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COMPUTING THE RECIPROCAL DISTANCE SIGNLESS LAPLACIAN EIGENVALUES AND ENERGY OF GRAPHS

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In this paper, we study the eigenvalues of the reciprocal distance signless Laplacian matrix of a connected graph and obtain some bounds for the maximum eigenvalue of this matrix. We also focus on bipartite graphs and find some bounds for the spectral radius of the reciprocal distance signless Laplacian matrix of this class of graphs. Moreover, we give bounds for the reciprocal distance signless Laplacian energy.

1. Introduction

In this paper, we consider only connected, undirected, simple and finite graphs, i.e, graphs on a finite number of vertices without multiple edges or loops. We use standard terminology; for concepts not defined here, we refer the reader to any standard graph theory monograph, such as [8], [16], [29] or [30]. A graph is (usually) denoted by $G = (V(G), E(G))$, where $V(G)$ is its vertex set and $E(G)$ its edge set. The *order* of G is the number $n = |V(G)|$ of its vertices and its *size* is the number $m = |E(G)|$ of its edges. The set of vertices adjacent to $v_i \in V(G)$, denoted by $N(v_i)$, refers to the *neighborhood* of v_i, and the *degree* of v_i means

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the cardinality of $N(v_i)$ and denoted by d_i or $deg_G(v_i)$. The *distance* between two vertices v_i and v_j, denoted by $d_G(v_i,v_j)$ or d_{ij}, is defined as the length of a shortest path between v_i and v_j in G. In particular, $d_G(u,u)=0$ for any vertex $u \in V(G)$. The *diameter* of G is the maximum distance between any pair of vertices and is denoted by $diam(G)$. The *distance matrix* of G is denoted by $D(G)$ and defined by $D(G)=[d_G(v_i,v_j)]_{v_i,v_j\in V(G)}$. The *transmission of a vertex* v, denoted by $Tr_G(v)$ is defined to be the sum of the distances from v to all other vertices in G, that is, $Tr_G(v)=\sum_{u\in V(G)} d_G(u,v)$. The *transmission of a connected graph* G, denoted by $\sigma(G)$, is the sum of the distances between all unordered pairs of vertices in G. Clearly, $\sigma(G)=\frac{1}{2}\sum_{v\in V(G)} Tr_G(v)$. A graph G is said to be *transmission regular* if $Tr_G(v)$ is a constant for each $v \in V(G)$. For $1 \le i \le n$ and $v_i \in V(G)$, one can easily see that $Tr_G(v_i)$ is just the i-th row sum of $D(G)$.

The *Harary matrix* $RD(G)$ of G, which is also called as the *reciprocal distance matrix*, is an $n\times n$ matrix (RD_{ij}), whose (i,j)-entry is equal to $\frac{1}{d_{ij}}$ if $i \neq j$ and 0 otherwise. The *Harary index* of a graph G, denoted by $H(G)$, is defined in [31] as

$$H(G)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} RD_{ij}=\sum_{i<j}\frac{1}{d_{ij}}.$$

In chemistry, in many instances the distant atoms influence each other much less than near atoms. The Harary matrix was introduced by Ivanciuc et al. [26] as an important molecular matrix to study this interaction. It was successfully used in a study concerning computer generation of acyclic graphs based on local vertex invariants and topological indices [27].

The reciprocal transmission $Tr'_G(v)$ of a vertex v is defined as

$$Tr'_G(v)=\sum_{u\in V(G)}\frac{1}{d_G(u,v)}, \quad u\neq v$$

and $Tr'(G)$ is the diagonal matrix

$$Tr'(G)=diag[Tr'_G(v_1),Tr'_G(v_2),\ldots,Tr'_G(v_n)].$$

For $1\le i\le n$, one can easily see that $Tr'_G(v_i)$ is just the i-th row sum of $RD(G)$. A graph G is said to be *reciprocal transmission regular* if for any pair of vertices u and v, we have $Tr'_G(v)=Tr'_G(u)$.

Clearly, $H(G)=\frac{1}{2}\sum_{v\in V(G)} Tr'_G(v)$.

We define the *reciprocal distance signless Laplacian matrix* as $RQ(G) = Tr'(G) + RD(G)$. In fact, $RQ(G) = [r_{ij}]$, where

$$r_{ij} = \begin{cases} \frac{1}{d_{ij}} & \text{if} \quad i \neq j \\ \sum_{j=1}^{n} \frac{1}{d_{ij}} & \text{if} \quad i = j. \end{cases}$$

The investigation of matrices related to various graphical structures is a very large and growing area of research. In particular, distance signless Laplacian matrix (spectrum) has attracted a good attention in the literature, since it has many useful applications [2]. The matrix $RQ(G)$ is irreducible, non-negative, symmetric and positive semidefinite. Let $\rho_i = \rho_i(G), i = 1, 2, \dots, n$ be the eigenvalues of the reciprocal distance signless Laplacian matrix $RQ(G)$ and they can be labeled in the non-increasing order as $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The collection of the eigenvalues is called the spectrum. The largest eigenvalue ρ_1 of $RQ(G)$ is called the reciprocal distance signless Laplacian *spectral radius* of G. By the Perron-Frobenius theorem, there is a unique normalized positive eigenvector of $RQ(G)$ corresponding to ρ_1, which is called the (reciprocal distance signless Laplacian) perron vector of G.

A column vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G) = \{v_1, v_2, \dots, v_n\}$, which maps a vertex v_i to x_i, i.e., $x(v_i) = x_i$ for $i = 1, 2, \dots, n$. Then,

$$x^T RQ(G) x = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{d_{uv}} (x(u) + x(v))^2, \tag{1}$$

and ρ is an eigenvalue of $RQ(G)$ corresponding to the eigenvector x if and only if $x \neq 0$ and for each $v \in V(G)$,

$$\rho x(v) = \sum_{u \in V(G)} \frac{1}{d_{uv}} (x(u) + x(v)). \tag{2}$$

These equations are called the (ρ, x)-eigenequations of G. For a normalized column vector $x \in \mathbb{R}^n$ with at least one non-negative component, by the Rayleigh's principle, we have

$$\rho(G) \geq x^T RQ(G) x,$$

with equality if and only if x is the principal eigenvector (Perron vector) of G, see [9].

The paper is organized as follows. In the Section 2, we get bounds for the eigenvalues of the reciprocal distance signless Laplacian matrix of graphs. In the Section 3, the bounds for the eigenvalues of the reciprocal distance signless Laplacian matrix of bipartite graphs are obtained. In the Section 4, we get the eigenvalues of the reciprocal distance signless Laplacian matrix of graphs obtained by some graph operations. In the Section 5, we get the bounds for the reciprocal distance signless Laplacian energy of graphs.

2. Bounds for the reciprocal distance signless Laplacian spectrum

Bounds for the spectral radius of graphs have been obtained in [40]. Spectral properties of the distance matrix are reported in [41]. Results on the eigenvalues of the Harary matrix have been done in [10, 12, 22, 26, 27, 43]. In this section, we obtain some lower and upper bounds for the maximum eigenvalue of the reciprocal distance signless Laplacian matrix. We start this section with the following simple lemma.

Lemma 2.1. *Let G be a connected graph on n vertices. Then,*

$$\rho_1(G) \geq \frac{4H(G)}{n},$$

with equality if and only if G is reciprocal transmission regular.

