LE MATEMATICHE
Vol. LXXIV (2019) – Issue I, pp. 95–108
doi: 10.4418/2019.74.1.6

ON THE CHOW RING OF CERTAIN HYPERSURFACES IN A GRASSMANNIAN

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This note is about Plücker hyperplane sections X of the Grassmannian $\mathrm{Gr}(3,V_{10})$. Inspired by the analogy with cubic fourfolds, we prove that the only non-trivial Chow group of X is generated by Grassmannians of type $\mathrm{Gr}(3,W_6)$ contained in X. We also prove that a certain subring of the Chow ring of X (containing all intersections of positive-codimensional subvarieties) injects into cohomology.

1. Introduction

Let $\mathcal{L}$ be the Plücker polarization on the complex Grassmannian $\mathrm{Gr}(3,V_{10})$, and let

$$X \in |\mathcal{L}|$$

Submission received : 28 July 2018

AMS 2010 Subject Classification: 14C15, 14C25, 14C30
Keywords: Algebraic cycles, Chow ring, motives, hyperkähler varieties, Beauville "splitting property"

be a smooth hypersurface in the linear system of $\mathcal{L}$. The Hodge diamond of the 20-dimensional variety X is

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ * \\ \vdots \\ * \\ 0 \quad \ldots \ \ldots \quad 0 \quad 1 \quad 30 \quad 1 \quad 0 \quad \ldots \ \ldots \quad 0 \\ * \\ \vdots \\ * \\ 3 \\ 2 \\ 1 \end{array}$$

(Here $*$ indicates some unspecified number, and all empty entries are 0. The Hodge numbers of the vanishing cohomology can be found in [4, Theorem 1.1]; alternatively, they can be computed using [7, Theorem 1.1].)

This looks much like the Hodge diamond of a cubic fourfold. To further this analogy, Debarre and Voisin [4] have constructed, for a general such hypersurface X, a hyperkähler fourfold Y that is associated (via an Abel–Jacobi isomorphism) to X. Just as in the famous Beauville–Donagi construction starting from a cubic fourfold [2], the hyperkähler fourfolds Y form a 20-dimensional family, deformation equivalent to the Hilbert square of a K3 surface. The analogy

$$\text{Plücker hypersurfaces in } \mathrm{Gr}(3,V_{10}) \ \leftrightsquigarrow\ \text{cubic fourfolds}$$

also exists on the level of derived categories [8, Section 4.4].

In this note we will be interested in the Chow ring $A^*(X)_{\mathbb{Q}}$ of the hypersurface X. Using her celebrated method of *spread* of algebraic cycles in families, Voisin [18, Theorem 2.4] (cf. also the proof of theorem 2.1 below) has already proven a form of the Bloch conjecture for X: one has vanishing

$$A^i_{hom}(X)_{\mathbb{Q}} = 0 \quad \forall\, i \neq 11$$

(where $A^i_{hom}(X)_{\mathbb{Q}}$ is defined as the kernel of the cycle class map to singular cohomology). This is the analogue of the well-known fact that the only non-trivial Chow group of a cubic fourfold is the Chow group of 1-cycles.

We complete Voisin's result, by describing the only non-trivial Chow group of X:

Theorem (=theorem 2.1)**.** Let $\mathcal{L}$ be the Plücker polarization on $\mathrm{Gr}(3,V_{10})$. Let $X \in |\mathcal{L}|$ be a smooth hypersurface for which the associated hyperkähler fourfold Y is smooth. Then $A^{11}_{hom}(X)_{\mathbb{Q}}$ is generated by Grassmannians $\mathrm{Gr}(3,W_6)$ contained in X.

This is the analogue of the well-known fact that for a cubic fourfold $V \subset \mathbb{P}^5(\mathbb{C})$, the Chow group $A^3(V)$ is generated by lines [12]. Theorem 2.1 is readily proven using the spread method of [17], [18], [19]; as such, theorem 2.1 could naturally have been included in [18].

The second result of this note concerns the ring structure of the Chow ring of X, given by the intersection product:

Theorem (=theorem 3.1)**.** Let $\mathcal{L}$ be the Plücker polarization on $\mathrm{Gr}(3,V_{10})$, and let $X \in |\mathcal{L}|$ be a smooth hypersurface. Let $R^{11}(X) \subset A^{11}(X)_{\mathbb{Q}}$ be the subgroup containing intersections of two cycles of positive codimension, the Chern class $c_{11}(T_X)$ and the image of the restriction map $A^{11}(\mathrm{Gr}(3,V_{10}))_{\mathbb{Q}} \to A^{11}(X)_{\mathbb{Q}}$. The cycle class map induces an injection

$$R^{11}(X) \hookrightarrow H^{22}(X,\mathbb{Q}) .$$

This is reminiscent of the famous result about the Chow ring of a K3 surface [2]. It is also an analogue of the fact that for a cubic fourfold V, the subgroup $A^2(V)_{\mathbb{Q}} \cdot A^1(V)_{\mathbb{Q}} \subset A^3(V)_{\mathbb{Q}}$ is one-dimensional. Theorem 3.1 suggests that the hypersurfaces X might have a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial [14]. This seems difficult to establish, however (cf. remark 3.6).

