decision problem. But soon it appeared clear that in order to combine such fragments into larger decidable theories, more sofisticated techniques were needed, for coping with the increasing mass of technicalities involved.

Such techniques have recently emerged in [6], [8], where a careful analysis of the formation process of disjoint families of sets has led to the characterization of suitable decidable conditions which are necessary and sufficient for the satisfiability of set-theoretic formulae belonging to specific collections. [2]

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[1] We recall that SETL has been developed at New York University under the direction of J.T. Schwartz and it has been used to implement the first certified compiler for the ADA programming language.

[2] Some of the ideas where already present in a rudimentary form in [2].

The paper is organized as follows. After a brief introduction to set theory and to the decision problem in set theory, respectively in Sections 2 and 3, in Section 4 we give a concise introduction to the basic concepts of transitive partitions, $\mathcal{P}$-graphs , and formative processes and then, in Section 5, we show how formative processes can be used to yield decision procedures in set theory. Section 6 concludes the paper with some final remarks and hints to future work.

2. Brief introduction to set theory.

We briefly recall here some of the basic set-theoretic concepts and notation used troughout the paper. In order to give a picture of the subject more familiar to graph theorists, we shall describe sets using downward growing trees (see [1]).

The reader is assumed to have some familiarity with the every day set-theoretic apparatus, as well as with some elementary notions from formal languages and computation theory.

Our considerations will take place in the standard axiom system *ZFC* , developed by Zermelo, Fraenkel, Skolem, and von Neumann (see [10]). In fact, for the sake of simplicity they will refer to a very specific model of *ZFC* — the von Neumann standard cumulative hierarchy of sets—though they could be fully formalizable in *ZFC* .

The definition of the von Neumann standard cumulative hierarchy is based on the principle of transfinite recursion on ordinals, which is a generalization of the recursion principle on integers.

Let us give a quick recollection of basic notions on ordinal numbers (for a more comprehensive presentation, see [10]).

Definition 1. A set T is said to be TRANSITIVE if $T \subseteq \mathscr{P}(T)$ or, equivalently, if $\bigcup T \subseteq T$. [3]

A set μ is said to be an ORDINAL NUMBER if μ is transitive and is linearly ordered (and hence well-ordered) by the membership relation $\in$.

As is well known, membership behaves as a well-ordering on the class $\mathcal{O}$ of all ordinals. One reason to be interested in ordinals is the following fundamental theorem:

Theorem 1. *Let* $\trianglelefteq$ *be a well-ordering on the set* x. *Then there exist, and are uniquely determined, an ordinal* ξ *and a function* $f \in x^{\xi}$ *such that* $f[\xi] = x$

[3] We recall that $\mathscr{P}(T)$ denotes the power set of T, wherease $\bigcup T$ denotes the union of all elements of T.

holds and, for any pair $\nu, \mu < \xi$ of ordinals:

$$f\nu \neq f\mu \text{ holds when } \nu \neq \mu, \text{ and moreover } f\nu \trianglelefteq f\mu \text{ when } \nu \leq \mu.$$

By virtue of the axiom of choice, a well-ordering can be imposed on any set. Therefore the following definition makes sense:

Definition 2. The CARDINALITY of a set x, to be denoted $|x|$, is the least ordinal ν such that there exists a function $f \in x^\nu$ with $f[\nu] = x$.[4] A CARDINAL NUMBER is an ordinal μ such that $\mu = |\mu|$.

It is rather easy to see in which sense ordinals are an extension of the natural numbers. Indeed, natural numbers, defined *à la* von Neumann by the rules

$$0 =_{\text{Def}} \emptyset, \quad i + 1 =_{\text{Def}} i \cup \{i\},$$

constitute the initial segment of the class of ordinals; their set, $\omega =_{\text{Def}} \{0, 1, 2, 3, \ldots\}$, is the first ordinal which exceeds all natural numbers, often denoted $\aleph_0$.

Even for ordinals (such as ω) which are not natural numbers, it is convenient to assign the meaning just indicated to the increment operation '+1': we will hence have, among ordinals, $\omega + 1$, $\omega + 1 + 1$, etc. The ordinals of the form $\mu + 1$ are called SUCCESSOR ORDINALS; all others, save 0, are called LIMIT ORDINALS. The latter comprise ω, $\omega + \omega$, $\underbrace{\omega + \ldots + \omega}_{\omega \text{ times}}$, etc. (we are making an appeal to the intuition of the reader).

All elements of $\omega + 1$ are cardinal numbers; but $\omega + 1$ itself is *not* such a number.

Definition 3. By a ξ-SEQUENCE, where ξ is an ordinal, we mean a function $\{Y_\mu\}_{\mu \in \xi}$, usually denoted $(Y_\mu)_{\mu < \xi}$, whose domain is ξ.