Proof. Let $\mathbf{1} = \frac{1}{\sqrt{n}}(1,1,\dots,1)^T \in \mathbb{R}^n$. Obviously, $\mathbf{1}$ is normalized. We have

$$\rho_1(G) \geq \mathbf{1}^T RQ(G)\mathbf{1} = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_{uv}} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}}\right)^2 = \frac{4H(G)}{n}$$

with equality if and only if $\mathbf{1}$ is the principal eigenvector of G, i.e., $Tr'_G(v)$ is a constant for each $v \in V(G)$. □

Corollary 2.2. *Let G be a connected graph with $n \geq 2$ vertices, m edges and diameter $d = diam(G)$. Then*

$$\rho_1(G) \geq \frac{4m}{n} + \frac{2}{d}\left(n - 1 - \frac{2m}{n}\right), \tag{3}$$

with equality if and only if $G = K_n$ or G is a regular graph of diameter $d = 2$.

Proof. If $G = K_n$ or G is a regular graph of diameter $d = 2$, then it is easy to see that (3) is an equality. Conversely, since there are m pair of vertices at distance 1 and the remaining $\frac{n(n-1)}{2} - m$ pair of vertices are at distance at most d, we have

$$H(G) \geq m + \frac{1}{d}\left(\frac{n(n-1)}{2} - m\right)$$

with equality if and only if $d \leq 2$. Then using Lemma 2.1, the result follows. □

Corollary 2.3. *Let G be a triangle-free and quadrangle-free connected graph with $n \geq 2$ vertices, m edges and diameter d. Let $M_1(G) = \sum_{u \in V(G)} (deg_G(u))^2$. Then*

$$\rho_1(G) \geq \frac{2(n-1)}{d} + \frac{2m}{n} + \frac{(d-2)}{nd} M_1(G),$$

with equality if and only if G is reciprocal transmission regular and $d \leq 3$.

Proof. By similar argument as in Corollary 2.2, the definition of Harary index and the Lemma 2.1, the result follows. □

Lemma 2.4. [3] *Let B be a non-negative irreducible matrix with row sums $B_1, B_2, \ldots, B_n$. If $\rho_1(B)$ is the largest eigenvalue of B, then* $\min_{1 \leq i \leq n} B_i \leq \rho_1(B) \leq \max_{1 \leq i \leq n} B_i$, *with either equality if and only if $B_1 = B_2 = \cdots = B_n$.*

Now, in the following result, we easily obtain a simple upper bound.

Theorem 2.5. *Let G be a connected graph with $n \geq 2$ vertices, diameter $d = diam(G)$ and maximum vertex degree Δ. Then*

$$\rho_1(G) \leq n - 1 + \Delta, \tag{4}$$

with equality if and only if G is a regular graph of diameter at most two.

Proof. It is easy to see that $RQ_i = \sum_{j=1}^{n} r_{ij} \leq 2d_i + n - 1 - d_i = n - 1 + d_i$, where $d_i = deg_G(v_i)$, with equality if and only if $d \leq 2$ for all j. Obviously, $RQ_1 = \cdots = RQ_n = n - 1 + d_i$ if and only if $d_1 = d_2 = \cdots = d_n$ and $d \leq 2$ for all i, j, i.e., G is a regular graph of diameter at most two. By the Lemma 2.4, the maximum eigenvalue of an irreducible non-negative matrix is at most the maximum row sum of the matrix, which is attained if and only if all the row sums are equal. Now the result follows easily. □

Following theorem gives another upper bound for $\rho_1(G)$ in terms of order, size and maximum vertex degree.

Theorem 2.6. *Let G be a connected graph with $n \geq 2$ vertices, m edges and maximum vertex degree Δ. Then*

$$\rho_1(G) \leq \sqrt{\frac{1}{2}[2(n-1)^2 + 6m + 3(n-2)\Delta]}, \tag{5}$$

with equality if and only if G is a complete graph K_n.

Proof. Let $X = (x_1, x_2, \dots, x_n)^T$ be an unit eigenvector corresponding to the largest eigenvalue $\rho_1(G)$ of $RQ(G)$. We have

$$RQ(G)X = \rho_1(G)X. \tag{6}$$

From the i-th equation of (6) we have

$$\begin{aligned} \rho_1(G)x_i &= \sum_{k:k\neq i} \frac{1}{d_{ik}}(x_k + x_i) \\ &\leq \sqrt{\sum_{k:k\neq i} \frac{1}{d_{ik}^2} \sum_{k:k\neq i} (x_k + x_i)^2} \end{aligned} \tag{7}$$

by Cauchy-Schwarz inequality. Let $T_i^* = \sum_{k:k\neq i}^n \frac{1}{d_{ik}^2}, i = 1, 2, \dots, n$ and $T_p^* = \max_{i\in V} T_i^*$. Squaring both sides in (7) and taking sum for $i = 1$ to n, we get

$$\begin{aligned} \rho_1^2(G) &\leq \sum_{i=1}^n T_i^*(1 - x_i^2 + (n-1)x_i^2 + 1 - x_i^2 + (n-1)x_i^2) && (8) \\ &= \sum_{i=1}^n T_i^*(2 + (2n-4)x_i^2) \\ &\leq 2\sum_{i=1}^n T_i^* + (2n-4)T_p^* \qquad \text{as} \quad \sum_{i=1}^n x_i^2 = 1. && (9) \end{aligned}$$

Since, $T_i^* = \sum_{k:k\neq i}^n \frac{1}{d_{ik}^2} \leq d_i + \frac{1}{4}(n-1-d_i) = \frac{1}{4}(n-1+3d_i)$, and $T_p^* \leq \frac{1}{4}(n-1+3\Delta)$, that is, $\sum_{i=1}^n T_i^* \leq \frac{1}{4}(n(n-1) + 3(2m))$. Thus we have

$$\rho_1^2(G) \leq \frac{1}{2}[2(n-1)^2 + 6m + 3(n-2)\Delta]. \tag{10}$$

Thus, we complete the first part of the proof.

Now suppose that equality holds in (5). Then all inequalities in the above argument must be equalities. From equality in (10), G has diameter at most 2 and $T_i^* = \frac{1}{4}(n-1+3d_i), i = 1, 2, \dots, n$. From equality in (9), we get $T_1^* = T_2^* = \cdots = T_n^*$. Then $d_1 = d_2 = \cdots = d_n$, that is, G is a regular graph. If $d = 1$, then $G \cong K_n$. Otherwise, $d = 2$ and hence we have $d_{i,j} = 1$ or $d_{i,j} = 2$, for all i, j. Without loss of generality, we can assume that the shortest distance between vertex v_1 and v_n is 2. From equality in (7) and (8), we get $d_{i,1}x_1 = d_{i,2}x_2 = \cdots = d_{i,i-1}x_{i-1} = d_{i,i+1}x_{i+1} = \cdots = d_{i,n}x_n, i = 1, 2, \dots, n$ and for $i = 1$ we get $x_k = 2x_n, k \in N(1)$ and $x_k = x_n, k \notin N(1), k \neq 1$. Similarly, $i = n$ we get $x_k = 2x_1, k \in N(n)$ and $x_k = x_1, k \notin N(n), k \neq n$. Thus we have $x_1 = x_n$ and two type of eigencomponents x_1 and $2x_1$ in eigenvector X, which is a contradiction as G is regular graph of diameter 2. Hence G is a complete graph K_n.