Conventions. In this note, the word *variety* will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. For a smooth variety X, we will denote by $A^j(X)$ the Chow group of codimension j cycles on X with $\mathbb{Q}$–coefficients.

The notations $A^j_{hom}(X)$, $A^i_{AJ}(X)$ will be used to indicate the subgroups of homologically trivial (resp. Abel–Jacobi trivial) cycles.

For a morphism between smooth varieties $f\colon X \to Y$, we will write $\Gamma_f \in A^*(X \times Y)$ for the graph of f, and ${}^t\Gamma_f \in A^*(Y \times X)$ for the transpose correspondence.

We will write $H^*(X) = H^*(X,\mathbb{Q})$ for singular cohomology with rational coefficients.

2. Generators for A^{11}

Theorem 2.1. Let $\mathcal{L}$ be the Plücker polarization on $\mathrm{Gr}(3,V_{10})$. Let $X \in |\mathcal{L}|$ be a smooth hypersurface for which there is an associated smooth hyperkähler fourfold Y. Then $A^{11}_{hom}(X)$ is generated by the classes of Grassmannians $\mathrm{Gr}(3,W_6)\subset X$ (where $W_6\subset V_{10}$ is a six-dimensional vector space).

Proof. As mentioned in the introduction, Voisin [18, Theorem 2.4] has proven that

$$A^i_{hom}(X)=0\quad \forall i>11\ .$$

Using the Bloch–Srinivas "decomposition of the diagonal" method [3], [19, Chapter 3] (in the precise form of [9, Theorem 1.7]), this implies that

$$\mathrm{Niveau}\big(A^*(X)\big)\le 2$$

in the language of [9], and also (using [9, Remark 1.8.1])

$$A^i_{AJ}(X)=0\quad \forall i\neq 11\ .$$

But all intermediate Jacobians of X are trivial (there is no odd-degree cohomology), and so

$$A^i_{hom}(X)=0\quad \forall i\neq 11\ .$$

That is, the 20-dimensional variety X motivically looks like a surface, and so in particular the Hodge conjecture is true for X [9, Proposition 2.4].

Let

$$\mathcal{X}\ \to\ B$$

denote the universal family of smooth hypersurfaces in the linear system $|\mathcal{L}|$. The base B is the Zariski open in $\mathbb{P}(\wedge^3 V_{10}^*)$ parametrizing 3-forms σ such that the corresponding hyperplane section

$$X_\sigma\ \subset\ \mathrm{Gr}(3,V_{10})\ \subset\ \mathbb{P}(\wedge^3 V_{10})$$

is smooth.

Let $B'\subset B$ be the Zariski open such that the fibre X_σ has an associated hyperkähler fourfold Y_σ, in the sense of [4]. That is, B' parametrizes 3-forms σ such that both X_σ and

$$Y_\sigma:=\big\{W_6\in \mathrm{Gr}(6,V_{10})\text{ such that }\sigma|_{W_6}=0\big\}\ \subset\ \mathrm{Gr}(6,V_{10})$$

are smooth of the expected dimension.

We rely on the spread result of Voisin's, in the following form:

Theorem 2.2 (Voisin [18])**.** Let $\Gamma\in A^{20}(\mathcal{X}\times_B\mathcal{X})$ be a relative correspondence with the property that

$$(\Gamma|_{X_\sigma\times X_\sigma})_* H^{11,9}(X_\sigma)=0\quad\text{for very general }\sigma\in B\ .$$

Then

$$(\Gamma|_{X_\sigma\times X_\sigma})_* A^{11}_{hom}(X_\sigma)=0\quad\text{for all }\sigma\in B\ .$$

(For basics on the formalism of relative correspondences, cf. [10, Section 8.1].) Since theorem 2.2 is not stated precisely in this form in [18], we briefly indicate the proof:

Proof. (of theorem 2.2) The assumption on Γ (plus the shape of the Hodge diamond of X_σ, and the truth of the Hodge conjecture for X_σ, as shown above) implies that for the very general $\sigma \in B$ there exist closed subvarieties $V^i_\sigma, W^i_\sigma \subset X_\sigma$ with $\dim V^i_\sigma + \dim W^i_\sigma = 20$, and such that

$$\Gamma|_{X_\sigma \times X_\sigma} = \sum_{i=1}^{s} V^i_\sigma \times W^i_\sigma \quad \text{in } H^{40}(X_\sigma \times X_\sigma)\ .$$