By SEQUENCE (without indication of ξ), one means ω-sequence.

In *ZFC* one has that a function rk exists that is univocally defined on all sets by the following recursive rule

$$\mathsf{rk}\ X = \bigcup \left\{ (\mathsf{rk}\ Y) + 1 \mid Y \in X \right\};$$

this function associates an ordinal to each set X, and is called the RANK function. Thanks to the axiom of choice, a well-ordering $\trianglelefteq$ can be imposed to any given set x so that

$$y \trianglelefteq z \text{ when } \mathsf{rk}\ y < \mathsf{rk}\ z \text{ and } y, z \in x.$$

[4] Given two sets A and B, by B^A we mean the collection of all functions from A to B.

We are now ready to define the von Neumann standard cumulative hierarchy $\mathcal{V}$ of all sets by the following transfinite induction on the class $\mathcal{O}$ of all ordinals:

$$\begin{array}{rcll} \mathcal{V}_0 & = & \emptyset & \\ \mathcal{V}_{\alpha+1} & = & \mathscr{P}(\mathcal{V}_\alpha)\,, & \text{for each ordinal } \alpha, \\ \mathcal{V}_\lambda & = & \bigcup_{\mu<\lambda} \mathcal{V}_\mu\,, & \text{for each limit ordinal } \lambda, \\ \mathcal{V} & = & \bigcup_{\mu \in On} \mathcal{V}_\mu\,. & \end{array}$$

The class $\mathcal{V}_\mu$ of all sets whose rank is smaller than μ is, for every ordinal μ, a *set*, which is easily recognized to be transitive. Among these sets, one has the family $\mathcal{V}_\omega$ of the HEREDITARILY FINITE sets, which are those sets that are *finite* and whose elements, elements of elements, etc., all are finite.

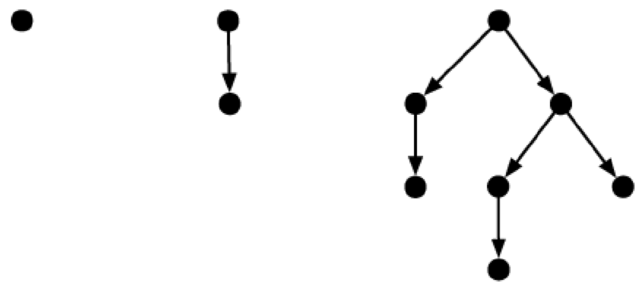
Example 1. Using the notation

$$\emptyset^n =_{\text{Def}} \underbrace{\{\{\ldots\{}_{n}\ \emptyset\ \underbrace{\}\ldots\}\}}_{n},$$

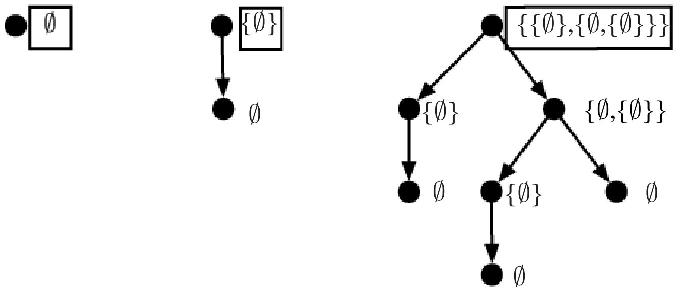
the first few layers of the von Neumann standard cumulative hierarchy are

$$\begin{array}{rcl} \mathcal{V}_0 & = & \emptyset \\ \mathcal{V}_1 & = & \{\emptyset\} \\ \mathcal{V}_2 & = & \{\emptyset,\ \emptyset^1\} \\ \mathcal{V}_3 & = & \{\emptyset,\ \emptyset^1,\ \emptyset^2,\ \{\emptyset,\ \emptyset^1\}\} \\ \mathcal{V}_4 & = & \{\emptyset,\ \emptyset^1,\ \emptyset^2,\ \emptyset^3,\ \{\emptyset,\ \emptyset^1\},\ \{\emptyset,\ \emptyset^2\},\ \{\emptyset^1,\ \emptyset^2\},\ \ldots \\ \vdots & \vdots & \vdots \end{array}$$

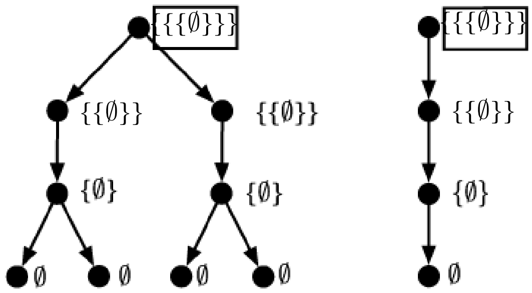
Sets of this kind can be represented as finite trees.



By traversing trees in bottom-up, we can compute the sets they represent, through a simple labeling process.