Conversely, one can see easily that the equality holds in (5) for complete graph K_n, and the proof is complete. □

A subset X of a vertex set $V(G)$ of a graph G is said to be an *independent set*, if no two vertices of X are adjacent in G. The *independence number* $\beta(G)$ of G is the maximum number of vertices in the independent sets in G. The following theorem gives the lower bound for the reciprocal distance signless Laplacian spectral radius in terms of the order of G and the independence number. The *clique* of a graph G is the maximal complete induced subgraph of G [21].

Theorem 2.7. *Let G be any connected graph of order n. If one of the following conditions holds, then*

$$\rho_1(G) \leq \frac{3n-s-3+\sqrt{s(2n-3s+2)+n(n-2)+1}}{2},$$

(i) *X be the maximum set with* $\beta(G)=|X|=s$*,*
(ii) *G is having a clique* K_s *of order s and* $G(n,s)$ *be the graph obtained from G by removing the edges of* K_s*, where* $0 \leq s \leq n-1$.

Proof. (i) Let X be the maximum independent set with $\beta(G)=|X|=s$ and suppose x be the principal eigenvector of G. It is easily seen that the components of x have the same value, say $x(v_1)$ for vertices in X and $x(v_n)$ for vertices in $V(G)\setminus X$. Then, by the $(\rho_1(G),x)$-eigenequations of G, we have

$$\rho_1(G)x(v_1) \leq \frac{1}{2}(s-1)(x(v_1)+x(v_1))+(n-s)(x(v_1)+x(v_n))$$

and

$$\rho_1(G)x(v_n) \leq (n-s-1)(x(v_n)+x(v_n))+s(x(v_1)+x(v_n)).$$

Thus $\rho_1=\rho_1(G)$ is the largest root of the equation

$$\rho_1^2+(s-3n+3)\rho_1+(2n^2+s^2-4n-2sn+s+2) \leq 0.$$

(ii) Let the vertices of G be $v_1,v_2,\ldots,v_n$. Without loss of generality, let the vertex set of the clique K_s of G be $S_1=\{v_1,v_2,\ldots,v_s\}$ and the remaining vertices of G are $S_2=\{v_{s+1},v_{s+2},\ldots,v_n\}$. Let x be the principal eigenvector of G. It is easily seen that the components of x have the same value, say $x(v_1)$ for vertices in S_1 and $x(v_n)$ for vertices in S_2. This is done analogously as in the proof (i) above, and the proof is complete. □

Theorem 2.8. *Let G be a connected graph with n vertices. Suppose that* $Tr'_1 \geq Tr'_2 \geq \cdots \geq Tr'_n$*, where* $Tr'_i = Tr'(v_i)$.
(i) *If* $Tr'_1 > Tr'_{l+1}$*, where* $1 \leq l \leq n-1$*, then*

$$\begin{aligned}\rho_1(G) &\leq \frac{1}{2}\Bigg(Tr'_1+2Tr'_{l+1}-1\\ &+ \sqrt{(2Tr'_{l+1}-Tr'_1)^2+4Tr'_{l+1}(1-2l)-2Tr'_1(1-4l)+1}\Bigg),\end{aligned}$$

with equality if and only if $l \leq n-2, G$ *is a graph with* l *vertices of degree* $n-1$ *and the remaining* $n-l$ *vertices have equal degree less than* $n-1$.
(ii) *If* $Tr'_{n-k} > Tr'_n > k-1$, *where* $1 \leq k \leq n-1$, *then*

$$\begin{aligned} \rho_1(G) &> \frac{1}{2}\Bigg(Tr'_n + 2Tr'_{n-k} - 1 \\ &+ \sqrt{(2Tr'_{n-k} - Tr'_n)^2 - 8k(Tr'_{n-k} - Tr'_n) + 2(2Tr'_{n-k} - Tr'_n) + 1}\Bigg). \end{aligned}$$

Proof. (i) Let $V_1 = \{v_1, \dots, v_l\}$ and $V_2 = V(G) \setminus V_1$. Then the reciprocal distance signless Laplacian matrix can be in the form

$$RQ(G) = \begin{pmatrix} RQ_{11} & RQ_{12} \\ RQ_{21} & RQ_{22} \end{pmatrix},$$

where RQ_{11} is an $l \times l$ matrix. For $y > 1$(to be determined),

$$B = \begin{pmatrix} RD_{11} & \frac{1}{y}RD_{12} \\ yRD_{21} & RD_{22} \end{pmatrix} + \begin{pmatrix} Tr'_{11} & 0 \\ 0 & Tr'_{22} \end{pmatrix}$$

is a non-negative irreducible matrix that has the same spectrum as RQ, where $Tr'_{ii} = \sum_{t=1}^{2} RD_{it}$. If $i = 1, \dots, l$, then we have,

$$\begin{aligned} B_i = \sum_{j=1}^{l} \frac{1}{d_{ij}} + \frac{1}{y} \sum_{j=l+1}^{n} \frac{1}{d_{ij}} + \sum_{j=1}^{n} \frac{1}{d_{ij}} &= (1 + \frac{1}{y})Tr'_i + (1 - \frac{1}{y}) \sum_{j=1}^{l} \frac{1}{d_{ij}} \\ &\leq (1 + \frac{1}{y})Tr'_1 + (1 - \frac{1}{y})(l-1). \end{aligned}$$

If $i = l+1, \dots, n$, we have

$$B_i = y\sum_{j=1}^{l} \frac{1}{d_{ij}} + \sum_{j=l+1}^{n} \frac{1}{d_{ij}} + \sum_{j=1}^{n} \frac{1}{d_{ij}} = 2Tr'_i + (y-1) \sum_{j=1}^{l} \frac{1}{d_{ij}} \leq 2Tr'_{l+1} + (y-1)l.$$

Let

$$\begin{aligned} y &= \frac{1}{2l}\Bigg(2l + Tr'_1 - 2Tr'_{l+1} - 1 \\ &+ \sqrt{(2Tr'_{l+1} - Tr'_1)^2 + 4Tr'_{l+1}(1-2l) - 2Tr'_1(1-4l) + 1)}\Bigg). \end{aligned}$$

Then

$$\begin{aligned} &(1 + \frac{1}{y})Tr'_1 + (1 - \frac{1}{y})(l-1) = 2Tr'_{l+1} + (y-1)l \\ &= \frac{1}{2}\Bigg(2l + Tr'_1 - 2Tr'_{l+1} - 1 \\ &+ \sqrt{(2Tr'_{l+1} - Tr'_1)^2 + 4Tr'_{l+1}(1-2l) - 2Tr'_1(1-4l) + 1}\Bigg). \end{aligned}$$

Since $Tr'_1 > Tr'_{l+1}$, we have $y > 1$. Thus by Lemma 2.4, we have

$$\begin{aligned}\rho_1(G) \le \max_{1\le i\le n} B_i &\le \frac{1}{2}\Big(Tr'_1 + 2Tr'_{l+1} - 1\\ &+ \sqrt{(2Tr'_{l+1} - Tr'_1)^2 + 4Tr'_{l+1}(1-2l) - 2Tr'_1(1-4l) + 1}\Big).\end{aligned}$$

Suppose that

$$\begin{aligned}\rho_1(G) &= \frac{1}{2}\Big(Tr'_1 + 2Tr'_{l+1} - 1\\ &+ \sqrt{(2Tr'_{l+1} - Tr'_1)^2 + 4Tr'_{l+1}(1-2l) - 2Tr'_1(1-4l) + 1}\Big).\end{aligned}$$

Then

$$B_1 = \cdots = B_n = (1+\frac{1}{y})Tr'_1 + (1-\frac{1}{y})(l-1) = 2Tr'_{l+1} + (y-1)l.$$

Thus $\frac{1}{d_{ij}} = 1$ for $i, j = 1, \ldots, l$ with $j \ne i$, and for $i = l+1, \ldots, n$ and $j = 1, \ldots, l$, which implies that every vertex in V_1 is adjacent to all other vertices of G, and then the diameter of G is 2. Since $Tr'_{l+1} = \cdots = Tr'_n$ and $Tr'_1 > Tr'_{l+1}$, every vertex in V_2 has the same degree, say s, and $k, s \le n-2$.