One has the Noether–Lefschetz property that $H^{20}(X_\sigma, \mathbb{Q})_{van} \cap F^{10} = 0$ for very general X_σ (this is because the Picard number of the very general Debarre–Voisin hyperkähler fourfold is 1). This implies that all the subvarieties V^i_σ, W^i_σ are obtained by restriction from subvarieties of $\mathrm{Gr}(3, V_{10})$, hence they exist universally. (Instead of evoking Noether–Lefschetz, one could also apply Voisin's Hilbert scheme argument [17, Proposition 3.7] to obtain that the V^i_σ, W^i_σ exist universally). That is, there exist closed subvarieties $\mathcal{V}^i, \mathcal{W}^i \subset \mathcal{X}$ with $\mathrm{codim} \mathcal{V}^i + \mathrm{codim} \mathcal{W}^i = 20$, and a cycle δ supported on $\cup \mathcal{V}^i \times_B \mathcal{W}^i$, such that

$$\big(\Gamma - \delta\big)|_{X_\sigma \times X_\sigma} = 0 \quad \text{in } H^{40}(X_\sigma \times X_\sigma)\ , \quad \text{for very general } \sigma \in B\ .$$

We now define a relative correspondence

$$R := \Gamma - \delta \quad \in A^{20}(\mathcal{X} \times_B \mathcal{X})\ .$$

For brevity, from now on let us write $M := \mathrm{Gr}(3, V_{10})$. Since M has trivial Chow groups (this is true for all Grassmannians, and more generally for *linear varieties*, cf. [15, Theorem 3]), and the hypersurfaces X_σ have non-zero primitive cohomology (indeed $h^{11,9}(X_\sigma) = 1$), we are in the set–up of [18]. As in loc. cit., we consider the blow-up $\widetilde{M \times M}$ of $M \times M$ along the diagonal, and the quotient morphism $\mu\colon \widetilde{M \times M} \to M^{[2]}$ to the Hilbert scheme of length 2 subschemes. Let $\bar{B} := \mathbb{P}H^0(M, \mathcal{L})$ and as in [18, Lemma 1.3], introduce the incidence variety

$$I := \big\{ (\sigma, y) \in \bar{B} \times \widetilde{M \times M} \mid s|_{\mu(y)} = 0 \big\}\ .$$

Since $\mathcal{L}$ is very ample on M, I has the structure of a projective bundle over $\widetilde{M \times M}$.

Next, let us consider

$$f\colon\ \widetilde{\mathcal{X} \times_B \mathcal{X}} \ \to\ \mathcal{X} \times_B \mathcal{X}\ ,$$

the blow-up along the relative diagonal $\Delta_{\mathcal{X}}$. There is an open inclusion

$$\widetilde{\mathcal{X}\times_B\mathcal{X}} \subset I .$$

Hence, given our relative correspondence $R\in A^n(\mathcal{X}\times_B\mathcal{X})$ as above, there exists a (non-canonical) cycle $\bar{R}\in A^n(I)$ such that

$$\bar{R}|_{\widetilde{\mathcal{X}\times_B\mathcal{X}}} = f^*(R) \quad \text{in } A^n(\widetilde{\mathcal{X}\times_B\mathcal{X}}) .$$

Hence, we have

$$\bar{R}|_{\widetilde{X_\sigma\times X_\sigma}} = \big(f^*(R)\big)|_{\widetilde{X_\sigma\times X_\sigma}} = (f_\sigma)^*(R|_{X_\sigma\times X_\sigma}) = 0 \quad \text{in } H^{40}(\widetilde{X_\sigma\times X_\sigma}) ,$$

for $\sigma\in B$ very general, by assumption on R. (Here, as one might guess, the notation

$$f_\sigma\colon \widetilde{X_\sigma\times X_\sigma} \to X_\sigma\times X_\sigma$$

indicates the blow-up along the diagonal Δ_{X_σ}.)

We now apply [18, Proposition 1.6] to the cycle $\bar{R}$. The result is that there exists a cycle $\gamma\in A^{20}(M\times M)$ such that there is a rational equivalence

$$R|_{X_\sigma\times X_\sigma} = (f_\sigma)_*(\bar{R}|_{\widetilde{X_\sigma\times X_\sigma}}) = \gamma|_{X_\sigma\times X_\sigma} \quad \text{in } A^{20}(X_\sigma\times X_\sigma) \quad \forall\sigma\in B .$$

We know that the restriction of γ acts as zero on $A^{11}_{hom}(X_\sigma)$. (Indeed, let $\iota\colon X_\sigma\to M$ denote the inclusion, and let $a\in A^{11}_{hom}(X_\sigma)$. With the aid of Lieberman's lemma [16, Lemma 3.3], one finds that

$$\big((\iota\times\iota)^*(\gamma)\big)_*(a) = \iota^*\gamma_*\iota_*(a) \quad \text{in } A^{11}_{hom}(X_\sigma) .$$

But $\iota_*(a)\in A^{12}_{hom}(M)=0$).

