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MAXIMAL SUBGROUPS OF FINITE CLASSICAL GROUPS AND THEIR GEOMETRY

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We survey some recent results on maximal subgroups of finite classical groups.

1. Introduzione.

The seminal contribution to the classification of the maximal subgroups of the finite classical groups was Aschbacher's theorem [1]. Aschbacher defines eight "geometric" classes $\mathcal{C}_1, \dots, \mathcal{C}_8$ of subgroups of the finite classical groups and proves that a maximal subgroup either belongs to one of these classes or has a non–abelian simple group as its generalized Fitting subgroup.

In their book, Kleidman and Liebeck [33] have identified the members of the eight classes for modules with dimension greater than 12, and Kleidman in his Ph.D. Thesis [32] completed the work for modules with dimension up to 12. However, their analysis relies heavily upon the classification of finite simple groups. Various authors, too many to be quoted here, have used Aschbacher's theorem to elucidate much of the maximal subgroup structure on the finite classical groups.

Li and his coworkers, obtained several results on maximal subgroups of classical groups, allowing the ground field to be infinite. Mainly he adopted an elementary but rather technical matrix approach, see for instance [35], [36], [37], [38].

At least seven of the eight Aschbacher's classes can be described as stabilizers of geometric configurations. Consequently, one might prefer a direct approach to the classification of maximal subgroups, which is free of the classification of finite simple groups, using the natural representations of classical groups.

Certainly, this is the approach adopted by R.H. Dye, O.H. King and A. Cossidente, elucidating in many cases how maximal subgroups of finite classical groups and geometry are closely related.

In this paper we survey some recent results on the determination of maximal subgroups of finite classical groups by making maximum use of the underlying geometry.

Our hope, one day, is to have in hands a complete geometric classification of maximal subgroups of finite classical groups.

2. Notation.

Let V be a vector space of dimension $n \geq 2$ over a field K with general linear group $GL_n(K)$ and special linear group $SL_n(K)$. The centre Z of $GL_n(K)$ is the group consisting of scalar matrices and the quotient group $GL_n(K)/Z$ is the projective general linear group $PGL_n(K)$ whose natural module is the projective space $PG(n-1, K)$ associated with V. For any subgroup G of $GL_n(K)$ its image in $PGL_n(K)$ under the canonical epimorphism will be denoted by $P(G)$.

In the sequel, Q will denote a quadratic form on V with associated symmetric bilinear form given by

$$B(x, y) = Q(x + y) - Q(x) - Q(y).$$

A and C will denote alternating and hermitian forms on V, respectively (often $(\ ,\ )$ will denote anyone of the three forms on $V \times V$). A form on V is said to be non–degenerate if $\{v \in V : (u, v) = 0, \forall u \in V\} = \{0\}$; we shall assume our forms to be non–degenerate. We shall also assume that there exist non–zero vectors v such that $Q(v) = 0$ or $C(v, v) = 0$. This is always the case if K is a finite field and $n \geq 3$ (for C, $n \geq 2$ suffices) but for infinite fields this restriction is necessary if V is to have a nice geometrical structure with respect to the given form. The orthogonal group $O_n(K)$, the symplectic group $Sp_n(K)$ and the unitary group $U_n(K)$ consists of all isomorphisms of V preserving Q, A and C respectively. The groups $SO_n(K)$ and $SU_n(K)$ are the special orthogonal and special unitary groups, and $\Omega_n(K)$ is the commutator subgroup of $O_n(K)$.

These groups together with $GL_n(K)$, $SL_n(K)$ and their projective counterparts, represents the *classical groups*.

The groups $PSL_n(K)$, $PSp_n(K)$, $PSU_n(K)$ and $P\Omega_n(K)$, apart from some few cases, are simple groups.

Given a subspace U of V, the orthogonal complem ent $U^{\perp}$ of U with respect to (,) is given by

$$U^{\perp} = \{v \in V : (u, v) = 0, \forall u \in V\}.$$

U is said to be

– non–isotropic if the restriction of (,) to U is non–degenerate, i.e., $U \cap U^{\perp} = \{0\}$;

– totally isotropic if $(u, v) = 0$ for all $u, v \in U$, i.e., $U \leq U^{\perp}$.

In the orthogonal case a vector $v \in V$ is said to be singular if $Q(v) = 0$ and a subspace U of V is said to be totally singular if all its vectors are singular; a totally singular subspace is necessarily totally isotropic. Indeed, totally singular subspaces and totally isotropic subspaces only differ over fields of even characteristic.

The totally singular subspaces form a geometry (polar geometry). They are permuted by $O_n(K)$, $Sp_n(K)$ and $U_n(K)$ and it is a consequence of Witt's Theorem that if we restrict their action to subspaces of a given dimension the action is transitive.

3. Aschbacher's Theorem.

The subgroup structure theorem for finite classical group is due to Aschbacher [1]. In [1], eight collections $\mathcal{C}_i$, $(1 \leq i \leq 8)$, of natural subgroups of a finite classical group G are described. A precise and concise definition of the members in $\mathcal{C}_i$ is quite difficult to give in limited space (this can be found for instance in [1] and [33]), so we content ourselves with a "rough" description:

- $(\mathcal{C}_1)$: stabilizers of totally singular or non–singular subspaces;
- $(\mathcal{C}_2)$: stabilizers of direct sum decompositions $V = \bigoplus_{i=1}^{b} V_i$, all V_i's having constant dimension;
- $(\mathcal{C}_3)$: stabilizers of extension fields of $GF(q)$ of prime degree;
- $(\mathcal{C}_4)$: stabilizers of tensor product decompositions $V = V_1 \otimes V_2$;
- $(\mathcal{C}_5)$: stabilizers of subfields of $GF(q)$ of prime index;
- $(\mathcal{C}_6)$: normalizers of extraspecial–type r–groups (r prime) in absolutely irreducible representations;

- ($\mathcal{C}_7$): stabilizers of decompositions $V = \bigotimes_{i=1}^{b} V_i$, all V_i's having constant dimension;
- ($\mathcal{C}_8$): classical subgroups.

It is quite evident that groups in classes $\mathcal{C}_i$, $i = 1, 2, 4, 7$, are stabilizers of geometric configurations. However, it should be noted that some restriction is in order on the nature of the subspaces when members in $\mathcal{C}_1$ are considered.

Indeed, assume $G = O_n(K)$ and U is a subspace of V. Then, the stabilizer of U in G is also the stabilizer of $U^\perp$ and of $U \cap U^\perp$. Thus, the stabilizer of U in G will only be maximal if $U \cap U^\perp = \{0\}$, or one of the two subspaces U and $U^\perp$ lies inside the other, i.e., either U is non–isotropic or one of U, $U^\perp$ is totally isotropic.

For groups in class $\mathcal{C}_3$ is assumed that V has the structure of vector space over an extension field $GF(q^t)$ of $GF(q)$ of prime degree t. Each 1–dimensional $GF(q^t)$–subspace of V corresponds to a t–dimensional $GF(q)$–subspace of V. These t–dimensional $GF(q)$–subspaces form a so–called spread of V (i.e., each non–zero vector lies in exactly one member of the spread, and the spread is the geometric configuration stabilized.

Groups in class $\mathcal{C}_5$ are normalizers of classical groups acting on the n–dimensional vector spaces V_F over maximal subfields $F = GF(q')$ of $GF(q)$ such that $V = V_F \otimes_F GF(q)$.

It is now clear that in this case the geometric configuration stabilized is a subgeometry: think for instance of a Baer subplane in a projective plane of square order. However, we shall see in the sequel, that at least in the unitary case, the more natural description of groups in class $\mathcal{C}_5$ is in terms of "commuting polarities".

There is no natural and "obvious" geometric configuration for groups in class $\mathcal{C}_6$. However, very recently such groups have played a crucial role in the theory of non–linear binary codes.

Groups in class $\mathcal{C}_8$ may be thought of as the stabilizers of the sets of singular 1–dimensional subspaces (for instance orthogonal groups may be thought of as stabilizing quadrics and unitary groups stabilizing Hermitian varieties).

As an example, consider the group $O_n(q)$, q even. In this case the bilinear form B stabilized by $O_n(q)$ is alternating as well and, as a consequence, $O_n(q)$ turns out to be a subgroup of the symplectic group $Sp_n(q)$ and indeed a maximal subgroup [17].

4. Some methods.

One geometric method of proving results on the structure of a given classical group is based on generating sets for that group. When we talk about generators we mean transvections, semi–transvections, symmetries or quasi–symmetries.

In $Sp_n(q)$, a *transvection* centered on a non–zero vector is given by

$$v \mapsto v + \lambda A(x, v)x,$$

for some $\lambda \in GF(q) \setminus \{0\}$.

In $U_n(q^2)$, a *transvection* centered on a non–zero singular vector is given by

$$v \mapsto v + \lambda C(x, v)x,$$

for some $\lambda \in GF(q^2) \setminus \{0\}$ such that $\lambda^q = -\lambda$. Such maps lie in $SU_n(q^2)$.

In $O_n(q)$, a *symmetry* or -1-*quasi symmetry* centered on a non–zero singular vector y is given by

$$v \mapsto v - [B(y, v)/Q(y)]y.$$

In $U_n(q)$, if $\lambda \in GF(q^2) \setminus \{1\}$ such that $\lambda\lambda^q = 1$, then the λ-*quasi symmetry* centered on a non–singular vector y is given by

$$v \mapsto v + (\lambda - 1)[C(y, v)/C(y, y)]y.$$

Let H be one of the groups $O_n(q)$ and $U_n(q^2)$. Let x be a non–zero singular vector in V, let $w \in \langle x\rangle^\perp$ and let $\rho_{x,w}$ be the isomorphism of $\langle x\rangle^\perp$ defined by

$$v \mapsto v + (w, v)x.$$

We call elements of H that extend $\rho_{x,w}$ (they exist by Witt's theorem) *semi-transvections* centered on x.

There are several theorems in literature specifying sets of such generators which generate a given finite classical group or at least an interesting subgroup. See, for instance, [15], [16] and [47].

As an easy example of the general method, let us consider the case of $H = Sp_n(q)$. Let U be a r–dimensional subspace of V. Let $G = \mathrm{Stab}_H(U)$. Assume that U is not a maximal totally isotropic subspace. It is straightforward to show that G has three orbits of non–zero vectors in V. The orbits consist of all non–zero vectors in U, all non–zero vectors in $U^\perp$ and all non–zero vectors in $V \setminus U \cup U^{\perp}$. Also it can be proved that any larger subgroup F of G must be transitive on non–zero vectors of V.

It is evident from the definition that if t is a transvection centered on a vector of U, then t stabilizes U. Let t be any transvection in H, centered on x say, and let $\alpha \in U$. Then, there exists $f \in F$ such that $f(x) = \alpha$, *i.e.*, $f(x) \in U$. Now, ftf^{-1} is again a transvection centered on $f(x)$, so it stabilizes U. Thus $ftf^{-1} \in G < F$, and so $t \in F$. Therefore, every transvection in H lies in F. It is known that H is generated by its transvections, [15], [16]. Hence $F = H$ and G is maximal in H.

Another very useful method in establishing maximality theorems is by induction, for instance, on the dimension of V. In this cases, sometimes, some results on small finite classical groups are involved: for instance, one can reduce to a subgroup of $PSL_2(q)$, $PSL_3(q)$ or $PSp_4(q)$, whose "geometric" subgroup structure is very well known, see [41], [14], [39], [40], [26].

In many other instances, generating sets or induction do not suffice and special geometric settings and techniques are needed.

Now, our main purpose is to illustrate the " state of the art " of the *geometric classification of maximal subgroups of finite classical groups*. We shall try to report, for each Aschbacher's class, what has been already done and what we need to do yet.

5. The classes $\mathcal{C}_i$, $i = 1, 2, 4, 7, 8$.

For these classes there is very little to say. The maximality of subgroups in the classes $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_8$ has been completely addressed by R.H. Dye and O.H. King. On the other hand, very little has been done for the classes $\mathcal{C}_4$ and $\mathcal{C}_7$, apart from a couple of papers by Li.