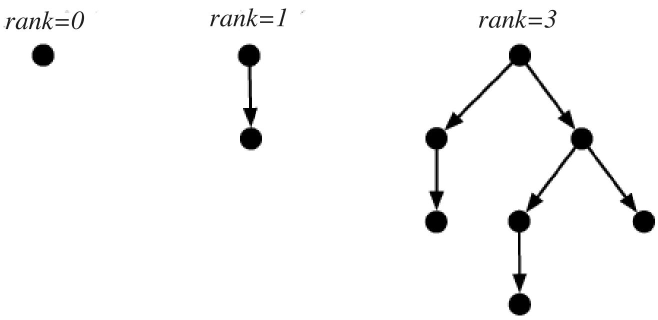


Unfortunately, different trees may represent the same set.



To prevent such pathology, it is possible to define a relation $\approx$ between trees, called MAXIMUM BISIMULATION, such that any set is represented by a unique $\approx$-equivalence class in the collection of all trees. However, an exhaustive treatment of this subject is out of our intentions (see [1] for a deeper presentation).

It can easily be checked that the rank of a hereditarily finite coincides with the height of any tree representing it.



3. The decision problem in set theory.

We are mainly interested in collections of propositional combinations of set-theoretic literals belonging to specific families. We will freely refer to such collections of formulae as THEORIES. For instance, the theory **MLSSP** is the collection of propositional combinations of literals of the following types:

$$
\begin{array}{lllll}
 & v = w, & v \neq w, & v = \emptyset, & v = u \cup w, \\
(1) & v = u \cap w, & v = u \setminus w, & v \subseteq u, & v \not\subseteq u, \\
 & v \in w, & v \notin w, & v = \mathscr{P}(w), & v = \{w_1, w_2, \ldots, w_H\}.
\end{array}
$$

An example of an **MLSSP**-formula is the following

$$(x \in y \wedge y \in z) \rightarrow (x = \emptyset \vee x = \mathscr{P}(z)).$$

To define the decision problem in set theory, we need the following further notions.

Definition 4. An ASSIGNMENT over a collection of variables V is any map from V into the universe of all sets $\mathcal{V}$.

Example 2. Let $V = \{x, y, z\}$ be a collection of variables. Then an example of an assignment over V is

$$\mathcal{M}x = \emptyset, \quad \mathcal{M}y = \{\emptyset\}, \quad \mathcal{M}z = \{\{\emptyset\}\}.$$

Definition 5. A set-theoretic formula φ is said to be SATISFIABLE if there exists an assignment $\mathcal{M}$ of sets from $\mathcal{V}$ over its free variables $x, y, z, \ldots$ such that the formula resulting from φ by

- substituting in it sets $\mathcal{M}x, \mathcal{M}y, \mathcal{M}z, \ldots$ in place of free occurrences of $x, y, z, \ldots$, and by
- interpreting the set-theoretic operators and predicates in it according to their standard meaning

is true (in the von Neumann universe $\mathcal{V}$ of all sets).

In this case $\mathcal{M}$ is said to be a MODEL for φ.

Example 3. Let φ be the formula $x \in y \wedge y \in z \wedge z \setminus x \neq \emptyset$. Then the assignment $\mathcal{M}$ defined by

$$\mathcal{M}x = \emptyset, \quad \mathcal{M}y = \{\emptyset\}, \quad \mathcal{M}z = \{\{\emptyset\}\}$$

clearly satisfies φ.

Definition 6. The DECISION PROBLEM for a theory $\mathcal{T}$ is the problem of establishing for any given formula φ in $\mathcal{T}$ whether φ is satisfiable.

Notice that a satisfiability test for a theory $\mathcal{T}$ can also be used to decide whether any given formula φ in $\mathcal{T}$ is TRUE (in the standard von Neumann model of set theory).[5] In fact, a formula φ is true if and only if its negation $\neg\varphi$ is unsatisfiable.

Example 4. The formula

$$x \in y \wedge y \in z \wedge z \setminus x = \emptyset$$

is unsatisfiable. Therefore the formula

$$(x \in y \wedge y \in z) \rightarrow z \setminus x \neq \emptyset$$

is true (regardless of the assignment).

Instead, the formula

$$(x \in y \wedge y \in z) \rightarrow z \setminus x = \emptyset$$

is not true, since the formula

$$x \in y \wedge y \in z \wedge z \setminus x \neq \emptyset$$

is satisfiable.

In several cases, the decidability of a given theory has been shown by proving that it enjoys the SMALL MODEL PROPERTY, defined as follows.

Definition 7. An assignment $\mathcal{M}$ is RANK-BOUNDED by k if $\mathsf{rk}\ \mathcal{M}x \leq k$, for each variable x in the domain of $\mathcal{M}$.