(ii) Let $l = n-k$ and $y = \frac{1}{x}$, $0 < x < 1$. If $i = 1, \ldots, n-k$, then, we have

$$\begin{aligned}B_i = \sum_{j=1}^{n-k}\frac{1}{d_{ij}} + x\sum_{j=n-k+1}^{n}\frac{1}{d_{ij}} + \sum_{j=1}^{n}\frac{1}{d_{ij}} &= 2Tr'_i + (x-1)\sum_{j=n-k+1}^{n}\frac{1}{d_{ij}}\\ &\ge 2Tr'_{n-k} + (x-1)k.\end{aligned}$$

If $i = n-k+1, \ldots, n$, then, we have

$$\begin{aligned}B_i = \frac{1}{x}\sum_{j=1}^{n-k}\frac{1}{d_{ij}} + \sum_{j=n-k+1}^{n}\frac{1}{d_{ij}} + \sum_{j=1}^{n}\frac{1}{d_{ij}} &= (1+\frac{1}{x})Tr'_i + (1-\frac{1}{x})\sum_{j=n-k+1}^{n}\frac{1}{d_{ij}}\\ &\ge (1+\frac{1}{x})Tr'_n + (1-\frac{1}{x})(k-1).\end{aligned}$$

Let

$$\begin{aligned}x &= \frac{1}{2k}\Big(2k + Tr'_n - 2Tr'_{n-k} - 1\\ &+ \sqrt{(2Tr'_{n-k} - Tr'_n)^2 - 8k(Tr'_{n-k} - Tr'_n) + 2(2Tr'_{n-k} - Tr'_n) + 1}\Big).\end{aligned}$$

Then

$$\begin{aligned} & 2Tr'_{n-k}+(x-1)k=(1+\frac{1}{x})Tr'_n+(1-\frac{1}{x})(k-1) \\ = \ & \frac{1}{2}\Bigg(Tr'_n+2Tr'_{n-k}-1 \\ + \ & \sqrt{(2Tr'_{n-k}-Tr'_n)^2-8k(Tr'_{n-k}-Tr'_n)+2(2Tr'_{n-k}-Tr'_n)+1}\Bigg). \end{aligned}$$

Since $Tr'_{n-k} > Tr'_n > k-1$, then by Lemma 2.4, we have

$$\begin{aligned} \rho_1(G) \ \geq \ & \min_{1\leq i\leq n} B_i > \frac{1}{2}\Bigg(Tr'_n+2Tr'_{n-k}-1 \\ + \ & \sqrt{(2Tr'_{n-k}-Tr'_n)^2-8k(Tr'_{n-k}-Tr'_n)+2(2Tr'_{n-k}-Tr'_n)+1}\Bigg). \end{aligned}$$

If

$$\begin{aligned} \rho_1(G) \ = \ & \frac{1}{2}\Bigg(Tr'_n+2Tr'_{n-k}-1 \\ + \ & \sqrt{(2Tr'_{n-k}-Tr'_n)^2-8k(Tr'_{n-k}-Tr'_n)+2(2Tr'_{n-k}-Tr'_n)+1}\Bigg), \end{aligned}$$

then

$$B_1=\cdots=B_n=2Tr'_{n-k}+(x-1)k=(1+\frac{1}{x})Tr'_n+(1-\frac{1}{x})(k-1),$$

and thus $d_{ij}=1$ for $i=1,\ldots,n-k$ and $j=n-k+1,\ldots,n$, and for $i,j=n-k+1,\ldots,n$ with $j\neq i$, which implies that every vertex in V_1 is adjacent to all other vertices of G, and we have $Tr'_{n-k+1}=\cdots=Tr'_n=n-1$, contradicting the assumption that $Tr'_{n-k}>Tr'_n$. □

In the following, we give the result for ρ_1 of the Nordhaus-Gaddum type. Note that letting G be any graph, $\overline{G}$ stands for its complement.

Theorem 2.9. *Let G be a connected graph on $n\geq 4$ vertices with a connected $\overline{G}$. Then*

$$2(n-1)\left(1+\frac{1}{k}\right)\leq \rho_1(G)+\rho_1(\overline{G})<4n-6, \tag{11}$$

where $k=\max\{d,\overline{d}\}$ and d, $\overline{d}$ are the diameter of G and $\overline{G}$, respectively. Moreover, the equality for the lower bound holds in (11) if and only if both G and $\overline{G}$ are regular graph of diameter 2.

Proof. Using the inequality (3) from Corollary 2.2, we arrive at

$$\rho_1(G)+\rho_1(\overline{G}) \geq \frac{4m+4\overline{m}}{n}+\frac{2(n(n-1)-2m)}{nd}+\frac{2(n(n-1)-2\overline{m})}{n\overline{d}}, \quad (12)$$

where $\overline{m}$ and $\overline{d}$ are, respectively, the number of edges and diameter of $\overline{G}$. Since $m+\overline{m}=\frac{n(n-1)}{2}$ and $k=\max\{d,\overline{d}\}$, we get (11) from (12). First part of the proof is over.

Now suppose that equality holds in (11). Then the equality holds in (12) and $k=d=\overline{d}$. From equality in (12), we get both G and $\overline{G}$ are regular graph of diameter 2, by Corollary 2.2. Hence both G and $\overline{G}$ are regular graph of diameter 2.

Conversely, let both G and $\overline{G}$ be regular graph of diameter 2. Then $\rho_1(G)=n+r-1$ and $\rho_1(\overline{G})=2(n-1)-r$. Hence $\rho_1(G)+\rho_1(\overline{G})=3(n-1)$.

Since both G and $\overline{G}$ are connected, we have $\max_{1\leq i\leq n} deg_G(v_i)\leq n-2$. By Lemma 2.5,

$$\begin{aligned}
\rho_1(G)+\rho_1(\overline{G}) &\leq (n-1+\max_{1\leq i\leq n} deg_G(v_i))+(n-1+n-1-\min_{1\leq i\leq n} deg_G(v_i))\\
&= 3(n-1)+\max_{1\leq i\leq n} deg_G(v_i)-\min_{1\leq i\leq n} deg_G(v_i)\\
&\leq 3(n-1)+(n-2-1)=4n-6.
\end{aligned}$$

If $\rho_1(G)+\rho_1(\overline{G})=4n-6$, then $\max_{1\leq i\leq n} deg_G(v_i)=n-2$ and $\min_{1\leq i\leq n} deg_G(v_i)=1$ and so G cannot be regular. But by Lemma 2.5, both G and $\overline{G}$ are regular graphs of diameter two, a contradiction. The right inequality in (11) follows. □

Here we give the upper bound for $\rho_1(G)+\rho_1(\overline{G})$ in terms of order n, maximum vertex degree Δ and minimum vertex degree δ.