6. The class $\mathcal{C}_3$: the class of spreads.

Let $GF(q^t)$ be an extension of the finite field $GF(q)$ of degree $t > 1$, where $t|n$. Then V acquires the structure of a $GF(q^t)$–vector space: there is a $GF(q)$–vector space isomorphism between V and an m–dimensional vector space over $GF(q^t)$, say W, where $m = n/t$, and so $GF(q^t)$ acts on V via this isomorphism. This way, $GF(q^t)$ embeds in $\mathrm{End}_{GF(q)}(V)$. Hence, in the simplest case when W is equipped with the zero–form, we obtain the embedding:

$$GL_m(q^t) \leq GL_n(q).$$

Typically, the normalizer of $GL_m(q^t)$ inside $GL_n(q)$ turns out to be maximal in $GL_n(q)$. This general case has been investigated by R.H. Dye in [24].

The focus now is the symplectic group $Sp_{2n}(q)$. In this case Aschbacher lists two subclasses: normalizers of $Sp_{2m}(q^t)$ where $n = mt$ and t is prime; and normalizers of $U_n(q^2)$. The first subclass was considered by R.H. Dye in [18], [19], [20], [21], [22], [23], where he proves the maximality in purely geometric terms. Our object is to do the same for the second subcase when q is odd.

Let $L = GF(q^2)$ and $K = GF(q)$, q odd. Let ω be an element of L such that $\omega^q = -\omega$. Then 1 and ω form a basis for L over K, and if $\theta \in L$, then $\theta = \alpha + \beta\omega$, with $\alpha, \beta \in GF(q)$. Let $\{e_1, \dots, e_n\}$ be a basis of L^n as a vector space over L.

Define a bijective map Φ from L^n to K^{2n} by the rule

$$(\theta_1, \dots, \theta_n) \mapsto (\alpha_1, \beta_1, \dots, \alpha_n, \beta_n),$$

where $\theta_i = \alpha_i + \beta_i\omega$, for each $i = 1, \dots, n$. We denote a vector of K^{2n} by $\bar{z}$ with the corresponding vector in L^n represented by z. The vectors of the 1–subspace $\langle z\rangle$ of L^n are K–linear combinations of the vectors z and ωz which correspond in K^{2n} to the vectors of a 2–dimensional subspace we call k_z. Since Φ is a bijection, each non–zero vector in K^{2n} lies in exactly one k_z. Passing to the projective space $PG(2n-1, q)$ whose underlying vector space is K^{2n}, the subspace k_z gives a line s_z in $PG(2n-1, q)$, and the set of all such lines gives a spread of lines (regular spread [27]) of $PG(2n-1, q)$.

Let H be a non–degenerate Hermitian form on L^n with isometry group $U_n(q^2)$. We can take $\{e_1, \dots, e_n\}$ to be an orthogonal basis for L^n with respect to H. Starting from H we can define a non–degenerate alternating form A on K^{2n} by

$$A(\bar{x}, \bar{y}) = Tr(\omega H(x, y)) = \omega H(x, y) + \omega^q H(x, y)^q,$$

for any $\bar{x}, \bar{y} \in K^{2n}$. In this setting isotropic 1–dimensional subspaces of L^n correspond to totally isotropic 2–dimensional subspaces of K^{2n}, and non–isotropic 1–dimensional subspaces of L^n correspond to non–isotropic 2–dimensional subspaces of K^{2n}. Any linear map on L^n preserving H gives rise to a linear map on K^{2n} preserving A.

We obtain an embedding

$$\iota : U_n(q^2) \to Sp_{2n}(q).$$

Let

$$\mathcal{K}_n = \{k_z : z \neq 0, H(\underline{z}) = 0\}; \qquad \mathcal{L}_n = \{k_z : H(z) \neq 0\},$$

$$\bar{\mathcal{K}}_n = \{s_z : z \neq 0, H(z) = 0\}; \qquad \bar{\mathcal{L}}_n = \{s_z : H(z) \neq 0\}.$$

We have $k = |\mathcal{K}_n| = (q^n - (-1)^n)(q^{n-1} + (-1)^n)/(q^2 - 1)$ and so $l = |\mathcal{L}_n| = (q^{2n} - 1)/(q^2 - 1) - k$. Of course $l > k$.

From our previous discussion, it follows that $\bar{\mathcal{K}}_n \cup \bar{\mathcal{L}}_n$ is a line–spread $\mathcal{S}$ of $PG(2n-1, q)$.

Let $\sigma : L \to L$ be the Frobenius automorphism of L: $\theta \mapsto \theta^q$, for each θ in L. Then σ gives rise to a semi–linear map: $\theta_i e_i \mapsto \theta_i^q e_i$ on L^n which corresponds to a linear map on K^{2n}. It turns out that $A(\sigma(\bar{x}), \sigma(\bar{y})) = -A(\bar{x}, \bar{y})$ and so σ multiplies A by -1. Hence σ is an element of $GSp_{2n}(q)$. If $\tau \in GU_n(q^2)$ is such that $\tau(e_i) = \lambda e_i$, $i = 1, \ldots n$, where $\lambda \in L$ and $\lambda^{q+1} = -1$, then it is easy to see that τ multiplies H by -1 and corresponds to an element of $GSp_{2n}(q)$ again multiplying A by -1. Thus $\tau\sigma \in Sp_{2n}(q)$; it has order 4 since its square is $-I_{2n}$, where I denotes the identity matrix.

We denote by G the group $\iota(\langle U_n(q^2), \tau\sigma\rangle)$ and often write $G = U_n(q^2)\cdot 2$. From our previous discussion it follows that G is contained in the stabilizer in $Sp_{2n}(q)$ of $\mathcal{K}_n \cup \mathcal{L}_n$. Since the subspaces in $\mathcal{K}_n$ are isotropic while those in $\mathcal{L}_n$ are non–isotropic it follows that G stabilizes each of $\mathcal{K}_n$ and $\mathcal{L}_n$. We shall prove that G is maximal in $Sp_{2n}(q)$ from which it follows that G is the stabilizer of $\mathcal{K}_n \cup \mathcal{L}_n$, and indeed the stabilizer of $\mathcal{K}_n$. Moreover G contains the centre of $Sp_{2n}(q)$ so an immediate consequence is the maximality of the image $P(G)$ of G in $PSp_{2n}(q)$.

We observe that $U_n(q^2)$ acts transitively on the 1–dimensional non–isotropic subspaces of L^n and transitively on the non–zero singular vectors of L^n [15], [16]. Hence G acts transitively on $\mathcal{L}_n$ and transitively on the non–zero vectors lying in members of $\mathcal{K}_n$. The stabilizer in $U_n(q^2)$ of a non–isotropic 1–dimensional subspace $\langle x\rangle$ of L^n is isomorphic to $U_1(q^2) \times U_{n-1}(q^2)$ acting on $\langle x\rangle \oplus \langle x\rangle^\perp$. Thus the stabilizer in G of k_x is isomorphic to $(U_1(q^2)\times U_{n-1}(q^2))\cdot 2$ and fixes the set $\mathcal{K}_{n-1} \cup \mathcal{L}_{n-1}$ where $\mathcal{K}_{n-1}$ (respectively $\mathcal{L}_{n-1}$) corresponds to the set of elements of $\mathcal{K}_n$ (respectively $\mathcal{L}_n$) contained in $k_x^\perp$.

In [7], Cossidente and King proved the following theorem.

Theorem 6.1. *Assume $n \geq 3$ and q odd. Then the group $G = U_n(q^2) \cdot 2$ is a maximal subgroup of $Sp_{2n}(q)$. If $n = 2$ and q is odd then $U_2(q^2) \cdot 2$ is a maximal subgroup of $Sp_4(q)$ except for $q = 3$. In the excepted case there is a single group $X \cong 2 \cdot 2^4 \cdot A_5$, such that $G < X < Sp_4(q)$.*

The group $Sp_{2n}(q)$ is transitive on the set of all isotropic 2–dimensional subspaces of K^{2n} so cannot stabilize $\mathcal{K}_n \cup \mathcal{L}_n$ or $\mathcal{K}_n$. It will be clear that in the excepted case, X does not stabilize $\mathcal{K}_2 \cup \mathcal{L}_2$ or $\mathcal{K}_2$ either. Thus we have the following theorem.

Theorem 6.2. *The stabilizer of $\bar{\mathcal{K}}_n \cup \bar{\mathcal{L}}_n$ in $Sp_{2n}(q)$ is the stabilizer of $\bar{\mathcal{K}}_n$, is* *isomorphic to $U_n(q^2) \cdot 2$ and is a maximal subgroup of $Sp_{2n}(q)$ except when $n = 2$ and $q = 3$.*

As we have already observed, G contains the centre of $Sp_{2n}(q)$. Thus we have the further theorem.

Theorem 6.3. *The stabilizer of the line spread $\bar{\mathcal{K}}_n \cup \bar{\mathcal{L}}_n$ of $PG(2n-1, q)$ in $PSp_{2n}(q)$ is the stabilizer of the partial spread $\bar{\mathcal{K}}_n$ and is a maximal subgroup of $PSp_{2n}(q)$ except when $n = 2$ and $q = 3$.*

The case $n = 2$ of Theorem 6. was proved in [40], see [7] for another proof. The case $n \geq 3$ is based on the following reduction argument.

Lemma 6.4. *Suppose that $G = U_n(q^2) \cdot 2 \leq F < Sp_{2n}(q)$. Then there is a non–isotropic 2–dimensional subspace k_x of $\mathcal{L}_n$ such that if F_1 and F_2 are the projections of $Stab_F(k_x)$ acting on k_x and $k_x^{\perp}$ respectively, then either $U_1(q^2) \cdot 2 < F_1$ or $U_{n-1}(q^2) \cdot 2 < F_2$ (or both).*

Assume $n \geq 3$ and $q \neq 3$. Here is a sketch of the proof of Theorem 6.1.

Assume as an inductive argument that $U_{n-1}(q^2) \cdot 2$ is a maximal subgroup of $Sp_{2n-2}(q)$. Note also that $U_1(q^2) \cdot 2$ is a maximal subgroup of $Sp_2(q)$. By Lemma 6.4, if $G < F \leq Sp_{2n}(q)$ then there is a non–isotropic 2–dimensional subspace $k_x \in \mathcal{L}_n$ such that if F_1 and F_2 are the projections of $\mathrm{Stab}_F(k_x)$ acting on k_x and $k_x^{\perp}$, respectively, then either $U_1(q^2) \cdot 2 < F_1$ or $U_{n-1}(q^2) \cdot 2 < F_2$ (or both). It follows that either $F_1 = Sp_2(q)$ or $F_2 = Sp_{2n-2}(q)$.

Suppose that $F_2 = U_{n-1}(q^2) \cdot 2$. Then $F_1 = Sp_2(q)$ and the subgroup $\{f_1 \in F_1 : (f_1, f_2) \in F$, for some $f_2 \in U_{n-1}(q^2)\}$ forms a subgroup of F_1 of index at most two, but $Sp_2(q)$ has no subgroup of index two. Furthermore $1 \times U_{n-1}(q^2) \leq G$ and so $Sp_2(q) \times 1 \leq F$. There exists $g \in G$ such that $g(k_x) = k_u \subseteq k_x^{\perp}$. Expressing V as $k_x \oplus (k_x^{\perp} \cap k_u^{\perp}) \oplus k_u$, we see that F contains $Sp_2(q) \times 1 \times 1$ and $g(Sp_2(q) \times 1 \times 1)g^{-1} = (1 \times 1 \times Sp_2(q))$. The last subgroup is contained in $\mathrm{Stab}_F(k_x)$ but not in $Sp_2(q) \times (U_{n-1}(q^2) \cdot 2)$. We conclude that F_2 cannot be just $U_{n-1}(q^2) \cdot 2$ and therefore $F_2 = Sp_{2n-2}(q)$.