Definition 8. A theory $\mathcal{T}$ enjoys the SMALL MODEL PROPERTY if there exists a computable function $f_{\mathcal{T}}$ such that any *satisfiable* formula φ of $\mathcal{T}$ is satisfied by some (finite) model which is rank-bounded by $f_{\mathcal{T}}(|\varphi|)$.[6]

[5] Moreover, when the arguments used to show that a formula is true are formalizable within the *ZFC* axiomatic system, a satisfiability test can be used also to establish theoremhood in *ZFC*.

[6] By $|\varphi|$ we mean the *size* of φ.

Plainly, if a theory $\mathcal{T}$ enjoys the small model property, then it has a solvable decision problem. Indeed, a possible, though very rough, satisfiability test for $\mathcal{T}$ would be the following. Let $f_{\mathcal{T}}$ be a computable function which rank-bounds the theory $\mathcal{T}$. Then, to check whether a given formula φ of $\mathcal{T}$ is satisfiable, one can systematically verify whether any of the finitely many assignments over the free variables of φ and rank-bounded by $f_{\mathcal{T}}(|\varphi|)$ satisfies φ.

Below is a very short list of some of the fragments of set theory which have a solvable decision problem: after each acronym, we list the operators and predicate symbols admitted in the fragment, and some references to the literature.

MLS:	$\cup, \cap, \setminus, \subseteq, =, \in$	(cf. [9])
MLSS:	$\cup, \cap, \setminus, \subseteq, =, \in, \{\cdot\}$	(cf. [9])
MLSSP:	$\cup, \cap, \setminus, \subseteq, =, \in, \{\cdot\}, \mathscr{P}$	(cf. [2] and [6])
MLSU:	$\cup, \cap, \setminus, \subseteq, =, \in, \bigcup$	(cf. [4])
...	...	...

The interested reader is referred to [3] and [5] for an extensive treatment of such results.

4. Transitive partitions, formative processes, and $\mathcal{P}$-graphs .

In this section we introduce the important notions of transitive partitions, formative processes, and $\mathcal{P}$-graphs .

4.1. Transitive partitions.

Definition 9. A TRANSITIVE PARTITION is a collection of pairwise disjoint nonempty sets, whose union is transitive.

A transitive partition Σ satisfies a formula φ if there exists a map

$$\Im : \mathrm{var}(\varphi) \to \mathscr{P}(\Sigma)$$

such that the induced assignment

$$\mathcal{M}x =_{\mathrm{Def}} \bigcup_{s \in \Im(x)} s$$

satisfies φ.

Given a finite transitive partition Σ and a formula φ, one can effectively verify whether Σ satisfies φ by simply checking all the possibilities.

Theorem 2. *Let Σ and $\widehat{\Sigma}$ be transitive partitions, and let $\beta : \Sigma \to \widehat{\Sigma}$ be a bijection. Moreover, let φ be a set-theoretic formula involving only unquantified variables and the symbols $\cup$, $\cap$, $\setminus$, $\subseteq$, $\not\subseteq$, $=$, $\neq$. If φ is satisfied by Σ, then it is also satisfied by $\widehat{\Sigma}$.*

Theorem 2, which can easily be deduced by a slight modification of Lemmas 3.3 and 10.1 in [6], yields the following immediate decision test for set-theoretic formulae involving only unquantified variables and the symbols $\cup$, $\cap$, $\setminus$, $\subseteq$, $\not\subseteq$, $=$, $\neq$:

> Given a formula φ in the above language, involving n distinct free variables, declare that φ is satisfiable provided that it is satisfiable by any (transitive) partition of size $2^n - 1$.

To handle new operators, we can add further suitable constraints to the bijection β. Here are some examples which have been treated in our recent research.

Theorem 3. *Let Σ and $\widehat{\Sigma}$ be transitive partitions, and let $\beta : \Sigma \to \widehat{\Sigma}$ be a bijection such that*

$$\bigcup \beta[X] \in \bigcup \beta[Y] \quad \textit{iff} \quad \bigcup X \in \bigcup Y\,, \qquad \textit{for all } X, Y \subseteq \Sigma$$

(i.e., $\widehat{\Sigma}$ $\in$-simulates Σ via β), and

$$\bigcup \beta[X] = \mathscr{P}(\bigcup \beta[Y]) \quad \textit{if} \quad \bigcup X \in \mathscr{P}(\bigcup Y)\,, \qquad \textit{for all } X, Y \subseteq \Sigma$$

(i.e., $\widehat{\Sigma}$ $\mathcal{P}$-simulates Σ via β).

Moreover, let φ be a set-theoretic formula involving only unquantified variables and the symbols $\cup$, $\cap$, $\setminus$, $\subseteq$, $=$, $\in$, $\notin$, $\mathscr{P}$.

If φ is satisfied by Σ, then it is also satisfied by $\widehat{\Sigma}$.