Theorem 2.10. *Let G be a connected graph on $n\geq 4$ vertices with a connected $\bar{G}$. Then*

$$\rho_1(G)+\rho_1(\bar{G}) \leq 2\sqrt{\frac{1}{2}[5(n-1)^2+\frac{3}{2}(n-2)(\Delta-\delta)]}. \quad (13)$$

Proof. Using the inequality (5) from Theorem 2.6, we arrive at

$$\begin{aligned}
&\rho_1(G)+\rho_1(\bar{G}) \leq\\
&\sqrt{\frac{1}{2}[2(n-1)^2+6m+3(n-2)\Delta]}+\sqrt{\frac{1}{2}[2(n-1)^2+6\bar{m}+3(n-2)\bar{\Delta}]}.\\
=\ &\sqrt{\frac{1}{2}[2(n-1)^2+6m+3(n-2)\Delta]}+\sqrt{\frac{1}{2}[8(n-1)^2-6m-3(n-2)\delta]}, \quad (14)
\end{aligned}$$

as $2\bar{m} = n(n-1) - 2m$, and $\bar{\Delta} = n-1-\delta$ where $\bar{m}$, is the number of edges of $\bar{G}$. Now we consider a function

$$f(m) = \sqrt{\frac{1}{2}[2(n-1)^2 + 6m + 3(n-2)\Delta]} + \sqrt{\frac{1}{2}[8(n-1)^2 - 6m - 3(n-2)\delta]}. \quad (15)$$

It is easy to show that

$$f(m) \le f(\frac{2(n-1)^2 - (n-2)(\delta+\Delta)}{4}) = 2\sqrt{\frac{1}{2}[5(n-1)^2 + \frac{3}{2}(n-2)(\Delta-\delta)]}.$$

From (14) and (15), we get the required result (13). □

3. On eigenvalues of the reciprocal distance signless Laplacian matrix of bipartite graphs

In this section, we will focus on bipartite graphs and find some bounds for the spectral radius of the reciprocal distance signless Laplacian matrix of this class of graphs.

Theorem 3.1. *Let G be a connected bipartite graph of order n and size m with bipartition of vertices as $V(G) = A \cup B$ where $|A| = p, |B| = q$. Then*

$$\rho_1(G) \le n - 1 + \sqrt{pq}, \quad (16)$$

with equality if and only if G is a complete bipartite graph $K_{p,q}$.

Proof. Let the vertex set of G can be partitioned as $A = \{v_1, v_2, \ldots, v_p\}$ and $B = \{v_{p+1}, v_{p+2}, \ldots, v_{p+q}\}$, where $p+q = n$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $RQ(G)$ corresponding to the maximum eigenvalue $\rho_1(G)$. We can assume that $x_i = \max_{v_k \in A} x_k$ and $x_j = \max_{v_k \in B} x_k$.
For $v_i \in A$,

$$\rho_1(G)x_i = \sum_{k=1, k\neq i}^{p} \frac{1}{d_{ik}}(x_k + x_i) + \sum_{k=p+1}^{p+q} \frac{1}{d_{ik}}(x_k + x_i) \le (p+q-1)x_i + qx_j. \quad (17)$$

For $v_i \in B$,

$$\rho_1(G)x_j = \sum_{k=1}^{p} \frac{1}{d_{jk}}(x_k + x_i) + \sum_{k=p+1, k\neq j}^{p+q} \frac{1}{d_{jk}}(x_k + x_i) \le px_i + (p+q-1)x_j. \quad (18)$$

Since G is a connected graph, $x_k > 0$ for all $v_k \in V$. From (17) and (18), we get $(\rho_1(G) - (p+q-1))(\rho_1(G) - (p+q-1)) \le pq$ as $x_i, x_j > 0$, that is

$$\rho_1^2(G) - 2(p+q-1)\rho_1(G) + (p^2 + q^2 + pq - 2p - 2q + 1) \le 0.$$

From this we get the required result (16).

Now, suppose that equality holds in (16). Then all inequalities in the above argument must be equalities. From equality in (17), we get

$$x_k = x_j \quad \text{and} \quad v_i v_k \in E(G) \quad \text{for} \quad \text{all} \quad v_k \in B.$$

From equality in (18), we get

$$x_k = x_i \quad \text{and} \quad v_j v_k \in E(G) \quad \text{for} \quad \text{all} \quad v_k \in A.$$

Thus each vertex in each set is adjacent to all the vertices on the other set and vice versa. Hence G is a complete bipartite graph $K_{p,q}$.

Conversely, one can easily see that (16) holds for $K_{p,q}$. □

Theorem 3.2. *Let G be a connected bipartite graph of order n and size m with bipartition of the vertex set as $V(G) = A \cup B$, where $|A| = p$ and $|B| = q$, $p+q = n$. Let Δ_A and Δ_B be the maximum degrees among vertices from A and B, respectively. Then*

$$\begin{aligned}\rho_1(G) \quad &\leq \quad \frac{2}{3}n + \frac{1}{3}(\Delta_A + \Delta_B) - 1 \\ &+ \quad \sqrt{\frac{1}{9}[n^2 + (\Delta_A + \Delta_B)^2 + 4(p\Delta_A + q\Delta_B)] - \frac{1}{3}(pq + 2(q\Delta_A + p\Delta_B))},\end{aligned}$$

with equality if and only if G is a complete bipartite graph $K_{p,q}$ or G is a semi-regular graph with every vertex eccentricity equal 3.

Proof. Let $A = \{v_1, v_2, \ldots, v_p\}$ and $B = \{v_{p+1}, v_{p+2}, \ldots, v_{p+q}\}$. Let also that $X = (x_1, x_2, \ldots, x_n)^T$ be a Perron eigenvector of $RQ(G)$ corresponding to the maximum eigenvalue $\rho_1(G)$ such that

$$x_i = \max_{v_k \in A} x_k \quad \text{and} \quad x_j = \max_{v_k \in B} x_k.$$

Then we have

$$\begin{aligned}\rho_1(G)x_i \quad &= \quad \sum_{k=1, k\neq i}^{p} \frac{1}{d_{ik}}(x_k + x_i) + \sum_{k=p+1}^{p+q} \frac{1}{d_{ik}}(x_k + x_i) \\ &\leq \quad (p + \frac{1}{3}(q + 2\Delta_A) - 1)x_i + \frac{1}{3}(q + 2\Delta_A)x_j.\end{aligned}$$

Analogously for the component x_j we have

$$\begin{aligned}\rho_1(G)x_j \quad &= \quad \sum_{k=1}^{p} \frac{1}{d_{jk}}(x_k + x_i) + \sum_{k=p+1, k\neq j}^{p+q} \frac{1}{d_{jk}}(x_k + x_i) \\ &\leq \quad (q + \frac{1}{3}(p + 2\Delta_B) - 1)x_j + \frac{1}{3}(p + 2\Delta_B)x_i.\end{aligned}$$

Combining these two inequalities, it follows

$$\left[\rho_1(G)-\left(p+\frac{1}{3}q+\frac{2}{3}\Delta_A-1\right)\right]\left[\rho_1(G)-\left(q+\frac{1}{3}p+\frac{2}{3}\Delta_B-1\right)\right] \leq \frac{1}{9}(q+2\Delta_A)(p+2\Delta_B).$$

Since $x_k > 0$ for $1 \leq k \leq p+q$,

$$\begin{aligned} &\rho_1(G)^2-\left(\frac{4}{3}n+\frac{2}{3}(\Delta_A+\Delta_B)-2\right)\rho_1(G) \\ + &\left(pq+\frac{1}{3}p^2+\frac{1}{3}q^2+\frac{2}{3}\Delta_B p+\frac{2}{3}\Delta_A q-\frac{2}{3}\Delta_A-\frac{2}{3}\Delta_B-\frac{4}{3}q-\frac{4}{3}p+1\right) \leq 0. \end{aligned}$$

From this inequality, we get the desired result.