The subgroup $\{f_2 \in F_2 : (1, f_2) \in F\}$ of F_2 is a normal subgroup of index at most $|Sp_2(q)|$, but $PSp_{2n-2}(q)$ is simple and the centre of $Sp_{2n-2}(q)$ has order 2 so $1 \times F_2 \leq F$. Utilizing u and g as above, F contains $g(1 \times Sp_{2n-2}(q))g^{-1} = Sp_{2n-2}(q) \times 1$ (where the first expression is acting on $k_x \oplus k_x^{\perp}$ and the second on $k_u^{\perp} \oplus k_u$). In particular F contains $Sp_2(q) \times 1 \times 1$ so contains $Sp_2(q) \times Sp_{2n-2}(q)$, the stabilizer of k_x in $Sp_{2n}(q)$. This stabilizer is maximal in $Sp_{2n}(q)$, [31, Section 3] but does not contain $Sp_{2n-2}(q) \times 1$ so $F = Sp_{2n}(q)$. Hence G is maximal in $Sp_{2n}(q)$. Ad hoc arguments for $q = 3$ complete the proof of Theorem 6.1. □

Now, we pass to investigate another $\mathcal{C}_3$–embedding: $O_n(q^2) \leq O_{2n}(q)$, q odd. In this case the group $G = O_n(q^2) \cdot 2$ is maximal in either the orthogonal group $O_{2n}(q)$ or the special orthogonal group $SO_{2n}(q)$. With the same geometric setting as in the symplectic case, the group G corresponds to the stabilizer of a line–spread $\mathcal{S}$ of the projective space $PG(2n-1,q)$ in which some lines lie on a quadric $\mathcal{Q}$, some are secant to the quadric and others are external to the quadric.

Notice that, the study of $O_n(q^2).2$ as a subgroup of $O_{2n}(q)$ makes sense when q is even, and a similar approach could work, but there are several significant geometric differences.

Let Q be a non–degenerate quadratic form on L^n with associated non–degenerate symmetric bilinear form B and isometry group $O_n(q^2)$. We can take $\{e_1, e_2, \ldots, e_n\}$ to be an orthogonal basis for L^n with respect to B. Moreover we can choose it so that $Q(e_i) = \xi$ for each $i < n$ and $Q(e_n) = 1$ or ξ. Starting from Q we can define a non–degenerate quadratic form $\bar{Q}$ on K^{2n} by

$$\bar{Q}(\bar{x}) = Tr(Q(x)) = Q(x) + Q(x)^q,$$

for any $\bar{x} \in K^{2n}$. In this setting isotropic 1–dimensional subspaces of L^n correspond to totally isotropic 2–dimensional subspaces of K^{2n}, and non–isotropic 1–dimensional subspaces of L^n correspond to non–isotropic 2–dimensional subspaces of K^{2n}. Any linear map on L^n preserving Q gives rise to a linear map on K^{2n} preserving $\bar{Q}$. We obtain an embedding $O_n(q^2) \leq O_{2n}(q)$.

Given a subspace U_z of K^{2n} with z non-isotropic we see that Q is either square or non-square in $GF(q^2)$ on the non-zero vectors. It follows that we can choose z so that $Q(z) = 1$ or ξ. We say that $Q(z)$ is the *type* of U_z and also of both the corresponding line $l \in \mathcal{S}$ and the 1-dimensional subspace $\langle z \rangle$ of L^n. Thus $\langle e_i \rangle$ has type ξ for each $i \leq n-1$. We say that $Q(e_n)$ is the type of Q. We write $\mathcal{S}_1$ for the lines in $\mathcal{S}$ of type 1 and $\mathcal{S}_\xi$ for the lines in $\mathcal{S}$ of type ξ. For completeness we write $\mathcal{S}_0$ for the lines in $\mathcal{S}$ corresponding to isotropic vectors in L^n. Thus $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_\xi$ with each of the three subsets non-empty when $n \geq 3$.

Consider the Frobenius automorphism of L: $\theta \mapsto \theta^q$, for each θ in L. There is a semi–linear involutory map ρ on L^n given by $\theta_i e_i \mapsto \theta_i^q e_i$ (if $Q(e_i) = 1$) and $\theta_i e_i \mapsto \theta_i^q \epsilon e_i$ (if $Q(e_i) = \xi$) which corresponds to a linear map on K^{2n}. It can be seen that $Q(\rho(x)) = Q(x)^q$ and $\bar{Q}(\rho(\bar{x})) = \bar{Q}(\bar{x})$, and clearly ρ preserves $\mathcal{S}$. Thus $O_{2n}(q)$ has a subgroup $\langle O_n(q^2), \rho \rangle$ with structure $O_n(q^2).2$ preserving $\mathcal{S}$.

We denote by H_2, H_1 and H_0 the groups $O_{2n}(q)$, $SO_{2n}(q)$ and $\Omega_{2n}(q)$ respectively, by G_2 the subgroup $O_n(q^2).2$ of H_2, and by G_1, and G_0 the subgroups $G_2 \cap H_1$ and $G_2 \cap H_0$ respectively. From our previous discussion it follows that G_2 is contained in the stabilizer in H_2 of $\mathcal{S}$. Since the subspaces in $\mathcal{S}_0$ are totally isotropic while those in $\mathcal{S}_1$ and $\mathcal{S}_\xi$ are non–isotropic of different types, it follows that G_2 stabilizes each of $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_\xi$. We shall prove that G_0 is maximal in H_0 from which it follows that G_0 is the stabilizer of $\mathcal{S}$, and indeed the stabilizer of $\mathcal{S}_0$. Similarly G_1 is maximal in H_1 and G_2 is maximal in H_2 or H_1. In fact it can be proved that G_2 is contained in H_1 precisely when Q has type ξ. Moreover G_2 contains the centre of H_2 so an immediate consequence is the maximality of the image of G_2 in $PO_{2n}(q)$ or $PSO_{2n}(q)$.

There are two conjugacy classes of symmetries, C_1 and C_ξ, the first corresponding to $Q(x)$ square and the other to $Q(x)$ non-square. We denote by $R^\xi O_n(q^2)$ the subgroup of $O_n(q^2)$ generated by C_ξ. Note that $SO_n(q^2)$ has index 2 in $O_n(q^2)$ and since $\Omega_n(q^2)$ is generated by pairs of conjugate symmetries it follows that $\Omega_n(q^2)$ has index 2 in $SO_n(q^2)$. It turns out that $\Omega_n(q^2)$ has index 2 in $R^\xi O_n(q^2)$ and $R^\xi O_n(q^2)$ has index 2 in $O_n(q^2)$ but $R^\xi O_n(q^2)$ is distinct from $SO_n(q^2)$. It turns out that $R^\xi O_n(q^2)$ is a subgroup of H_0 so lies inside G_0 but that symmetries in C_1 lie in $H_1 \setminus H_0$. The group $O_n(q^2).2$ lies in H_1 if Q has type ξ but not if Q has type 1. It now follows that if Q has type 1, then $G_1 = O_n(q^2)$ and $G_0 = R^\xi O_n(q^2)$, while if Q has type ξ, then $G_1 = G_2$ and $G_0 = R^\xi O_n(q^2).2$ (not a subgroup of $O_n(q^2)$).

In [10], Cossidente and King proved the following theorem.

Theorem 6.5. *If $n \geq 3$, then G_0 is maximal in H_0 and G_1 is maximal in H_1. Moreover, if Q has type 1, then G_2 is maximal in H_2. If Q has type ξ, then $G_2 \leq SO_{2n}(q)$. The groups G_0, G_1 and G_2 are the stabilizers of $\mathcal{S}$ in H_0, H_1 and H_2.*

The centre of H_2 consists of the matrices $\pm I_{2n}$, both of which lie in G_1. Hence, by the standard isomorphism theorem for subgroups of quotient groups, we can deduce the following from the main theorem:

Theorem 6.6. *If $n \geq 3$, then the stabilizer of the line spread $\mathcal{S}$ of $PG(2n-1, q)$ in $PSO_{2n}(q)$ is the stabilizer of the partial spread $\mathcal{S}_0$ and is a maximal subgroup of $PSO_{2n}(q)$. If Q has type 1, then the stabilizer of $\mathcal{S}$ in $PO_{2n}(q)$ is the stabilizer of $\mathcal{S}_0$ and is a maximal subgroup of $PO_{2n}(q)$. If Q has type ξ, then the stabilizer lies in $PSO_{2n}(q)$.*

The proof of Theorem 6.5 is again by induction on the vector space dimension n, with $n = 3$ as an initial step. We omit the details.

Remark 6.7. If $n = 3$ or 4, there are connections between the partial spread $\mathcal{S}_0$ and certain combinatorial configurations in $PG(2n-1, q)$.

When $n = 3$ and $\bar{Q}$ is elliptic, the lines of the partial spread $\mathcal{S}_0$ of $Q^-(5, q)$ is a so–called *BLT-set* (see [45]), that is, a set of q^2+1 mutually disjoint lines of $Q^-(5, q)$ with the property that every line of $Q^-(5, q)$, which is not a member of $\mathcal{S}_0$ meets non–trivially exactly two or none of the lines in $\mathcal{S}_0$. The partial spread $\mathcal{S}_0$ is the only known example of BLT–set of $\mathbf{Q}^-(5, q)$.

A 1–system $\mathcal{M}$ of $Q^-(7, q)$ is a set of $q^4 + 1$ lines $\ell_0, \ell_1, \dots, \ell_{q^4}$ of $Q^-(7, q)$ such that every plane of $Q^-(7, q)$ containing a line $\ell_i \in \mathcal{M}$ has an empty intersection with $(\ell_0, \ell_1, \dots, \ell_{q^4}) \setminus \ell_i$, see [44] for more details. There is just one 1–system of $Q^-(7, q)$ known, both for q even and for q odd. This is the *classical* 1–system, which arises from the so–called trace trick applied to $Q^-(3, q^2)$ considered as an ovoid of itself.

Other $\mathcal{C}_3$–embeddings remain to be investigated such as unitary embeddings and certain special orthogonal embeddings: this is work in progress!

7. The class $\mathcal{C}_5$.

We recall that for a classical group G acting on an n–dimensional vector space V over a field F, the class $\mathcal{C}_5$ is the collection of normalizers of the classical groups acting on the n–dimensional vector spaces V_K over maximal subfields K of F such that $V = F \otimes_K V_K$.

Apart form the work of Kleidman and Liebeck, very little has been done for subgroups belonging to this class. As far as we know there are just three papers by Li [35], [36], [37] and Li and Zha [38] devoted to this case.

When the ground field is finite and G is a unitary group then there seems to be a close connection between subgroups in the class $\mathcal{C}_5$ and the geometry of commuting polarities as it has been introduced by Segre [42].

Here, we start with certain symplectic subgroups belonging to the class $\mathcal{C}_5$ of the unitary group $PSU_n(K)$, $n \geq 4$ even, where K now is any field admitting a non-trivial involutory automorphism. When the ground field is finite, the same result has also been obtained by Li and Zha in [38] using suitable subgroups of unitary transvections.

7.1. Symplectic subgroups of unitary groups.

Let K be a commutative field admitting a non-trivial involutory automorphism $\lambda \mapsto \bar{\lambda}$, with K_0 the fixed subfield.

Suppose that V is an n–dimensional vector space over K_0 and A is a non–degenerate alternating bilinear form on V. Let ω be an element of $K \setminus K_0$. Then $K = K_0 \oplus K_0\omega$ and there is a vector space $W = V \otimes_{K_0} K = \{(\alpha + \beta\omega)v | \alpha, \beta \in K_0, v \in V\}$. Any vector $w \in W$ can be written as $w = \sum v_i \otimes (a_i + b_i\omega) =$

$\sum(v_i \otimes 1)a_i + \sum(v_i \otimes \omega)b_i = (\sum v_i a_i) \otimes 1 + (\sum v_i b_i) \otimes w = w_1 + w_2\omega$. [Also if $\omega^2 = \gamma\omega + \delta$, then $(\alpha + \beta\omega)(w_1 + w_2\omega) = (\alpha w_1 + \beta\delta w_2) + (\beta w_1 + \beta\gamma w_2 + \alpha w_2)\omega$]. There is a natural extension of A to an anti-hermitian form C on W given by:

$$C(w_1+w_2\omega, v_1+v_2\omega) = A(w_1, v_1)+\omega\bar{\omega}A(w_2, v_2)+\omega A(w_2, v_1)+\bar{\omega}A(w_1, v_2).$$

If $char\, K = 2$, then C is already an hermitian form. In all cases there exists a $\tau \in K$ such that $\bar{\tau} = -\tau$ (as follows from Hilbert's Theorem 90) and τC is a hermitian form with the same group as C. We write H for τC, $U_n(K)$ for the unitary group of H, $Sp_n(K_0)$ for the symplectic group of A. We obtain the embedding $Sp_n(K_0) \leq SU_n(K)$. Note that H does not depend on the choice of ω. Factoring out scalars, we get the embedding $PSp_n(K_0) \leq PSU_n(K)$.