Theorem 3 can easily be deduced by a slight modification of Lemmas 3.3 and 10.1 in [6].

The last example involves finite enumerations.

Theorem 4. *Let φ be a set-theoretic formula involving only unquantified variables and the symbols $\cup$, $\cap$, $\setminus$, $\subseteq$, $=$, $\in$, $\notin$, $\mathscr{P}$, $\{\cdot\}$, where each term of type $\{w_1, \ldots, w_H\}$ in φ is such that $H \le L$.*

Let Σ and $\widehat{\Sigma}$ be transitive partitions, and let $\beta : \Sigma \to \widehat{\Sigma}$ be a bijection such that

$$\bigcup \beta[X] = \left\{\bigcup \beta[Y_1], \ldots, \bigcup \beta[Y_H]\right\} \quad \textit{if} \quad \bigcup X = \left\{\bigcup Y_1, \ldots, \bigcup Y_H\right\}\,,$$

for all $X, Y_1, \ldots, Y_H \subseteq \Sigma$, $H \leq L$ (i.e., $\widehat{\Sigma}$ L-simulates Σ via β) and such that $\widehat{\Sigma}$ $\in$-simulates and $\mathcal{P}$-simulates Σ via β.

If φ is satisfied by Σ, then it is also satisfied by $\widehat{\Sigma}$.

Theorem 4 follows from Lemmas 3.3 and 10.1 in [6].

The above Theorems 3 and 4 are at the base of a decision test for **MLSSP**, provided that we are able to prove that there exists a computable function f such that for any given finite transitive partition Σ and any $L \in \mathbb{N}$ there exists a "small" transitive partition $\widehat{\Sigma}$, rank-bounded by $f(|\Sigma|)$, which $\in$-, $\mathcal{P}$-, and L-simulates Σ via some bijection $\beta : \Sigma \to \widehat{\Sigma}$. It is therefore important to study the combinatorial properties of transitive partitions.

4.2. Formative processes.

A transitive partition Σ can be constructed by monotone sequences of approximations:

$$\emptyset = p^{(0)} \subseteq p^{(1)} \subseteq p^{(2)} \subseteq \ldots \subseteq p^{(\xi)} = p,$$

for each $p \in \Sigma$, which can be recursively defined as follows.

Firstly, let us put $p^{(0)} = \emptyset$, for each $p \in \Sigma$.

Next, let us assume that after μ steps we have constructed a set $p^{(\mu)} \subseteq p$, for each $p \in \Sigma$. If $\{p^{(\mu)} : p \in \Sigma\} = \Sigma$, then we are done. Otherwise, let us select $c \in \bigcup_{p\in\Sigma}(p \setminus p^{(\mu)})$ of minimal rank. By the transitivity of Σ and the minimality of rank(c), we have $c \subseteq \bigcup_{p\in\Sigma} p^{(\mu)} \subseteq \bigcup_{p\in\Sigma} p$.

Let $A_\mu =_{\mathrm{Def}} \{p \in \Sigma : c \cap p \neq \emptyset\}$ and let $q \in \Sigma$ be such that $c \in q$ (A_μ is called a $\mathcal{P}$-NODE and q is a TARGET of A_μ).

The $\mathcal{P}$-node A_μ is the subset of Σ which is needed to build "new" sets at stage μ of the construction, whereas the targets are the elements of Σ which receive such "new" sets.

At each step, constructions are carried out through the operator

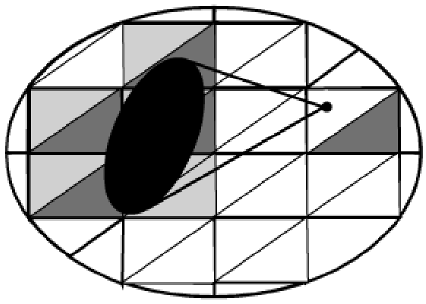
$$\mathscr{P}^*(X) =_{\mathrm{Def}} \{s \subseteq \bigcup X : s \cap x \neq \emptyset, \text{ for each } x \in X\}$$

which, given a collection X, constructs the family of all subsets of $\bigcup X$ which intersect all sets in X.

In fact, it turns out that

$$c \in \mathscr{P}^*(A_\mu^{(\mu)}) \setminus \bigcup_{p\in\Sigma} p^{(\mu)}, \tag{2}$$

where $A_\mu^{(\mu)} =_{\mathrm{Def}} \{q^{(\mu)} \ : \ q \in A_\mu\}$, i.e., $A_\mu^{(\mu)}$ is the $\mathscr{P}$-node A_μ at stage μ. Roughly speaking, (2) indicates that the element c appears in the construction process at step $\mu + 1$, for the first time.