For the case of equality, we have $x_i = x_k$ for $k = 1,2,\ldots,p$ and $x_j = x_k$ for $k = p+1, p+2, \ldots, p+q$. This means that the eigenvector x has at most two different coordinates, the degrees of vertices in A are equal to Δ_A, and the degrees of vertices in B are equal to Δ_B, implying that G is a semi-regular graph. If G is not a complete bipartite graph, it follows from $p\Delta_A = q\Delta_B$ that $\Delta_A < q$ and $\Delta_B < p$ and the eccentricity of every vertex must be equal to 3. □

4. Eigenvalues of the reciprocal distance signless Laplacian matrix of graphs obtained by some graph operations

The distance spectra of the graph composition has reported in [23]. In this section, we compute eigenvalues of the reciprocal distance signless Laplacian matrix with respect to some graph operations. The following lemma will be helpful in the sequel.

Lemma 4.1. [11] *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a symmetric 2×2 *block matrix. Then the spectrum of* A *is the union of the spectra of* $A_0 + A_1$ *and* $A_0 - A_1$.

The graph $G \nabla G$ is obtained by joining every vertex of G to every vertex of another copy of G.

Theorem 4.2. *Let* G *be a connected r-regular graph on n vertices. If* $r, \lambda_2, \lambda_3, \ldots, \lambda_n$ *are the eigenvalues of the adjacency matrix of* G, *then the eigenvalues of the reciprocal distance signless Laplacian matrix of* $G \nabla G$ *are*

$$\begin{aligned}
&3n+r-1, && \\
&n+r-1, && \textit{and} \\
&\tfrac{1}{2}(\lambda_i+3n+r-2), && 2 \textit{ times}, \quad i=2,3,\dots,n.
\end{aligned}$$

Proof. As G is an r-regular graph, the reciprocal distance signless Laplacian matrix of $G\nabla G$ can be written as

$$\begin{bmatrix} A+\frac{1}{2}\overline{A}+\left(\frac{3n-1+r}{2}\right)I & J \\ J & A+\frac{1}{2}\overline{A}+\left(\frac{3n-1+r}{2}\right)I \end{bmatrix},$$

where A is the adjacency matrix of G, $\overline{A}$ is the adjacency matrix of $\overline{G}$, J is a matrix whose all entries are equal to 1 and I is an identity matrix. Since $\overline{A}=J-I-A$, then by applying Lemma 4.1, we get the result. □

Definition 4.3. [25] Let G be a graph with vertex set $V(G)=\{v_1,v_2,\dots,v_n\}$. Take another copy of G with the vertices labeled by $\{u_1,u_2,\dots,u_n\}$ where u_i corresponds to v_i for each i. Make u_i adjacent to all the vertices in $N(v_i)$ in G, for each i. The resulting graph, denoted by D_2G is called the double graph of G.

Theorem 4.4. *Let G be a connected r-regular graph on n vertices with diameter 2 and let $r,\lambda_2,\lambda_3,\dots,\lambda_n$ be the eigenvalues of the adjacency matrix of G. Then the eigenvalues of the reciprocal distance signless Laplacian matrix of D_2G are*

$$\begin{aligned}
&2n+2r-1, && \\
&n+r-1, && n \textit{ times}, \ \textit{and} \\
&\lambda_i+n+r-1, && i=2,3,\dots,n.
\end{aligned}$$

Proof. By definition of D_2G, the reciprocal distance signless Laplacian matrix of D_2G is of the form

$$\begin{bmatrix} A+\frac{1}{2}\overline{A}+\left(n+r-\frac{1}{2}\right)I & A+\frac{1}{2}\overline{A}+\frac{1}{2}I \\ A+\frac{1}{2}\overline{A}+\frac{1}{2}I & A+\frac{1}{2}\overline{A}+\left(n+r-\frac{1}{2}\right)I \end{bmatrix},$$

where A is the adjacency matrix of G, $\overline{A}$ is the adjacency matrix of $\overline{G}$ and I is an identity matrix. Since $\overline{A}=J-I-A$, then by applying Lemma 4.1, the result follows. □

Definition 4.5. [8] Let G and H be two graphs on vertex sets $V(G)$ and $V(H)$, respectively. Then their lexicographic product $G[H]$ is a graph with vertex set

$V(G[H]) = V(G) \times V(H)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G[H]$ if and only if either
(i) u_1 is adjacent to v_1 in G or
(ii) $u_1 = v_1$ and u_2 is adjacent to v_2 in H.

Theorem 4.6. *Let G be a k-transmission regular graph of order p. Let H be an r-regular graph on n vertices with its adjacency eigenvalues* $r, \lambda_2, \ldots, \lambda_n$. *Let also that* $\mu_1, \mu_2, \ldots, \mu_p$ *be the eigenvalues of the* $RD(G)$. *Then the eigenvalues of the reciprocal distance signless Laplacain matrix of* $G[H]$ *are*

$$n\mu_i + kn + n + r - 1, \qquad i = 1, 2, \ldots, p \text{ and}$$

$$\tfrac{1}{2}(\lambda_j + n + r) + kn - 1, \quad p \text{ times}, \ j = 2, 3, \ldots, n.$$

Proof. By a suitable ordering of vertices of $G[H]$, its RQ-matrix F, can be written in the form as

$$F = RD_G \otimes J_n + I_p \otimes (A + \frac{1}{2}\overline{A} + \frac{1}{2}(2kn + n - 1)I + \frac{1}{2}Diag(Deg)),$$

where $\overline{A}$ denote the adjacency matrix of $\overline{G}$.
Since H is r-regular, the all one column vector $\mathbf{1}$ of order $n \times 1$ is an eigenvector of A with an eigenvalue r. Then, the all one vector $\mathbf{1}$ is an eigenvector of $A + \frac{1}{2}\overline{A} + \frac{1}{2}(2kn + n - 1)I + \frac{1}{2}Diag(Deg)$ with an eigenvalue $kn + n + r - 1$. Similarly if λ_j is any other eigenvalue of A with eigenvector Y_j, then Y_j is an eigenvector of $A + \frac{1}{2}\overline{A} + \frac{1}{2}(2kn + n - 1)I + \frac{1}{2}Diag(Deg)$ with eigenvalue $\frac{1}{2}(\lambda_i + n + r) + kn - 1$ and that Y_j is orthogonal to $\mathbf{1}$.