Let $x = w_1 + w_2\omega \in W$. Then, with respect to H, x is isotropic if and only if $C(x, x) = 0$, i.e., if and only if $\omega A(w_2, w_1) + \bar{\omega}A(w_1, w_2) = 0$, i.e., if and only if $\omega A(w_2, w_1) = \bar{\omega}A(w_2, w_1)$, i.e., if and only if $A(w_1, w_2) = 0$. In particular every vector in V is isotropic with respect to H. Suppose that $0 \neq v \in V$ and that t is a unitary transvection centred on v. Then $t : x \mapsto x + \lambda H(x, v)v$ for some $\lambda \in K$ such that $\bar{\lambda} = -\lambda$. If $x \in V$, then $t(x) = x + \lambda\tau A(x, v)v$ with $\lambda\tau \in K_0$, so t fixes V globally and the restriction of t to V is a symplectic transvection, i.e., $t \in Sp_n(K_0)$.

Let $\mathcal{H}$ be the Hermitian variety of $PG(n-1, K)$ associated with H. Let Σ be the set of points of the $PG(n-1, K_0)$ corresponding to V, considered as a subset of $\mathcal{H}$ inside $PG(n-1, K)$. We can regard $Sp_n(K_0)$ and $SU_n(K)$ as acting on $PG(n-1, K)$. Then $Sp_n(K_0)$ fixes $\mathcal{H}$ globally and has Σ as one orbit. Suppose that x and y are isotropic vectors in W corresponding to points of $\mathcal{H} \setminus \Sigma$. Then $x = w_1 + w_2\omega$, $y = v_1 + v_2\omega$, for some linearly independent $w_1, w_2 \in V$ and some linearly independent $v_1, v_2 \in V$ and by Witt's Theorem there is an element of $Sp_n(K_0)$ taking w_i to v_i for each i, i.e., taking x to y. Hence $Sp_n(K_0)$ has exactly two orbits on $\mathcal{H}$.

Let G_n denote the stabilizer of Σ in $SU_n(K)$ and let F be a subgroup of $SU_n(K)$ such that $G_n < F$. Then F has a single orbit of points on $\mathcal{H}$. If t is any unitary transvection in $SU_n(K)$, centred on y say, then there exists $f \in F$ such that $f(y) \in V$ and ftf^{-1} is a transvection centred on $f(y)$. Thus $ftf^{-1} \in Sp_n(K_0)$ and $t \in F$. It is well known that $SU_n(K)$, $n \geq 4$, is generated by its transvections [15], [16] and so $F = SU_n(K)$, and G_n is maximal in $SU_n(K)$. By the standard theorem for subgroups of quotients groups, the stabilizer $P(G_n)$ of Σ in $PSU_n(K)$ is maximal in $PSU_n(K)$.

It is of some interest to know the structure of G_n. Suppose that $g \in G_n$ and that $v_1, \ldots, v_m, v_{m+1}, \ldots, v_n$ (with $n = 2m$) is a symplectic basis for V

with respect to A (i.e., $A(v_i, v_{m+j}) = \delta_{ij}$). Then $A(g(v_i), g(v_{m+j})) = 0$ if and only if $i \neq j$ and by Witt's Theorem there exists $h_1 \in Sp_n(K_0)$ such that $h_1 g(v_i) = \lambda_i v_i$ for some $\lambda_i \in K$ ($1 \leq i \leq n$). As $h_1 g$ fixes Σ, it follows that for all $i > 1$, $\lambda_i = \beta_i \lambda_1$ for some $\beta_i \in K_0$. Hence $h_1 g = \lambda_1 I_n h_2$, where $h_2 \in GL_n(K_0)$ and fixes Σ, i.e., $h_2 \in GSp_n(K_0)$ (the general symplectic group, consisting of elements of $GL_n(K_0)$ that preserve A up to a scalar). It is now clear that g can be expressed as the product of a scalar matrix and an element of $GSp_n(K_0)$. Indeed all such products stabilize Σ. Hence G_n consists of all such products lying in $SU_n(K)$. The image $P(G_n)$ of G_n in $PGL_n(K)$ is then simply $PGSp_n(K_0) \cap PSU_n(K)$.

We have the following theorem, see [5].

Theorem 7.1. *G_n is a maximal subgroup of $SU_n(K)$ containing $Sp_n(K_0)$ and $G_n = (GSp_n(K_0).G_K) \cap SU_n(K)$ where G_K is the group of scalar matrices in $GL_n(K)$. The stabilizer $P(G_n)$ of Σ in $PSU_n(K)$ is a maximal subgroup of $PSU_n(K)$ containing $PSp_n(K_0)$ and $P(G_n) = PGSp_n(K_0) \cap PSU_n(K)$.*

7.2. Commuting polarities.

When the ground field K is finite, a natural approach in proving the maximality of $P(G_n)$ in $PSU_n(q^2)$ seems to be Segre's theory on commuting polarities as described in his celebrated paper [42]. Here, we briefly discuss this theory.

In $PG(n-1, q^2)$ a *non–singular Hermitian variety* is defined to be the set of all absolute points of a non–degenerate unitary polarity, and is denoted by $\mathcal{H}(n-1, q^2)$.

For an Hermitian variety $\mathcal{H} = \mathcal{H}(n-1, q^2)$ we have that ([42])

1. the number of points is $[q^n + (-1)^{n-1}][q^{n-1} - (-1)^{n-1}]/(q^2 - 1)$.
2. the number of generators (maximal totally singular subspaces) is $(q^3 + 1)(q^5 + 1)\ldots(q^{2m+1} + 1)$, if $n = 2m + 1$, and $(q + 1)(q^3 + 1)\ldots(q^{2m+1} + 1)$, if $n = 2m + 2$.

Let $\mathcal{A}$ be a symplectic polarity commuting with the Hermitian polarity $\mathcal{U}$ associated with $\mathcal{H}(n-1, q^2)$. Set $\mathcal{V} = \mathcal{A}\mathcal{U} = \mathcal{U}\mathcal{A}$. Then $\mathcal{V}$ is a non–linear collineation and $\mathcal{V}$, $\mathcal{A}$ and $\mathcal{U}$ together with the identity map form a four–group. From [42] pg. 132, the points and lines fixed by $\mathcal{V}$ form a configuration $\mathcal{W}$ on $\mathcal{H}(n-1, q^2)$. As B. Segre pointed out [42], p. 128, 132, $\mathcal{V}$ fixes $(q^{n+1}-1)/(q-1)$ points on $\mathcal{H}(n-1, q^2)$ but no point outside $\mathcal{H}(n-1, q^2)$, and leaves $((q^n - 1)(q^{n/2} - 1))/((q - 1)(q^2 - 1)$ lines of $\mathcal{H}(n-1, q^2)$ invariant so that each fixed point is incident with $(q^{n/2} - 1)/(q - 1)$ invariant lines and each invariant line is incident with $q + 1$ fixed points. This symmetric configuration extends to a $(n-1)$-dimensional projective space $\Sigma \cong PG(n-1,q)$. In this context, Σ is naturally equipped with the symplectic polarity $\mathcal{A}$ whose absolute lines are the lines of the above symmetric configuration. If $\mathcal{H}(n-1,q^2)$ has canonical equation $\sum_{i=0}^{n-1} X_i^{q+1} = 0$, then Σ can be described as the subset of points of $\mathcal{H}(n-1,q^2)$ whose coordinates are of the form $x_{2i-1} = \rho x_{2i}^q$, $i = 0,1,\dots,(n-2)/2$, where $\rho \in GF(q^2)$ such that $\rho^{q+1} = -1$. Finally, since $\mathcal{A}$ and $\mathcal{U}$ commute, a $\mathcal{V}$–fixed point P on $\mathcal{H}(n-1,q^2)$ admits the same conjugate hyperplane $P^{\perp}$ with respect to both $\mathcal{A}$ and $\mathcal{U}$.

The stabilizer of Σ in $PSU_n(q^2)$ turns out to be $PSp_n(q)\cdot((2,q-1)(q+1,n/2))/((q+1,n))$.

Now, we study some geometry of the embedding $P(G_4) \leq PSU_4(q^2)$. This gives us information on possible intersection sizes of two symplectic groups $PSp_4(q)$ inside the unitary group $PSU_4(q^2)$. Notice that, if $n=4$, there are $q^2(q^3+1)$ symplectic subgeometries embedded in $\mathcal{H}(3,q^2)$ [42].

The next lemma allows us to have information on possible intersection sizes of two copies of $PSp_4(q)$ inside $PSU_4(q^2)$.

Lemma 7.2. *Two symplectic subgeometries in $\mathcal{H}(3,q^2)$ meet in 0, $q+1$ or $2(q+1)$ points. In the case of $q+1$ points, the points lie on a totally isotropic line. In the case of $2(q+1)$ points, the points lie on a hyperbolic pair. If $q=2$, no two disjoint symplectic subgeometries exist.*

We have the following theorem.

Theorem 7.3. *Let $P(G_4)$, $P(G_4)'$ be the stabilizers in $PSU_4(q^2)$ of two symplectic geometries embedded in $\mathcal{H}(3,q^2)$. Set $K = P(G_4)\cap P(G_4)'$. Then one of the following cases occur. K is either the stabilizer of an elliptic congruence or, the stabilizer of a totally isotropic line or, the stabilizer of an hyperbolic pair. In all cases K is a maximal subgroup of $P(G_4)$.*

7.3. Orthogonal subgroups of unitary groups.

Here we discuss the maximality of certain orthogonal subgroups of the finite unitary group $PSU_n(q^2)$ for $n \geq 3$. Our main result is expressed again in terms of, and our approach to the proof depends on, the geometry of commuting polarities in projective spaces, described in [42].

In [9] Cossidente and King proved the following theorem.

Theorem 7.4. *Suppose that $n \geq 3$ and that q is odd. If $\mathcal{U}$ a non–degenerate unitary polarity on $PG(n-1,q^2)$ and if $\mathcal{B}$ is an orthogonal polarity commuting with $\mathcal{U}$, then the set of absolute points of $\mathcal{U}$ fixed by the non-linear collineation $\mathcal{V} = \mathcal{U}\mathcal{B}$ forms a non-degenerate quadric in a subgeometry $PG(n-1,q)$. The* *stabilizer of the quadric in $PSU_n(q^2)$ is maximal except when $n = 3$ and $q = 3$ or 5 and when $n = 4$, $q = 3$ and the quadric is hyperbolic.*

The approach here is essentially an induction argument in which a reduction to lower dimension is achieved via “hyperbolic rotations”. These are elements of order $q - 1$ that “rotate” most of the points on a hyperbolic line and leave fixed two points of the line and the points on its orthogonal complement.

We assume that q is a power of an odd prime p and that $n \geq 3$. It is perhaps appropriate to comment that over even order fields, the orthogonal groups have symplectic groups as overgroups and so cannot be maximal.

Here below is the geometric setting. Let us adopt the notation of the above subsection.

Let $\mathcal{B}$ be an orthogonal polarity commuting with the unitary polarity $\mathcal{U}$ associated with $\mathcal{H}$. Set $\mathcal{V} = \mathcal{B}\mathcal{U} = \mathcal{U}\mathcal{B}$. Then $\mathcal{V}$ is a non–linear collineation and from [42], the fixed points of $\mathcal{V}$ on $\mathcal{H}$ form a non–degenerate quadric $\mathcal{Q}$. Moreover, the complete set of points of Σ fixed by $\mathcal{V}$ forms a subgeometry Σ_0 isomorphic to $PG(n-1, q)$ such that $\mathcal{Q} = \Sigma_0 \cap \mathcal{H}$. Notice that the points of Σ fixed under $\mathcal{V}$ are those admitting the same tangent or polar space with respect to both the unitary polarity and the orthogonal polarity.

Remark 7.5. In the geometric setting of quadrics commuting with a Hermitian surface of PG$(3, q^2)$, q odd, a class of hemisystems on the Hermitian surface $\mathcal{H}(3, q^2)$ admitting the group $P\Omega_4^-(q)$ has been constructed in [12].

In a very recent paper jointly with A. Siciliano [13] we proved the following theorem which is based on the classification of irreducuble classical groups generated by elations due to A. Wagner [49].