Therefore we put

$$p^{(\mu+1)} =_{\mathrm{Def}} p^{(\mu)} \cup (p \cap \mathscr{P}^*(\{p^{(\mu)}) \ : \ p \in A_\mu\})), \ \text{for } p \in \Sigma$$

$$T_\mu =_{\mathrm{Def}} \{p \in \Sigma \ : \ p^{(\mu+1)} \neq p^{(\mu)}\}$$

(T_μ is the set of TARGETS at step μ).
At any limit ordinal λ, we put

$$p^{(\lambda)} =_{\mathrm{Def}} \bigcup_{\nu<\lambda} p^{(\nu)}, \quad \text{for } p \in \Sigma.$$

Such process terminates after ξ steps, where ξ is the minimum ordinal such that $p^{(\xi)} = p$, for each $p \in \Sigma$.

The sequence $(\{p^{(\mu)}\}_{p\in\Sigma})_{\mu\leq\xi}$ is called a FORMATIVE PROCESS for Σ.

The sequence $(A_\mu, T_\mu)_{\mu<\xi}$ is the HISTORY of the formative process $(\{p^{(\mu)}\}_{p\in\Sigma})_{\mu\leq\xi}$.

4.3. $\mathscr{P}$-graphs

Definition 10. Let Σ be a transitive partition and let $(\{p^{(\mu)}\}_{p\in\Sigma})_{\mu\leq\xi}$ be a formative process for Σ, with history $(A_\mu, T_\mu)_{\mu<\xi}$. The (expanded) $\mathscr{P}$-PGRAPH Γ of Σ is defined by putting:

$$\mathit{nodes}(\Gamma) = \{A_\mu \ : \ \mu < \xi\} \cup \Sigma$$

$$\mathit{edges}(\Gamma) = \{(A_\mu, q) \ : \ \mu < \xi \ , \ q \in T_\mu\} \cup \{(p, A_\mu) \ : \ \mu < \xi \ , \ p \in A_\mu\}.$$

(Notice that $|\mathit{nodes}(\Gamma)| \leq |\Sigma| + 2^{|\Sigma|}$.)

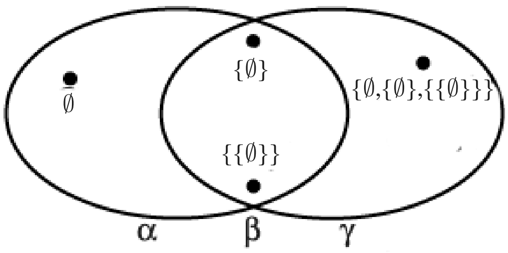
The transitive partition Σ is a said to be a REALIZATION of the induced $\mathscr{P}$-PGRAPH Γ.

The $\wp$-pgraph Γ is a kind of flow graph, whose source is the $\mathcal{P}$-node $\emptyset$.

Example 5. Consider the following transitive partition $\Sigma = \{\alpha, \beta, \gamma\}$, where

$$\begin{aligned}
\alpha &= \{\emptyset\} \\
\beta &= \{\underline{\{\emptyset\}}, \underline{\{\{\emptyset\}\}}\} \\
\gamma &= \{\underline{\{\emptyset, \underline{\{\emptyset\}}, \underline{\{\{\emptyset\}\}}\}}\}
\end{aligned}$$

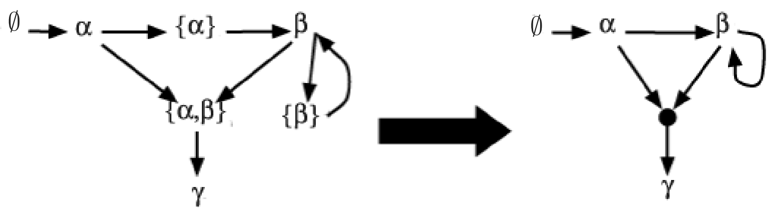
(underlines are an aid to the reader to match parentheses).



We have

$$\begin{aligned}
&\mathscr{P}^*(\emptyset) \cap \alpha \neq \emptyset \\
&\mathscr{P}^*(\{\alpha\}) \cap \beta \neq \emptyset \\
&\mathscr{P}^*(\{\beta\}) \cap \beta \neq \emptyset \\
&\mathscr{P}^*(\{\alpha, \beta\}) \cap \gamma \neq \emptyset
\end{aligned}$$

Below is the corresponding $\mathcal{P}$-pgraph both in extended and contracted form. [7]



[7] The contracted form of a $\mathcal{P}$-pgraph retains all information present in the corresponding extended form. We refrain from giving here a formal definition, since it can easily be deduced from the picture.