Let $X_i = [x_1^i \quad x_2^i \quad \ldots \quad x_p^i]^T$ be an eigenvector corresponding to the eigenvalue μ_i of RD_G. Therefore

$$RD_G.X_i = \mu_i X_i$$

Now

$$\begin{aligned}
F.(X_i \otimes \mathbf{1_n}) &= (RD_G \otimes J_n \\
&+ I_p \otimes (A + \frac{1}{2}\overline{A} + \frac{1}{2}(2kn + n - 1)I + \frac{1}{2}Diag(Deg))(X_i \otimes \mathbf{1_n}) \\
&= (RD_G.X_i) \otimes (J_n.\mathbf{1_n}) \\
&+ (I_p.X_i) \otimes (A + \frac{1}{2}\overline{A} + \frac{1}{2}(2kn + n - 1)I + \frac{1}{2}Diag(Deg)).\mathbf{1_n} \\
&= \mu_i X_i \otimes n\mathbf{1_n} + X_i \otimes (kn + n + r - 1)\mathbf{1_n} \\
&= n\mu_i(X_i \otimes \mathbf{1_n}) + (kn + n + r - 1)(X_i \otimes \mathbf{1_n}) \\
&= (n\mu_i + kn + n + r - 1)(X_i \otimes \mathbf{1_n})
\end{aligned}$$

Therefore $n\mu_i + kn + n + r - 1$ is an eigenvalue of F with eigenvector $X_i \otimes \mathbf{1_n}$. As Y_j is orthogonal to $\mathbf{1}$, we have $J_n.Y_j = 0$ for each $j = 2, 3, \ldots, n$. Let $\{Z_k\}, k = 1, 2, \ldots, p$ be the family of p linearly independent eigenvectors associated with the eigenvalue 1 of I_p. Then for each $j = 2, 3, \ldots, n$, the p vectors $Z_k \otimes Y_j$ are eigenvectors of F with eigenvalue $\frac{1}{2}(\lambda_i + n + r) + kn - 1$. For

$$\begin{aligned}
F.(Z_k \otimes Y_j) &= (RD_G \otimes J_n \\
&+ I_p \otimes (A + \frac{1}{2}\overline{A} + \frac{1}{2}(2kn + n - 1)I + \frac{1}{2}Diag(Deg))(Z_k \otimes Y_j) \\
&= (RD_G.Z_k) \otimes (J_n.Y_j) \\
&+ (I_p.Z_k) \otimes (A + \frac{1}{2}\overline{A} + \frac{1}{2}(2kn + n - 1)I + \frac{1}{2}Diag(Deg)).Y_j \\
&= 0 + Z_k \otimes (\frac{1}{2}(\lambda_i + n + r) + kn - 1)Y_j \\
&= (\frac{1}{2}(\lambda_i + n + r) + kn - 1).(Z_k \otimes Y_j).
\end{aligned}$$

Also the pn vectors $X_i \otimes \mathbf{1_n}$ and $Z_k \otimes Y_j$ are linearly independent. As the eigenvectors belonging to different eigenvalues are linearly independent and as F has a basis consisting entirely of eigenvectors, the theorem follows. □

5. Bounds for the reciprocal distance signless Laplacian energy

The ordinary *graph energy* $E_\pi(G)$ was defined by Gutman [15] in 1978 as the sum of the absolute values of the eigenvalues of the adjacency matrix of G. It has an application in the total π-electron energy of non-saturated hydrocarbons as calculated with the Huckel Molecular Orbital Method in quantum chemistry [18]. Looking to the success of ordinary graph energy, the new graph energies were introduced by many scholars. Nowdays in the mathematical literature there exists over 100 different graph energies [17]. Some of these are distance energy [24], Laplacian energy [19], Randić energy [13], skew energy [1], degree sum energy [38], Harary energy [14], distance Laplacain energy [42], terminal distance energy [37], reciprocal complementary distance energy [39], complementary distance signless Laplacian energy [32], Seidel energy [20], Seidel Laplacian energy [35] and Seidel signless Laplacian energy [33]. More deatils about the different graph energies can be found in the books [16, 29].

The *Harary energy* of a graph G, denoted by $E_H(G)$, is defined as [14]

$$E_H(G) = \sum_{i=1}^{n} |\mu_i|,$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the reciprocal distance matrix of G. The eigenvalues of the reciprocal distance matrix of a graph G satisfies the relations

$$\sum_{i=1}^{n} \mu_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \mu_i^2 = 2 \sum_{1 \leq i < j \leq n} \left(\frac{1}{d_{ij}} \right)^2 \tag{19}$$

Recent results on the Harary energy can be found in [4, 7, 34, 36].

The *distance Laplacian matrix* of a connected graph G is defined as [2]

$$D^L(G) = \operatorname{diag}(Tr_G(v_i)) - D(G),$$

where $D(G)$ is the distance matrix of G.

The *distance Laplacian energy* of G denoted by $LE_D(G)$ is defied as [42]

$$LE_D(G) = \sum_{i=1}^{n} \left| \delta_i - \frac{1}{n} \sum_{j=1}^{n} Tr_G(v_j) \right|,$$

where $\delta_1, \delta_2, \ldots, \delta_n$ are the eigenvalues of the distance Laplacian matrix of G.

To preserve the main features of the Harary energy and distance Laplacian energy and bearing in mind the Eq. (19), we define here

$$\xi_i = \rho_i - \frac{1}{n} \sum_{j=1}^{n} Tr'_G(v_j), \quad i = 1, 2, \ldots, n,$$

where ρ_i, $i = 1, 2, \ldots, n$ are the eigenvalues of $RQ(G)$.

Definition 5.1. Let G be a connected graph of order n. Then the reciprocal distance signless Laplacian energy of G, denoted by $E_{RQ}(G)$ is defined as

$$E_{RQ}(G) = \sum_{i=1}^{n} |\xi_i| = \sum_{i=1}^{n} \left| \rho_i - \frac{1}{n} \sum_{j=1}^{n} Tr'_G(v_j) \right|.$$

The results of this section are analogous to the results obtained in [42].

First we give the following simple lemma.

Lemma 5.2. *Let G be a connected graph of order n. Then*

$$\sum_{i=1}^{n} \xi_i = 0 \quad \textit{and} \quad \sum_{i=1}^{n} \xi_i^2 = 2S,$$

where

$$S=\sum_{1\le i<j\le n}\left(\frac{1}{d_{ij}}\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left[Tr'_G(v_i)-\frac{1}{n}\sum_{j=1}^{n}Tr'_G(v_j)\right]^2.$$

Proof. Clearly,

$$\sum_{i=1}^{n}\rho_i=\text{trace}[RQ(G)]=\sum_{i=1}^{n}Tr'_G(v_i)$$

and

$$\sum_{i=1}^{n}\rho_i^2=\text{trace}[(RQ(G))^2]=2\sum_{1\le i<j\le n}\left(\frac{1}{d_{ij}}\right)^2+\sum_{i=1}^{n}(Tr'_G(v_i))^2.$$