Let V be an n-dimensional vector space, $n > 2$, over the finite field GF(q), $q = p^h$, p prime. Let GF(q_0) be the subfield of index r in GF(q), that is, $q_0 = q^{1/r}$, $r \geq 2$ prime. Let V_0 be the GF(q_0)-span of a GF(q)-basis β of V.

Theorem 7.6. *Let $G(q)$ be one of the groups $SL_n(q)$, $SU_n(q)$, $Sp_n(q)$. Let F be the form stabilized by $G(q)$. Let Σ_0 be the lattice of totally isotropic subspaces with respect to the restriction $F_{|V_0}$ of F to V_0 and let G_0 denote the stabilizers of Σ_0 in $G(q)$. If $q_0 > 2$, then G_0 is maximal in $G(q)$.*

Another $\mathcal{C}_5$–embedding remains to be investigated, namely, orthogonal subgroups of orthogonal groups: this is work in progress!

8. The class $\mathcal{C}_6$: any hope?

There is very little to say about this class. What we can observe is that there seems to be a close connection between extraspecial 2–groups and certain non–linear binary codes.

9. The class $\mathcal{C}_9$: the Steinberg's geometry.

Let G be a finite classical group with natural module V_0 of dimension $n \geq 2$ over the Galois field $GF(q^t)$. Let $V_0^{\psi^i}$ denote the G–module V_0 with group action given by $v \cdot g = vg^{\psi^i}$, where g^{ψ^i} denotes the matrix g with its entries raised to the q^i–th power, $i = 0, \ldots, t-1$. Then one can form the tensor product module $V_0 \otimes V_0^{\psi} \otimes \ldots \otimes V_0^{\psi^{t-1}}$, a module which can be realized over the field $GF(q)$. This gives rise to an embedding of the group G in a classical group having an n^t–dimensional natural module over $GF(q)$, yielding an absolutely irreducible representation of the group G. Also let V_0^* denote the G–module with group action given by $v \cdot g = vg^*$, where g^* is the inverse–transpose of g. For t even, there is a similar module given by $V_0 \otimes V_0^{*\psi} \otimes V_0^{\psi^2} \otimes \ldots \otimes V_0^{*\psi^{t-1}}$, realizable over $GF(q^2)$. Such representations are given by Steinberg ([46]) and further studied by Seitz ([43]). As Seitz observed, the normalizers of such "twisted tensor product groups" might easily be considered a ninth Aschbacher class [1].

As we have seen, the geometry of maximal subgroups in the Aschbacher classes is well understood (with the possible exception of the class $\mathcal{C}_6$). Our main purpose is to describe the geometry of subgroups lying outside the Aschbacher classes, little being known at present.

In the first part we concentrate on classical groups of low dimension, namely with $t = 2$ and $n = 3$, and study the embeddings $PGL(3, q^2)$ in $PGL(9, q)$, $PGL(3, q^2)$ in $PU(9, q^2)$ and $\Omega(3, q^2)$ in $\Omega(9, q)$; in the last case q is odd. We identify the normalizers of the embedded groups as (in most cases) maximal subgroups and stabilizers of geometrical configurations: hermitian veroneseans, twisted hermitian veroneseans and rational curves.

In the second part we study the geometry of two other classes of twisted tensor product groups: $PSL_2(q^t) \leq P\Omega^+_{2^t}(q)$; and $PSp_{2m}(q^t) \leq P\Omega^{\epsilon}_{(2m)^t}(q)$. Throughout this second part we shall assume that q is even and that $t \geq 2$. We will see that the last embeddings are closely related to partial ovoids of quadrics.

An *ovoid* $\mathcal{O}$ in a classical polar space [28,Chapter 26] is a set of singular points such that every maximal totally singular subspace contains just one point of $\mathcal{O}$. The points of $\mathcal{O}$ are pairwise non-orthogonal. More generally a *partial* *ovoid* is a set of pairwise non-orthogonal singular points. A partial ovoid is said to be *complete* if it is maximal with respect to set–theoretic inclusion.

The possibility of the existence of ovoids in polar spaces of various dimensions has been studied extensively, for both odd and even q (although the results referred to here are solely for even q). On the one hand there are known to be ovoids in $PG(7, q)$, both infinite families such as the unitary ovoids and the Desarguesian ovoids and individual ovoids such as Dye's ovoid, and although the 2-transitive ovoids have been classified by Kleidman in [34], there is no general classification of ovoids. On the other hand J.A. Thas [48] has shown that quadrics in $PG(2n, q)$ and elliptic quadrics in $PG(2n + 1, q)$ have no ovoids if $n \geq 4$ and Kantor [29], [30], has shown that hyperbolic quadrics in $PG(2n + 1, 2)$ have no ovoids if $n \geq 4$. Further, Blokhuis and Moorhouse ([2]) established an upper bound for the size of a partial ovoid of a polar space, a consequence of which is the non-existence of ovoids of hyperbolic quadrics in $PG(2n + 1, q)$ if $n \geq 4$. There are known to be examples of partial ovoids on quadrics in $PG(4n+3, 8)$ whose size meets the Blokhuis-Moorhouse bound ([25]) for all values of n; also in [25] there are examples of complete partial ovoids on quadrics in $PG(4n+1, 8)$ whose size falls just short of the Blokhuis-Moorhouse bound.

We find that our embedding of $PSL(2, q^t)$ is associated with an embedding of $PG(1, q^t)$ as a partial ovoid of a quadric in $PG(2^t - 1, q)$; if $t \geq 3$, then the quadric is hyperbolic. In $PG(2^t - 1, q)$ with q even, the Blokhuis–Moorhouse bound is given by $q^t + 1$. We thus have a family of partial ovoids whose size attains the Blokhuis–Moorhouse bound. In particular when $t = 3$ and $q \geq 4$ the embedding yields a nice description of a Desarguesian ovoid of the hyperbolic quadric of $PG(7, q)$ ([29], [30]) as the image of a projective line in much the same way as an elliptic quadric of $PG(3, q)$ is the image of a projective line. Similarly our embedding of $PSp_{2m}(q^t)$ in $P\Omega^{\epsilon}_{(2m)^t}(q)$ has a particular application when $m = 2$ in the embedding of ovoids of $PG(3, q^t)$ as partial ovoids of $PG(4^t - 1, q)$ again with size attaining the Blokhuis–Moorhouse bound. The families of complete partial ovoids arising from Suzuki-Tits ovoids are not equivalent to those arising from elliptic quadrics or projective lines; the partial ovoids given by Dye in [19] are different again.

10. $n = 3$, $t = 2$ and some generalizations.

10.1. The Hermitian Veronesean of PG(2, q^2).

10.1.1. Tensored spaces.

Let V_i, $1 \leq i \leq t$ be vector spaces of dimension n_i over the Galois field

$GF(q)$. Then $V = V_1 \otimes \ldots \otimes V_t$ is a vector space of dimension $\prod_{i=1}^{t} n_i = n$.

Assuming that $m_i = n_i - 1 \geq 1$ for each i, let $PG(m_1, q)$, $PG(m_2, q)$, $\ldots$, $PG(m_t, q)$ be the projective spaces over $GF(q)$ corresponding to V_1, V_2, $\ldots$, V_t. The set of all vectors in V of the form $v_1 \otimes \ldots \otimes v_t$ with $0 \neq v_i \in V_i$ corresponds to a set of points in $PG(n-1, q)$ known as the *Segre variety*, $S_{m_1,\ldots,m_r}$, of $PG(m_1, q), \ldots, PG(m_r, q)$, [28], 25.5.

10.1.2. A representation of GL(3, q²).

Let $G = GL_3(q^2)$ and let $\psi : GF(q^2) \to GF(q^2)$ be the Frobenius automorphism of $GF(q^2)$ given by $x \mapsto x^q$; we sometimes write $\bar{x}$ for x^q. Let V_0 be the natural module for $GL_3(q^2)$ over $GF(q^2)$. Let V_0^{ψ} be the G–module with group action given by $v \cdot g = vg^{\psi}$, where vg^{ψ} denotes the matrix g with its entries raised to the q-th power and let $V = V_0 \otimes V_0^{\psi}$. Then we have a representation $\rho : G \to GL_{3^2}(q^2)$ with $\rho(g) = g \otimes g^{\psi} \in GL_3(q^2) \otimes GL_3(q^2)$. This representation of $GL_3(q^2)$ is absolutely irreducible (c.f. [46]). The two representations ρ and $\rho\psi$ are isomorphic, so this representation of G on V can be written over $GF(q)$ (c.f. [1, 26.3]). Moreover if ψ_0 is the Frobenius automorphism of $GF(q^2)$ given by $x \mapsto x^{q_0}$ for any $q_0 \leq q$, then ρ and $\rho\psi_0$ are not isomorphic (c.f. [46]) and so ρ cannot be written over $GF(q_0)$.

We can give a concrete construction of a $GF(q)$-subspace of V fixed by $\rho(G)$. If v_1, v_2, v_3 is a basis for V_0 and $\alpha \in GF(q^2) \setminus GF(q)$ is fixed, then the vectors $v_i \otimes v_i$, $v_i \otimes v_j + v_j \otimes v_i$ and $\alpha v_i \otimes v_j + \alpha^q v_j \otimes v_i$ $(i \leq j)$ form a basis for an 3^2- dimensional $GF(q)$-subspace V_q of V fixed by G. There is an involution $\theta \in GL(3^2, q^2)$ on V that takes $v_i \otimes v_j$ to $v_j \otimes v_i$ for each i, j. We see that θ fixes V_q and normalizes $\rho(G)$; it is not difficult to show that θ does not lie in $\rho(G)$. Factoring out scalars we get an embedding of $PGL_3(q^2)$ in $PGL_{3^2}(q)$. Restricting to matrices with determinant one, we find $\rho(SL_3(q^2)) \leq SL_{3^2}(q)$ so that $PSL_3(q^2)$ is embedded in $PSL_{3^2}(q)$. The involution $-\theta$ lies in $SL_{3^2}(q)$ and normalizes $\rho(SL_3(q^2))$.

The realization over $GF(q)$ can be seen in another way. Let $\phi : V \to V$, $\lambda u_1 \otimes u_2 \to \lambda^q u_2 \otimes u_1$, with each u_i being one of v_1, v_2, v_3, extended linearly over $GF(q)$. Then ϕ is a semi–linear map that commutes with $\rho(G)$. Let W be the set of all vectors in V that are fixed by ϕ. Then for all $u \in W$, $g \in G$, $\phi(g(u)) = g(\phi(u)) = g(u)$, and so $g(u) \in W$. Thus the set W is fixed by G and it is a $GF(q)$–subspace of V. We observe that W contains all the vectors in V_q above. Moreover $GF(q)$–linearly independent vectors in W are linearly independent over $GF(q^2)$. For otherwise, consider a minimally–sized counterexample: $w_1, \ldots, w_r$ are linearly independent over $GF(q)$ but not over $GF(q^2)$. Then, there are scalars $\mu_1, \ldots, \mu_r \in GF(q^2)$ such that $\sum_{i=1}^{r} \mu_i w_i = 0$, with not all μ_i in $GF(q)$, and we may assume, without loss of generality that $\mu_r = 1$. Now $\sum_{i=1}^{r} \mu_i^q w_i = 0$ and so $\sum_{i=1}^{r-1} (\mu_i^q - \mu_i) w_i = 0$. We get a contradiction to r minimal. Given the absolute irreducibility of $\rho(G)$ we conclude that W has dimension 3^2 over $GF(q)$. Thus $W = V_q$.