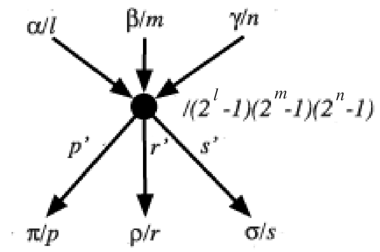
Summing up, we have the following formative process

$$\begin{pmatrix} \alpha^{(\mu)} \\ \beta^{(\mu)} \\ \gamma^{(\mu)} \end{pmatrix}_{\mu \le 4} = \left(\begin{array}{c|c|c|c|c} \emptyset & \{\emptyset\} & \{\emptyset\} & \{\emptyset\} & \{\emptyset\} \\ \emptyset & \emptyset & \{\{\emptyset\}\} & \{\{\emptyset\}, \underline{\{\{\emptyset\}\}}\} & \{\{\emptyset\}, \underline{\{\{\emptyset\}\}}\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \underline{\{\{\emptyset, \{\emptyset\}, \underline{\{\{\emptyset\}\}}\}\}} \end{array}\right)$$

with history

$$\begin{pmatrix} A_\mu \\ T_\mu \end{pmatrix}_{\mu < 4} = \left(\begin{array}{c|c|c|c} \emptyset & \{\alpha\} & \{\beta\} & \{\alpha, \beta\} \\ \{\alpha\} & \{\beta\} & \{\beta\} & \{\gamma\} \end{array}\right).$$

Remark 1. During the "flow" propagation in $\mathscr{P}$-graphs , one can focus just on cardinalities of sets, rather than on the actual sets which are propagated (with the only exception of the last element propagated during the final discharge).



For instance, the above pictures illustrates the situation in which at a certain stage μ the place α has received l elements, the place β has received m elements, etc. The intermediate unlabeled node can distribute up to $(2^l - 1) \cdot (2^m - 1) \cdot (2^n - 1)$ elements (this is the cardinality of $\mathscr{P}^*(\{\alpha^\mu, \beta^\mu, \gamma^{mu}\})$), p' of which have been assigned to the target π, r' have been assigned to the target ρ, and s' have been assigned to the target σ.

A more detailed and different use of formative processes can be found in [7] and [11], where they are used to enlarge a given transitive partition.

5. Thinning of formative processes as a technique for deciding certain classes of formulae.

As argued in Section 4, the capability of L-simulating a given finite transitive partition Σ by means of another partition $\widehat{\Sigma}$ having finite rank, bounded by a computable function in L and $|\Sigma|$ (the small model property), is crucial in order to solve the decision problem for collections of literals, such as **MLSSP**. Therefore, we need a technique which allows to prune those formative processes which exceed the desired bound.

Definition 11. Let Σ be a finite transitive partition and let $(\{p^{(\mu)}\}_{p\in\Sigma})_{\mu\leq\xi}$ be a formative process for Σ, with history $(A_\mu, T_\mu)_{\mu<\xi}$.

Let $L \in \mathbb{N}$ and let $\rho \in \mathbb{N}$ be such that $2^{\rho-1} > \max(\rho \cdot \Sigma, L)$.

The ν-th step of $(\{p^{(\mu)}\}_{p\in\Sigma})_{\mu\leq\xi}$ is SALIENT if

- $|p^{(\nu)}| < \rho$, for each $p \in A_\nu$, or
- for some $q \in \Sigma$ such that $q^{(\nu+1)} \neq q^{(\nu)}$ we have either $|q^{(\nu)}| < \rho$ or $q^{(\nu)} \cap \mathscr{P}^*(\{p^{(\nu)} : p \in A_\nu\}) = \emptyset$, or
- $\{p^{(\nu)} : p \in A_\nu\} = A_\nu$ and $\bigcup A_\nu \in \bigcup \Sigma$.

We denote the collection of all salient steps by Sal.

All the proofs of the results stated below can be found in [6]. The following result gives a numerical bound on the number of relevant steps to build a new smaller transitive partition from a given one.

Theorem 5. $|Sal| < \rho \cdot |\Sigma| \cdot 2^{|\Sigma|} + 3$.

Let $\mu_0, \mu_1, \ldots, \mu_\ell$ be such that

- $\mu_0 < \mu_1, \ldots < \mu_\ell$, and
- $\{\mu_0, \mu_1, \ldots, \mu_\ell\} = Sal \cup \{\xi\}$.