Therefore

$$\sum_{i=1}^{n}\xi_i=\sum_{i=1}^{n}\left[\rho_i-\frac{1}{n}\sum_{j=1}^{n}Tr'_G(v_j)\right]=\sum_{i=1}^{n}\rho_i-\sum_{j=1}^{n}Tr'_G(v_j)=0$$

and

$$\begin{aligned}\sum_{i=1}^{n}\xi_i^2 &= \sum_{i=1}^{n}\left[\rho_i-\frac{1}{n}\sum_{j=1}^{n}Tr'_G(v_j)\right]^2\\ &= \sum_{i=1}^{n}\rho_i^2-\frac{2}{n}\sum_{j=1}^{n}Tr'_G(v_j)\sum_{i=1}^{n}\rho_i+\frac{1}{n}\left(\sum_{j=1}^{n}Tr'_G(v_j)\right)^2\\ &= 2\sum_{1\le i<j\le n}\left(\frac{1}{d_{ij}}\right)^2+\sum_{i=1}^{n}(Tr'_G(v_i))^2-\frac{2}{n}\left(\sum_{j=1}^{n}Tr'_G(v_j)\right)^2\\ &+ \frac{1}{n}\left(\sum_{j=1}^{n}Tr'_G(v_j)\right)^2=2\sum_{1\le i<j\le n}\left(\frac{1}{d_{ij}}\right)^2\\ &+ \sum_{i=1}^{n}\left[Tr'_G(v_i)-\frac{1}{n}\sum_{j=1}^{n}Tr'_G(v_j)\right]^2=2S\end{aligned}$$

□

Corollary 5.3. *Let G be a connected graph of order n and size m with diameter less than or equal to 2. Then*

$$\sum_{i=1}^{n}\xi_i^2=\frac{1}{4}[6m+n(n-1)+M_1(G)]-\frac{m^2}{n},$$

where $M_1(G)=\sum_{i=1}^{n}(deg_G(v_i))^2$.

Proof. If diameter of G is less than or equal to 2, then G has m pairs of vertices which are at distance 1 and the remaining $\left(\binom{n}{2}-m\right)$ pairs of vertices are at distance 2. Therefore

$$\sum_{1\le i<j\le n}\left(\frac{1}{d_{ij}}\right)^2=\frac{1}{8}[6m+n(n-1)]$$

and

$$Tr'_G(v_i)=\sum_{j=1}^{n}\frac{1}{d_{ij}}=\frac{1}{2}[n-1+deg_G(v_i)].$$

Therefore

$$\begin{aligned}\sum_{i=1}^{n}\xi_i^2 &= 2\sum_{1\le i<j\le n}\left(\frac{1}{d_{ij}}\right)^2+\sum_{i=1}^{n}\left[Tr'_G(v_i)-\frac{1}{n}\sum_{j=1}^{n}Tr'_G(v_j)\right]^2\\ &= 2\left[\frac{6m+n(n-1)}{8}\right]+\sum_{i=1}^{n}\left[\frac{deg(v_i)}{2}-\frac{m}{n}\right]^2\\ &= \frac{1}{4}[6m+n(n-1)+M_1(G)]-\frac{m^2}{n}.\end{aligned}$$

□

Theorem 5.4. *Let G be a connected graph of order n. Then*

$$2\sqrt{S}\le E_{RQ}(G)\le\sqrt{2nS}.$$

Proof. By direct calculation, the non-negative term

$$\begin{aligned}T &= \sum_{i=1}^{n}\sum_{j=1}^{n}(|\xi_i|-|\xi_j|)^2\\ &= 2n\sum_{i=1}^{n}|\xi_i|^2-2\left(\sum_{i=1}^{n}|\xi_i|\right)\left(\sum_{j=1}^{n}|\xi_j|\right)\\ &= 4nS-2(E_{RQ}(G))^2.\end{aligned}$$

Since $T\ge 0$, $E_{RQ}(G)\le\sqrt{2nS}$.
Now $(\sum_{i=1}^{n}\xi_i)^2=0$. This implies

$$\sum_{i=1}^{n}\xi_i^2+2\sum_{1\le i<j\le n}(\xi_i\xi_j)=0.$$

Hence

$$2S = -2\sum_{1\le i<j\le n}(\xi_i\xi_j) \le 2\left|\sum_{1\le i<j\le n}(\xi_i\xi_j)\right| \le 2\sum_{1\le i<j\le n}|\xi_i||\xi_j|.$$

Therefore

$$\begin{aligned}(E_{RQ}(G))^2 &= \left(\sum_{i=1}^{n}|\xi_i|\right)^2\\ &= \sum_{i=1}^{n}|\xi_i|^2 + 2\sum_{1\le i<j\le n}|\xi_i||\xi_j|\\ &\ge 2S+2S = 4S,\end{aligned}$$

which leads to the lower bound $E_{RQ}(G) \ge 2\sqrt{S}$.

□

Corollary 5.5. *Let G be a connected graph of order n and diameter d. Then*

$$E_{RQ}(G) \ge \frac{1}{d}\sqrt{2n(n-1)}.$$

Proof. Using Theorem 5.4 and since $d_{ij} \le d$, for $i, j = 1, 2, \ldots, n$, we have

$$\begin{aligned}E_{RQ}(G) &\ge 2\sqrt{\sum_{1\le i<j\le n}\left(\frac{1}{d_{ij}}\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left[Tr'_G(v_i) - \frac{1}{n}\sum_{j=1}^{n}Tr'_G(v_j)\right]^2}\\ &\ge 2\sqrt{\sum_{1\le i<j\le n}\left(\frac{1}{d_{ij}}\right)^2} \ge 2\sqrt{\sum_{1\le i<j\le n}\left(\frac{1}{d^2}\right)} \ge \frac{1}{d}\sqrt{2n(n-1)}.\end{aligned}$$

□

By Corollary 5.3 and Theorem 5.4, we get the following corollary.

Corollary 5.6. *Let G be a connected graph with n vertices, m edges. Let the diameter of G be less than or equal to 2. Then*

$$\sqrt{\frac{1}{2}T - \frac{2m^2}{n}} \le E_{RQ}(G) \le \sqrt{\frac{n}{4}T - m^2},$$

where $T = 6m + n(n-1) + M_1(G)$ *and* $M_1(G) = \sum_{i=1}^{n}(deg_G(v_i))^2$.

Lemma 5.7. [28] *Let* $a_1, a_2, \ldots, a_n$ *be non-negative numbers. Then*

$$nM \leq n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \leq n(n-1)M,$$

where $M = \frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}$.

In the following, we give another bounds for the reciprocal distance signless Laplacian energy of G.

Theorem 5.8. *Let* G *be a connected graph with* n *vertices,* I_n *the unit matrix of order* n *and*

$$\Gamma = \left|\det\left(RQ(G) - \frac{1}{n}\sum_{i=1}^{n} Tr'_G(v_i) I_n\right)\right|.$$

Then

$$\sqrt{2S + n(n-1)\Gamma^{2/n}} \leq E_{RQ}(G) \leq \sqrt{2(n-1)S + n\Gamma^{2/n}}.$$

Proof. Let $a_i = |\xi_i|^2$, $i = 1, 2, \ldots, n$ and

$$K = n\left[\frac{1}{n}\sum_{i=1}^{n} |\xi_i|^2 - \left(\prod_{i=1}^{n} |\xi_i|^2\right)^{1/n}\right] = n\left[\frac{2S}{n} - \left(\prod_{i=1}^{n} |xi_i|\right)^{2/n}\right] = 2S - n\Gamma^{2/n}.$$

By Lemma 5.7, we have

$$K \leq n\sum_{i=1}^{n} |\xi_i|^2 - \left(\sum_{i=1}^{n} |\xi_i|\right)^2 \leq (n-1)K,$$

that is

$$2S - n\Gamma^{2/n} \leq 2nS - (E_{RQ}(G))^2 \leq (n-1)[2S - n\Gamma^{2/n}].$$

Simplification of above equation leads to the desired result.

□

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