10.2. The Hermitian veronesean embedding and its automorphism group.

Every element $z \in GF(q^2)$ has a unique representation as $x + \alpha y$ with $x, y \in GF(q)$ and $\bar{z} = x + \bar{\alpha} y$. Let $PG(2, q^2)$ denote the projective plane over $GF(q^2)$ and consider the map $\varphi : PG(2, q^2) \to PG(8, q^2)$ defined as follows:

$$(X_0, X_1, X_2) \to (X_0^{q+1}, X_1^{q+1}, X_2^{q+1}, X_0 X_1^q, X_0^q X_1, X_0 X_2^q, X_0^q X_2, X_1 X_2^q, X_1^q X_2).$$

The map φ is well–defined and injective. φ is called the *Hermitian veronesean embedding of* $PG(2, q^2)$ and we denote by $\widehat{H}$ the image of such a correspondence in $PG(8, q^2)$. We note that $\widehat{H}$ is contained in the Segre variety $S_{2,2} \simeq PG(2, q^2) \times PG(2, q^2)$. In fact $\widehat{H} = \{(P, \bar{P})f : P \in PG(2, q^2)\}$, where f is the Segre map sending $PG(2, q^2) \times PG(2, q^2)$ onto $S_{2,2}$. Indeed, the co-ordinate system for $PG(8, q^2)$ corresponds to the basis $v_i \otimes v_j$ $(1 \leq i \leq 3, 1 \leq j \leq 3)$ for V and the points of $\widehat{H}$ all lie in the Baer subgeometry of $PG(8, q^2)$ determined by the subset $V_q = W$ of V. The point–set $\widehat{H}$ is a variety of the Baer subgeometry known as *the Hermitian Veronesean of* $PG(2, q^2)$ [35], [13]. We denote the variety $\mathcal{H}$ when regarding it as a variety in $PG(8, q)$.

The variety $\mathcal{H}$ can also be described in terms of a normal line spread of $PG(5, q)$ [35]. If $\tau : PG(5, q^2) \to PG(5, q^2)$ is the map sending the point $P(X_0, \ldots, X_5)$ to $P(\bar{X}_3, \bar{X}_4, \bar{X}_5, X_0, X_1, X_2)$, then the points fixed by τ form a subgeometry $\mathcal{G}$ of $PG(5, q^2)$ isomorphic to $PG(5, q)$. If π is the plane with equations $X_3 = X_4 = X_5 = 0$, then the plane $\bar{\pi}$ with equations $X_0 = X_1 = X_2$ is disjoint from π. The set of lines of $PG(5, q^2)$ joining a point $P \in \pi$ with the point $\bar{P} \in \bar{\pi}$ is a normal line spread of $\mathcal{G}$ which can be represented on the Grassmannian $G_{1,5}$ of lines of $PG(5, q)$ by the variety $\mathcal{H}$. The variety $\mathcal{H}$ is a $(q^4 + q^2 + 1)$-cap of $PG(8, q)$ and it is not contained in any proper subspace of $PG(8, q)$ [13].

Let $G(\mathcal{H}) = \{\zeta \in PGL_9(q) : \zeta(\mathcal{H}) = \mathcal{H}\}$. The group $G(\mathcal{H})$ is a subgroup of $PGL_9(q)$ containing $PGL_3(q^2)$. Given a projectivity ξ of $PG(2, q^2)$, the corresponding projectivity of $G(\mathcal{H}) \leq PGL_9(q)$, denoted by ξ, is called *the Hermitian lifting of* ξ, or briefly the *$\mathcal{H}$–lifting of* ξ [13].

Let ξ be a linear collineation of $PG(2, q^2)$ with matrix representation $A = (a_{ij})$, $i, j = 0, 1, 2$. The matrix representation of the $\mathcal{H}$–lifting $xi^{\mathcal{H}}$ of ξ is the matrix whose generic column is

$$(\bar{a}_{0i} a_{0j}, \bar{a}_{0i} a_{1j}, \bar{a}_{0i} a_{2j}, \bar{a}_{1i} a_{0j}, \bar{a}_{1i} a_{1j}, \bar{a}_{1i} a_{2j}, \bar{a}_{2i} a_{0j}, \bar{a}_{2i} a_{1j}, \bar{a}_{2i} a_{2j})$$

with $0 \leq i, j \leq 2$. In particular, $xi^{\mathcal{H}}$ is the collineation induced by the Kronecker product $A \otimes A^{\psi}$. Hence, the embedding $PGL_3(q^2) \leq PGL_9(q)$ gives the representation of the group $PGL_3(q^2)$ as an automorphism group of the Hermitian Veronesean $\mathcal{H}$. Notice that the involutory Frobenius automorphism of $GF(q^2)$ induces a collineation of $PG(8, q)$ fixing $\mathcal{H}$ (actually, it interchanges the planes π and $\bar{\pi}$).

Theorem 10.1. *The full stabilizer H of the Hermitian Veronesean $\mathcal{H}$ in $PSL_9(q)$ is almost simple and is induced by an absolutely irreducible subgroup of $SL_9(q)$ modulo scalars.*

Corollary 10.2. *H is isomorphic to $PSL_3(q^2) \cdot [(q-1, 3)^2/(q-1, 9)] \cdot C_2$ and is a maximal subgroup of $PSL_9(q)$.*

10.2.1. Generalizations.

Here we discuss a generalization of the ideas above in which we consider mappings from $GL_n(q^t)$ to $GL_{n^t}(q)$.

Remark 10.3. The concrete realization over $GF(q)$ described above can be extended to a more general setting. Let $G = GL_n(q^t)$ and let $\psi : GF(q^t) \rightarrow GF(q^t)$ be the Frobenius automorphism of $GF(q^t)$ given by $x \mapsto x^q$. Let V_0 be the natural module for $GL_n(q^t)$ over $GF(q^t)$ with $V_0^{\psi^i}$ the G–module with group action given by $V \cdot g = vg^{\psi^i}$, and let $V = V_0 \otimes V_0^{\psi} \otimes V_0^{\psi^2} \ldots \otimes V_0^{\psi^{t-1}}$. Then we have a representation $\rho : G \rightarrow GL_{n^t}(q^t)$ with $\rho(g) = g \otimes g^{\psi} \otimes \ldots \otimes g^{\psi^{t-1}}$. As with the specific case above, this representation of $GL_n(q^t)$ is absolutely irreducible, can be written over $GF(q)$ but over no subfield of $GF(q)$. This time let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V_0 and let $\phi : V \rightarrow V$, $\lambda u_1 \otimes u_2 \otimes \ldots \otimes u_t \rightarrow \lambda^q u_t \otimes u_1 \otimes \ldots \otimes u_{t-1}$, with each u_i being one of $v_1, v_2, \ldots, v_n$, extended linearly over $GF(q)$. The set W of all vectors in V that are fixed by ϕ is fixed by G and is a $GF(q)$–subspace of V. Moreover $GF(q)$–linearly independent vectors in W are linearly independent over $GF(q^t)$ and we conclude that W has dimension n^t over $GF(q)$. We return to this later.

10.3. The Twisted Hermitian Veronesean of PG(2, q²): the geometry of flags of PG(2, q²).

10.3.1. Embedding PGL₃(q²) in PU₉(q²).

The notation here is similar to that used earlier, with $G = GL_3(q^2)$, ψ the Frobenius automorphism of $GF(q^2)$ and V_0 the natural module for $GL_3(q^2)$ over $GF(q^2)$. Let V_0^* be the dual module of V_0 (with group action given by $v \cdot g = vg^* = v(g^T)^{-1}$) and let $V = V_0^* \otimes V_0^{\psi}$. Then we have an absolutely irreducible representation $\rho^* : G \to GL_{3^2}(q^2)$ with $\rho^*(g) = g^* \otimes g^\psi \in GL_3(q^2) \otimes GL_3(q^2)$ [46]. The module presented here is dual to $V_0 \otimes V_0^{\psi *}$ but is a more convenient setting from our point of view. The modules $V^* = V_0 \otimes (V_0^{\psi *})$ and $V^\psi = (V_0^{\psi *}) \otimes V_0$ are isomorphic and so $\rho^*(G)$ fixes a Hermitian form on V. In general such a representation cannot be realized over a subfield of $GF(q^2)$ (see [1], [33, theorem 5.4.5]). Indeed, suppose $V_0^* \otimes V_0^\psi$ can be realized over a proper subfield $GF(q_0)$ of $GF(q^2)$. Then $V_0^* \otimes V_0^\psi \simeq V_0^{\psi_0 *} \otimes V_0^{\psi\psi_0}$, where ψ_0 is the automorphism $x \mapsto x^{q^0}$ of $GF(q^2)$. By [46] these two representations are equivalent if and only if, either $V_0^* \simeq V_0^{\psi_0 *}$ (i.e., $V_0 \simeq V_0^{\psi_0}$), which is not possible, or $V_0^* \simeq V_0^{\psi\psi_0}$ and $V_0^\psi \simeq V_0^{\psi_0 *}$. The latter can happen if and only if $\psi_0 = \psi$ and $V_0 \simeq V_0^*$, which in turn is possible if and only if $GL_3(q^2)$ fixes a symmetric or symplectic bilinear form on V_0. As $GL_3(q^2)$ fixes no such form on V_0, its representation on V cannot be realized over a proper subfield of $GF(q^2)$. The same applies to $SL_3(q^2)$.

The representation of $GL_3(q^2)$ may be stated explicitly as follows. Assume that we have a fixed basis v_1, v_2, v_3 for V_0 as in the previous section. A non-degenerate Hermitian form is defined by $(u \otimes v, w \otimes z) = (uz^{\psi T}).(w^\psi v^T)$ and this is preserved by $\rho^*(g) = (g^T)^{-1} \otimes g^\psi$ for all $g \in G$. It follows that $PGL_3(q^2)$ can be embedded in $PU_9(q^2)$. Recall that the involution θ of $V(9, q^2)$ takes $v_i \otimes v_j$ to $v_j \otimes v_i$ for each i, j; we now observe that θ lies in $U_9(q^2)$ and normalizes (but does not lie in) $\rho^*(G)$. We find that $\rho^*(SL_3(q^2)) \le SU_9(q)$ with $PSL_3(q^2)$ embedded in $PSU_9(q^2)$; $-\theta \in SU_9(q^2)$ and normalizes $\rho^*(SL_3(q^2))$. We shall shortly see that the image of $PGL_3(q^2)$ is an automorphism group of a variety that we call the *Twisted Hermitian Veronesean* of $PG(2, q^2)$ and denote by $\mathcal{H}^*$.

10.3.2. The Twisted Hermitian Veronesean.

In considering the action of $G = GL_3(q^2)$ on $V(9, q^2)$, we see that one orbit is given by $\{(v_1 \otimes v_2)\rho^*(g) : g \in GL_3(q^2)\}$ and this orbit consists of singular vectors. The corresponding orbit in $PG(8, q^2)$ is preserved by (the image of) $PGL_3(q^2)$. Let $\mathcal{R}$ be the set of non-zero singular vectors of the form $u \otimes v$. For any $u \otimes v \in \mathcal{R}$ and any $g \in G$ we see that $(u \otimes v)g = ug^* \otimes vg^\psi$ is singular and so lies in $\mathcal{R}$. It is straightforward to calculate that $u \otimes v$ is singular if and only if $u.w^{\psi T} = 0$, so singular vectors of the form $v_1 \otimes w$ are precisely the vectors given by $w = \lambda v_2 + \mu v_3$ where $\lambda, \mu \in GF(q^2)$; such a singular vector is mapped to $v_1 \otimes v_2$ by the inverse of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^q & \mu^q \\ 0 & \nu & \zeta \end{pmatrix},$$

where $\nu, \zeta \in GF(q^2)$ such that the matrix is non-singular. Thus G is transitive on $\mathcal{R}$, i.e., $\mathcal{R}$ is precisely the orbit that we initially identified. The involution $-\theta$ preserves the Hermitian form and preserves the tensor product $V_0 \otimes V_0$ so it preserves $\mathcal{R}$. Hence the stabilizer in $U_9(q^2)$ of $\mathcal{R}$ has a subgroup isomorphic to $GL_3(q^2) \cdot C_2$.

Let $\mathcal{H}^*$ be the set of points in $PG(8, q^2)$ corresponding to $\mathcal{R}$. We call this the *Twisted Hermitian Veronesean* of $PG(2, q^2)$. This set is the intersection of the Hermitian variety corresponding to the given Hermitian form and the Segre variety $S_{2,2}$. As we have seen above, the points of $\mathcal{H}^*$ corresponding to $v_1 \otimes w$ for various w are just $P(v_1 \otimes (\lambda v_2 + \mu v_3))$, i.e., are the points on a line. It follows that $\mathcal{H}^*$ consists of $q^4 + q^2 + 1$ disjoint lines of the form $u \otimes L$. At the same time $\mathcal{H}^*$ can be expressed as the disjoint union of lines of the form $L \otimes u$.