Using the original formative process as an oracle, it is possible, to extract effectively a shorter formative process which has, as final product, a new transitive partition that L-simulates the original one.

$$(A_0, T_0)\,,\ (A_1, T_1)\,,\ (A_2, T_2)\,,\ (A_3, T_3)\,,\ (A_4, T_4)\,,\ (A_5, T_5)\,,\ (A_6, T_6)\,, \ldots$$

History

$$\boxed{(A_0, T_0)},\ (A_1, T_1)\,,\ \boxed{(A_2, T_2)},\ (A_3, T_3)\,,\ (A_4, T_4)\,,\ (A_5, T_5)\,,\ \boxed{(A_6, T_6)}, \ldots$$

Salient Steps

$$\boxed{(A_0, T_0)},\ \xcancel{(A_1, T_1)}\,,\ \boxed{(A_2, T_2)},\ \xcancel{(A_3, T_3)}\,,\ \xcancel{(A_4, T_4)}\,,\ \xcancel{(A_5, T_5)}\,,\ \boxed{(A_6, T_6)}, \ldots$$

History of a thinner formative process

Theorem 6. *There exists a formative process* $(\{\widehat{p}^{(j)}\}_{p\in\Sigma})_{j\leq\ell}$ *(with history* $(A_{\mu_j}, T_{\mu_j})_{j<\ell}$*) for a transitive partition* $\widehat{\Sigma}$ *which* L*-simulates* Σ.

Then, in view of the results in Section 4, we can conclude

Theorem 7. *Let φ be a satisfiable* **MLSSP** *conjunction with m variables. Then φ is satisfiable by a transitive partition of rank at most*

$$\left\lceil \frac{25}{24}m + 5 \right\rceil \cdot 2^{2^m+m} + 3$$

and size at most $2^m - 1$.

The above theorems are based on the following elementary combinatorial property of formative processes.

Lemma 1. *Let Σ be a finite transitive partition and let $(\{p^{(\mu)}\}_{p\in\Sigma})_{\mu\leq\xi}$ be a formative process for Σ, with history $(A_\mu, T_\mu)_{\mu<\xi}$.*

Let $\rho \in \mathbb{N}$ such that $2^{\rho-1} > \rho \cdot |\Sigma|$.

If for some $\overline{\mu} < \xi$ and $\overline{q} \in A_{\overline{\mu}}$ we have $|\overline{q}^{(\overline{\mu})}| \geq \rho$ and

$$\mathscr{P}^*(\{p^{(\overline{\mu})} \,:\, p \in A_{\overline{\mu}}\}) \subseteq \bigcup \Sigma\,,$$

then

$$\left| \bigcup_{p\in T_{\overline{\mu}}} (\overline{p}^{(\mu+1)} \setminus p^{(\overline{\mu})}) \right| > \rho \cdot |\Sigma|,$$

and therefore there must exist an $r \in T_{\overline{\mu}}$ such that $|\overline{r}^{(\mu+1)} \setminus r^{(\overline{\mu})}| \geq \rho$.

A quick and meaningful exposition of the above facts can be found in [8].

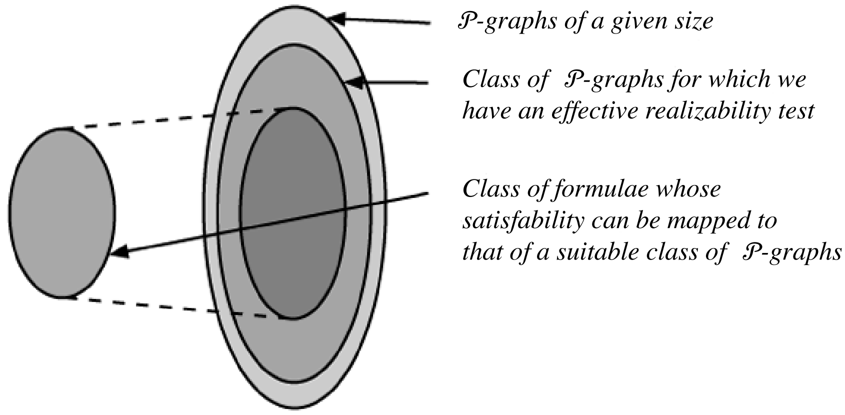
6. Conclusions and future work.

Decidability of **MLSSP** is $\mathcal{NP}$-hard. Hence, there is no hope to find a polynomial-time decision test for it. Nevertheless, it appears that its $\mathcal{NP}$-hardness is caused by very particular phenomena. It would therefore be very convenient to be able to characterize, for any given formula φ, the collection of compatible $\mathcal{P}$-graphs , namely those $\mathcal{P}$-graphs which are induced by models of φ. Then select those $\mathcal{P}$-graphs which are realizable by some formative process and try to construct a model for φ using such $\mathcal{P}$-graphs as oracles.

Having such approach in mind, it is rather clear why it is important

- to establish in an effective way whether a given $\mathcal{P}$-pgraph is realizable by a formative process (even in the presence of constraints);

- to find the length of a shortest formative process which realizes a given realizable constrained $\mathcal{P}$-pgraph.
- to single out classes of constrained $\mathcal{P}$-graphs which admit an effective realizability test;
- to develop new techniques to describe formative processes through finite skeletons.



It would also be important to develop new techniques for the realizability of $\mathcal{P}$-graphs endowed with specific constraints which could yield the decidability of new extensions of **MLSS** (e.g., **MLSS** plus the cartesian product).

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