Theorem 10.4. *The full stabilizer H^* of the Twisted Hermitian Veronesean $\mathcal{H}^*$ in $PSU_9(q^2)$ is almost simple and is induced by an absolutely irreducible subgroup of $SU_9(q^2)$ modulo scalars.*

Corollary 10.5. *H^* is isomorphic to $PSL_3(q^2)[(q+1,3)^2/(q+1,9)] \cdot C_2$ and is a maximal subgroup of $PSU_9(q^2)$.*

10.3.3. Generalizations.

As before, we discuss a generalization of the ideas above and consider mappings from $GL_n(q^2)$ to $U_{n^2}(q^2)$.

Remark 10.6. From [33], Lemma 2.10.15 ii, Theorem 5.4.5, there is an absolutely irreducible representation ρ^* of the group $G = GL_n(q^2)$ on $V = V_0^* \otimes V_0^\psi$ over $GF(q^2)$ that fixes a Hermitian form, not generally realizable over a subfield of $GF(q^2)$. As argued above, ρ^* can be realized over a subfield of $GF(q^2)$ if and only if $GL_n(q^2)$ fixes a symmetric or symplectic bilinear form on V_0, and this can never happen. However, when we consider $SL_n(q^2)$, we find that it fixes a non-degenerate symplectic bilinear form precisely when $n = 2$. In this one case, $\rho^*(SL_2(q^2))$ can be realized over $GF(q)$, effectively we have the well known isomorphism between $PSL_2(q^2)$ and $\Omega_4^-(q)$. The non-degenerate Hermitian form defined by $(u \otimes v, w \otimes z) = (uz^{\psi T}).(w^\psi v^T)$ is preserved by $\rho^*(G)$. It now follows that $PGL_n(q^2)$ can be embedded in $PU_{n^2}(q^2)$. The involution θ lies in $U_{n^2}(q^2)$ and normalizes (but does not lie in) $\rho^*(G)$. We find that for $n \geq 3$ the image of $PGL_n(q^2)$ acts transitively on the intersection of a Hermitian variety and a Segre variety, the automorphism group of this intersection contains $PGL_n(q^2) \cdot C_2$ and so the full automorphism group is absolutely irreducible. This intersection can be expressed as the disjoint union of subspaces of (projective) dimension $n - 2$ in two ways.

10.4. $\mathbf{PSL_2(q^2) \simeq \Omega_3(q^2) \leq \Omega_9(q)}$, q odd, as the stabilizer of a rational curve.

10.4.1. Embedding $\mathbf{\Omega_3(q^2)}$ in $\mathbf{\Omega_9(q)}$.

Now suppose that q is odd, that $H \leq GL_3(q^2)$ and that H fixes a non–degenerate symmetric bilinear form f_0 on V_0. Then one can define a non–degenerate symmetric bilinear form $f = f_0 \otimes f_0$ on V by $f(u_1 \otimes u_2, w_1 \otimes w_2) = f_0(v_1, w_1).f_0(v_2, w_2)$, fixed by $\rho(H)$. Assume that the basis $\{v_1, v_2, v_3\}$ chosen for V_0 is such that $f_0(v_i, v_j) \in GF(q)$ for each i, j. Recall the semilinear map ϕ introduced before (with W its space of fixed vectors). Then for any $u, v \in W = V_q$ we have $f(u, v) = f(\phi(u), \phi(v)) = f(u, v)^q$. Hence $f(u, v) \in GF(q)$ for all $u, v \in W$. If $H = O_3(q^2)$, then $\rho(H)$ is absolutely irreducible on V and therefore the restriction of f to W is non-degenerate. Thus $\rho(O_3(q^2)) \leq O_9(q)$. Indeed (considering commutator subgroups) $\rho(\Omega_3(q^2)) \leq \Omega_9(q)$ and the restriction of ρ to $\Omega_3(q^2)$ is injective.

Let us specifically choose the basis v_1, v_2, v_3 for V_0 so that the quadratic form corresponding to f_0 is given by $Q_0(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) = \lambda_3^2 - \lambda_1 \lambda_2$. Then the points on the conic $\mathcal{C}_0$ of Q_0 can be represented by $(1, t^2, t) : t \in GF(q^2)$ together with $(0, 1, 0)$. The image $\mathcal{X}$ of $\mathcal{C}_0$ in the Hermitian Veronesean $\mathcal{H}$ is then given by

$$\{P(1, t^{2q+2}, t^{q+1}, t^{2q}, t^2, t^q, t, t^{2+q}, t^{2q+1}) : t \in GF(q^2)\} \cup \{P(0, 1, 0, \ldots, 0)\}.$$

Thus $\mathcal{X}$ is a rational curve, all of whose points lie in a Baer subgeometry. Put another way, $\mathcal{X}$ is just the orbit of $\rho(SO_3(q^2))$ on $PG(8, q^2)$ given by $\{P(v_1 g \otimes v_1 g^{\psi}) : g \in SO_3(q^2)\}$. A point $x \otimes x^{\psi}$ of $\mathcal{H}$ is singular precisely when x is singular. Hence if $\mathcal{Q}$ is the quadric corresponding to the bilinear form f the points of $\mathcal{Q}$ lying on $\mathcal{H}$ are precisely the points of $\mathcal{X}$, i.e., $\mathcal{X}$ is the intersection of $\mathcal{H}$ and $\mathcal{Q}$. No two points of $\mathcal{X}$ are orthogonal so $\mathcal{X}$ is a partial ovoid.

There is a further geometric description. Take a conic C in π and $\bar{C}$ in $\bar{\pi}$. The lines joining a point on C with its conjugate on $\bar{C}$ form a set $\mathcal{Y}$ of $q^2 + 1$ lines defined over $GF(q)$, and it lies in the subgeometry $\mathcal{G}$ of $PG(5, q^2)$. The image of $\mathcal{Y}$ on the Grassmannian $G_{1,5}$ of lines of $PG(5, q)$, under the Plücker map, is the curve $\mathcal{X}$.

Proposition 10.7. *Let X be the full stabilizer of the rational curve $\mathcal{X}$ in $\Omega_9(q)$ (q odd), then X contains a subgroup isomorphic to $PSL_2(q^2) \cdot C_2$.*

Theorem 10.8. *The full stabilizer X of the rational curve $\mathcal{X}$ is almost simple and is an absolutely irreducible subgroup of $\Omega_9(q)$.*

Corollary 10.9. *Assume that $q \neq 3$. X is isomorphic to $PSL_2(q^2) \cdot C_2$ and is maximal in $P\Omega_9(q)$.*

In considering the case $q = 3$ we find the following.

Corollary 10.10. *If $q = 3$ then X is isomorphic to A_{10} and is maximal in $P\Omega_9(3)$.*

10.4.2. Generalizations.

Let us consider possible generalizations of the ideas above. On this occasion we consider different forms as well as mappings from subgroups of $GL_n(q^t)$ to $GL_{n^t}(q)$, and we consider possible embeddings of alternating groups.

If $O_n(q^t)$ is the orthogonal group of a non-degenerate symmetric bilinear form f_0 on $V(n, q^t)$ (with q odd) and if ρ is the representation of $GL_n(q^t) \to GL_{n^t}(q^t)$ described in Subsection 10.2.1, then $\rho(O_n(q^t))$ preserves a non-degenerate symmetric bilinear form f on $V(n^t, q)$. If f_0 can be given by a matrix with entries in $GF(q)$, then $f = f_0 \otimes \cdots \otimes f_0$ (t copies of f_0); in other cases some care is required in writing down f. If an appropriate basis is chosen for V_0, then f is defined on $V_q = W$ over $GF(q)$ and $\rho(O(n, q^t)) \leq O_{n^t} q)$. If we assume $n \geq 3$ and exclude the case $O_4^+(q^t)$, the subgroup $\rho(\Omega_n(q^t))$ is absolutely irreducible and cannot be written over a subfield of $GF(q)$.

If $Sp_n(q^t)$ is the symplectic group of a non-degenerate alternating form f_0 on $V(n, q^t)$ (with n even but q odd or even), then $\rho(Sp_n(q^t))$ preserves the tensor product form f. If t is odd, then f is an alternating form and we find that $\rho(Sp_n(q^t))$ is a subgroup of $Sp_{n^t}(q)$. If t is even and q is odd, then f is a symmetric bilinear form and $\rho(Sp_n(q^t))$ is a subgroup of $O_{n^t}(q)$. If q is even (and n must then be even), then $O_n(q^t)$ maybe regarded as a subgroup of $Sp_n(q^t)$ so $\rho(O_n(q^t)) \leq Sp_{n^t}(q)$, but more than this $\rho(Sp_n(q^t))$ preserves a quadratic form on $V_q = W$ so $\rho(O_n(q^t)) \leq \rho(Sp_n(q^t)) \leq O_{n^t}(q)$. If $U_n(q^t)$ is the unitary group of a non-degenerate Hermitian form f_0 on $V(n, q^t)$ (with q square and t odd), then the tensor product form f is an Hermitian form preserved by $\rho(U_n(q^t))$ and $\rho(U_n(q^t)) \leq U_{n^t}(q)$. [Except in the case of $O_4^+(q^t)$, the image under ρ is absolutely irreducible and cannot be written over a subfield of $GF(q)$.]

It is worth noting that the restrictions on n mean that there is no irreducible subgroup $\rho(SL_3(q^2))$ of $SL_9(q)$ and thus, for q even, no irreducible subgroup $\rho(O_3(q^2))$ of $SL_9(q)$. The restriction on t for $U_n(q^t)$ is more subtle. Steinberg's Tensor Product Theorem leads us to believe that for t even $\rho(U_n(q^t))$ is not absolutely irreducible. Indeed for the case $t = 2$ it is known that $\rho(U_n(q^2))$ is reducible, indeed $\rho(U_n(q^2))$ fixes all vectors in a 1-dimensional subspace of

$V(n^2, q^2)$; moreover the restriction of the Hermitian form f to $V_q = W$ is actually a symmetric bilinear form so $\rho(U_n(q^2))$ is a subgroup of $O_{n^2}(q)$ (for q odd) or $Sp_{n^2}(q)$ (for q even).

11. Embedding $\mathbf{Sp_{2m}(q^t)}$ in $\Omega^{\epsilon}_{(2m)^t}(q)$.

11.1. Introduction.

Let V_0 now be a $2m$–dimensional vector space over $GF(q^t)$. Then $V = V_0 \otimes V_0 \otimes \ldots \otimes V_0$ (t copies of V_0) is a vector space of dimension $(2m)^t$. If f_0 is a non-degenerate alternating form on V_0 then one can define a non-degenerate alternating form on V as follows:

$$f(u_1 \otimes \ldots \otimes u_t, w_1 \otimes \ldots \otimes w_t) = \prod_{i=1}^{t} f_0(u_i, w_i).$$

Moreover there exists a unique quadratic form Q on V such that $Q(u_1 \otimes \ldots \otimes u_t) = 0$ for all $u_i \in V_0$ and such that f is the bilinear form associated with Q ([1]). If U is an m–dimensional totally isotropic subspace of V_0, then $U \otimes V_0 \otimes \ldots \otimes V_0$ is a totally singular subspace of V of dimension $(2m)^t/2$, so Q is a quadratic form of maximal Witt index.

Let G be the group $Sp_{2m}(q^t)$ acting on V_0 and preserving f_0. Let $x_1, x_2, \ldots, x_m,\ y_1, y_2, \ldots, y_m$ be a symplectic basis for V_0 (so $f_0(x_i, x_j) = f_0(y_i, y_j) = 0$ and $f_0(x_i, y_j) = \delta_{ij}$). The action of G on V, as described in the introduction, is given by $u_1 \otimes \ldots \otimes u_t \mapsto u_1 g \otimes u_2 g^{\psi} \otimes \ldots \otimes u_t g^{\psi^{t-1}}$ with g^{ψ^i} preserving f_0 for each i. Thus g preserves Q on V. Taking note of the fact that $Sp_{2m}(q^t)$ is perfect, we thus have a representation $\rho : G \to \Omega^{+}_{(2m)^t}(q^t)$.

We introduced a semi–linear map ϕ on V and the subset W of V consisting of all vectors fixed by ϕ. Here we list a number of properties of ϕ and W that we shall need to refer to. We recall that $\phi(\lambda u_1 \otimes u_2 \otimes \ldots \otimes u_t) = \lambda^q u_t \otimes u_1 \otimes \ldots \otimes u_{t-1}$, whenever each u_i is one of $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ and $\lambda \in GF(q^t)$ and then ϕ is extended linearly over $GF(q)$. If we write $v^{\psi} = \sum(\lambda_i^q x_i + \mu_i^q y_i)$ when $v = \sum(\lambda_i x_i + \mu_i y_i) \in V_0$, then $\phi(v_1 \otimes v_2 \otimes \ldots \otimes v_t) = v_t^{\psi} \otimes v_1^{\psi} \otimes \ldots \otimes v_{t-1}^{\psi}$. We have not explicitly referred to the case: $m = 1$ here. However, given any basis x_1, y_1 for V_0 when $m = 1$ we can define an alternating form f_0 on V_0 such that $f_0(x_1, y_1) = 1$ and the symplectic group $Sp_2(q^t)$ is just $SL_2(q^t)$.

Lemma 11.1.

(i) The semilinear map ϕ commutes with $\rho(G)$ on V;

(ii) *The set* $W = \{w \in V : \phi(w) = w\}$ *is a* $GF(q)$*–subspace of* V*, (globally) stabilized by* $\rho(G)$*;*
(iii) *Any vectors in* W *that are linearly independent over* $GF(q)$ *are linearly independent over* $GF(q^t)$*;*
(iv) W *has dimension* $(2m)^t$ *over* $GF(q)$ *and spans* V *over* $GF(q^t)$*.*
(v) $Q(w) \in GF(q)$ *for each* $w \in W$ *and* Q *is non–degenerate on restriction on* W*.*

Since $Sp_{2m}(q^t)$ has a trivial centre and is simple, ρ is injective. Thus we may regard $Sp_{2m}(q^t)$ as a subgroup of $\Omega^{\epsilon}_{2^t}(q)$, and equivalently in the projective context, $PSp_{2m}(q^t)$ as a subgroup of $P\Omega^{\epsilon}_{(2m)^t}(q)$.

11.2. The nature of the quadrics.

It is of some interest to know the nature of the quadratic form on W described above. The investigation is complex but yields the following.

Theorem 11.2. *If* t *is odd, then* Q *has maximal Witt index on* W*. If* t *is even, then* Q *has maximal Witt index on* W *except when* $t = 2$ *and* $m = n/2$ *is odd. In the exceptional case,* Q *has non-maximal Witt index on* W*.*

11.3. Embedding PG(2m − 1, q^{t}) in PG((2m)^{t} − 1, q).

11.3.1. Embedding the projective line and partial ovoids.

Given $0 \neq v \in V_0$ let us denote by $\underline{v}$ the vector $v \otimes v^{\psi} \otimes \ldots \otimes v^{\psi^{t-1}} \in V$; recall that all such vectors lie in W and are singular. Observe that for $0 \neq \lambda \in GF(q^t)$ we have $\underline{\lambda v} = (\lambda\lambda^q \ldots \lambda^{q^{t-1}})\underline{v}$ with $\lambda\lambda^q \ldots \lambda^{q^{t-1}} \in GF(q)$. Hence the injective map: $V_0 \to W$, $v \to \underline{v}$ leads to an injective map $\varphi : PG(2m-1, q^t) \to PG((2m)^t - 1, q)$. Suppose that $x, y \in V_0$ such that $f_0(x, y) \neq 0$. Then

$$\begin{aligned} f(\underline{x}, \underline{y}) &= f_0(x, y) \cdot f_0(x^{\psi}, y^{\psi}) \cdot \ldots \cdot f_0(x^{\psi^{t-1}}, y^{\psi^{t-1}}) \\ &= f_0(x, y) \cdot f_0(x, y)^q \cdot \ldots \cdot f_0(x, y)^{q^{t-1}} \neq 0\,. \end{aligned}$$

This leads to the following theorem:

Theorem 11.3. *Suppose that* $\mathcal{L}$ *is either the projective line* $PG(1, q^t)$ *or a partial ovoid of a symplectic polarity of* $PG(2m-1, q^t)$ *with* $m \geq 2$*, and that* $\mathcal{P} = \varphi(\mathcal{L})$ *in* $PG((2m)^t - 1, q)$*. Then* $\mathcal{P}$ *is a partial ovoid of a non-degenerate quadric in* $PG((2m)^t - 1, q)$ *having the same size as* $\mathcal{L}$*. The quadric is hyperbolic unless* $t = 2$ *and* $m = n/2$ *is odd, in which case it is elliptic.*

11.3.2. The Blokhuis-Moorhouse bound.

In their 1995 paper [2] Blokhuis and Moorhouse give an upper bound for size of a partial ovoid of a classical polar space in $PG(k, q)$. If $q = p^e$ where p is prime, then the size of a partial ovoid is no greater than $\binom{k+p-1}{k}^e + 1$. If $p = 2$, then this bound becomes simply $(k+1)^e + 1$. If, in addition, $k + 1 = 2^t$ for some t, then the bound is $2^{te} + 1 = q^t + 1$. In particular we get the same value for the bound in $PG(2^a - 1, 2^{et})$ and $PG(2^{at} - 1, 2^e)$ for any $a \geq 1$.

The following theorem is a corollary to 11.3. It demonstrates that, for $p = 2$, the Blokhuis-Moorhouse bound is sharp for arbitrarily large dimension of the form $2^t - 1$.

Theorem 11.4. *If $\mathcal{L}$ is the projective line $PG(1, q^t)$, then $\varphi(\mathcal{L})$ is a partial ovoid of a non-degenerate quadric in $PG(2^t - 1, q)$ whose size attains the Blokhuis-Moorhouse bound. If $\mathcal{L}$ is a partial ovoid of a symplectic polarity of $PG(2^a - 1, q^t)$ (with $a \geq 2$ and $q = 2^e$) whose size attains the Blokhuis-Moorhouse bound, then $\varphi(\mathcal{L})$ is a partial ovoid of a non-degenerate quadric in $PG(2^{at} - 1, q)$ whose size attains the Blokhuis-Moorhouse bound.*

11.3.3. Complete partial ovoids from embeddings of PG(1, q^t) and ovoids of PG(3, q^t).

Now let us consider complete partial ovoids in $PG(2^t - 1, q)$ and $PG(4^t - 1, q)$ arising as images of projective lines and of elliptic quadrics or Suzuki-Tits ovoids of $PG(3, q^t)$ respectively. We identify the stabilizers of these partial ovoids using the classification of finite 2-transitive groups. In turn this relies on the Classification of Finite Simple Groups. Our source for the list of finite 2-transitive groups is [3] where the groups are listed in Tables 7.3 and 7.4. Our interest is in groups having permutation degree $2^k + 1$ for some $k \geq 1$.

Result 11.5. *A 2-transitive permutation group of degree 2^k+1 for some $k \geq 1$ is almost simple with unique minimal normal subgroup one of the following: A_M with $M = 2^k + 1$, $SL_2(2^k)$, $PSU_3(2^{2k/3})$, $Sz(2^{k/2})$ where $k/2$ an odd integer.*

We consider three possibilities:

- Case PL: $\mathcal{L}$ is the projective line $PG(1, q^t)$ with $t \geq 3$, $G = SL_2(q^t)$, $2^k = q^t$, we write $\Omega = \Omega^+_{2^t}(q)$ and $PW = PG(2^t - 1, q)$.
- Case EQ: $\mathcal{L}$ is an elliptic quadric of $PG(3, q^t)$, $G = \Omega^-_4(q^t)$, $2^k = q^{2t}$, we write $\Omega = \Omega^+_{4^t}(q)$ and $PW = PG(4^t - 1, q)$.
- Case STO: $\mathcal{L}$ is a Suzuki-Tits ovoid of $PG(3, q^t)$ with q an odd power of 2 and t odd, $G = Sz(q^t)$, $2^k = q^{2t}$, we write $\Omega = \Omega^+_{4^t}(q)$ and $PW = PG(4^t - 1, q)$.

We write $\mathcal{P} = \varphi(\mathcal{L})$, $\tilde{F} = \rho(F)$ for any subgroup F of G and let $\tilde{H}$ be the stabilizer of $\mathcal{P}$ in Ω. Then $\tilde{H}$ contains $\tilde{G}$ and acts 2-transitively on $\mathcal{P}$. The action of $\tilde{G}$ on the vector space W is irreducible by Steinberg's Tensor Product Theorem ([46, Theorem 7.4, Theorem 12.2]) and so the points of $\mathcal{P}$ span the corresponding projective space. As the number of points exceeds the vector space dimension and $\tilde{G}$ acts 2-transitively, the only transformations fixing each point of $\mathcal{P}$ are scalar maps, here the identity is the only possibility. Thus the action of $\tilde{H}$ is faithful and we have:

Proposition 11.6. *$\tilde{H}$ is almost simple with unique minimal normal subgroup X being isomorphic to one of the following: A_M with $M = 2^k + 1$, $SL_2(2^k)$, $PSU_3(2^{2k/3})$, $Sz(2^{k/2})$ where $k/2$ is an odd integer.*

The subgroup $X \cap \tilde{G}$ is normal in $\tilde{G}$, but $\tilde{G}$ is simple so either $X \cap \tilde{G} = 1$ or $\tilde{G} \leq X$.

Proposition 11.7. $\tilde{G} \leq X$.

Proposition 11.8. *$X = \tilde{G}$ except when $q = 2$, in which case $X = A_M$ with $M = 2^k + 1$.*

Proposition 11.9. *The normalizer of X stabilizes $\mathcal{P}$.*

In summary, the results above give:

Theorem 11.10.

(i) Suppose that $q \geq 2$ is even and that $t \geq 3$. If $\mathcal{L} = PG(1, q^t)$, then the stabilizer of $\mathcal{P}$ in $\Omega^+_{2^t}(q)$ is the normalizer of $\rho(SL_2(q^t))$.

(ii) Suppose that $q \geq 2$ is even and that $t \geq 2$. If $\mathcal{L}$ is an elliptic quadric in $PG(3, q^t)$, then the stabilizer of $\mathcal{P}$ in $\Omega^+_{4^t}(q)$ is the normalizer of $\rho(\Omega^-_4(q^t))$.

(iii) Suppose that $q \geq 2$ is an odd power of 2 and that $t \geq 3$ is odd. If $\mathcal{L}$ is a Suzuki-Tits ovoid in $PG(3, q^t)$, then the stabilizer of $\mathcal{P}$ in $\Omega^+_{4^t}(q)$ is the normalizer of $\rho(Sz(q^t))$.

(iv) If $q = 2$, then in all three cases $\mathcal{P}$ is a polygon whose stabilizer is the alternating group on the points.

Remark 11.11. Comparing the partial ovoids arising as images of Suzuki-Tits ovoids with those arising as the images of elliptic quadrics, the difference in the structure of the stabilizers shows that the two families of partial ovoids are not equivalent. The cases where $\mathcal{L} = PG(1, q^t)$ (with $t \geq 2$ even) and where $\mathcal{L}$ is an elliptic quadric of $PG(3, q^t)$ lead to stabilizers with the same structure, but it is not clear whether or not the partial ovoids are equivalent. Indeed, the embedding procedure allows for the possibility of a stepped embedding of $PG(1, q^{abcd\dots})$ as a partial ovoid in $PG(2^a - 1, q^{bcd\dots})$ that is then embedded in $PG(2^{ab} - 1, q^{cd\dots})$ and so on, and it is not clear that the resulting partial ovoid is necessarily equivalent to that obtained from a single step embedding of $PG(1, q^{abcd\dots})$ in $PG(2^{abcd\dots} - 1, q)$.

For more details see [8], [11].

11.3.4. Dye's partial ovoids.

In [18], R.H. Dye constructed new ovoids on a hyperbolic quadric in $PG(7, 8)$. Later, in [25], he constructed partial ovoids on quadrics (sometimes elliptic, sometimes hyperbolic) in $PG(2m + 1, 8)$ whose size attains the Blokhuis–Moorhouse bound. The construction of ovoids in $PG(7, 8)$ involved an initial construction of nine points of a polygon, followed by the addition of points lying on conics formed by the intersection of various planes with the quadric. This approach formed the basis of the constructions of partial ovoids in the later paper. We are led to ask whether the partial ovoids we have constructed in $PG(2^t - 1, 8)$ could be those constructed by Dye. The answer is no for the reason that Dye's partial ovoids, by construction, meet some planes in conics, whereas our partial ovoids never meet a plane in a conic.

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