Thus, it follows that

$$\big(R|_{X_\sigma\times X_\sigma}\big)_* = 0\colon\ A^{11}_{hom}(X_\sigma) \to A^{11}_{hom}(X_\sigma) \quad \forall\sigma\in B .$$

For any given $\sigma\in B$, one can construct the subvarieties $\mathcal{V}^i,\mathcal{W}^i\subset\mathcal{X}$ in the above argument in such a way that they are in general position with respect to the fibre X_σ. This implies that the restriction

$$\delta|_{X_\sigma\times X_\sigma} \in A^{20}(X_\sigma\times X_\sigma)$$

is a *completely decomposed cycle*, i.e. a cycle supported on a union of subvarieties $V_j^\sigma\times W_j^\sigma\subset X_\sigma\times X_\sigma$ with $\operatorname{codim}(V_j^\sigma)+\operatorname{codim}(W_j^\sigma)=20$. But completely decomposed cycles do not act on $A^*_{hom}()$ [3], and so

$$\big(\Gamma|_{X_\sigma\times X_\sigma}\big)_* = \big((R+\delta)|_{X_\sigma\times X_\sigma}\big)_* = 0\colon\ A^{11}_{hom}(X_\sigma) \to A^{11}_{hom}(X_\sigma) \quad \forall\sigma\in B .$$

This ends the proof of theorem 2.2. □

Let us now pick up the thread of the proof of theorem 2.1. As in [4, Section 2], for any 3-form $\sigma \in B'$ let

$$G_\sigma := \Big\{ (W_3, W_6) \ \in \mathrm{Gr}(3, V_{10}) \times \mathrm{Gr}(6, V_{10}) \ \Big| \ W_3 \subset W_6 \ , \ \sigma|_{W_6} = 0 \Big\}$$

denote the incidence variety, with projections

$$\begin{array}{ccc} G_\sigma & \xrightarrow{p_\sigma} & X_\sigma \\ \downarrow q_\sigma & & \\ Y_\sigma \ . & & \end{array}$$

The fibres of q_σ are 9-dimensional Grassmannians $\mathrm{Gr}(3, W_6)$.

Let $\mathcal{Y} \to B'$ denote the universal family of Debarre–Voisin fourfolds (i.e., $\mathcal{Y} \subset \mathrm{Gr}(6, V_{10}) \times B'$ is the subvariety of pairs (W_6, σ) such that $\sigma|_{W_6} = 0$), and let $\mathcal{G} \to B'$ be the relative version of G_σ, with projections

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{p} & \mathcal{X} \\ \downarrow q & & \\ \mathcal{Y} \ . & & \end{array}$$

We will rely on an Abel–Jacobi type result from [4], concerning the *vanishing cohomology* defined as

$$\begin{aligned} H^{20}(X_\sigma, \mathbb{Q})_{van} &:= \ker\big(H^{20}(X_\sigma, \mathbb{Q}) \ \to \ H^{22}(\mathrm{Gr}(3, V_{10}), \mathbb{Q}) \big) \ , \\ H^{2}(Y_\sigma, \mathbb{Q})_{van} &:= \ker\big(H^{2}(Y_\sigma, \mathbb{Q}) \ \to \ H^{42}(\mathrm{Gr}(6, V_{10}), \mathbb{Q}) \big) \ . \end{aligned}$$

Lemma 2.1. *Let $\sigma \in B'$ be very general. Then there is an isomorphism*

$$(q_\sigma)_* (p_\sigma)^* : \ \ H^{20}(X_\sigma, \mathbb{Q})_{van} \ \xrightarrow{\cong} \ H^2(Y_\sigma, \mathbb{Q})_{van} \ .$$

The inverse isomorphism is given by

$$H^2(Y_\sigma, \mathbb{Q})_{van} \ \xrightarrow{\cdot \frac{1}{\mu} g^2} \ H^6(Y_\sigma, \mathbb{Q})_{van} \ \xrightarrow{(p_\sigma)_* (q_\sigma)^*} \ H^{20}(X_\sigma, \mathbb{Q})_{van} \ .$$

(Here $\mu \in \mathbb{Q}$ is some non-zero number independent of σ, and $g \in A^1(Y_\sigma)$ is the Plücker polarization.)

Proof. The first part (i.e. the fact that $(q_\sigma)_* (p_\sigma)^*$ is an isomorphism on the vanishing cohomology) is [4, Theorem 2.2 and Corollary 2.7]. For the second part, we observe that the dual map (with respect to cup product)

$$(p_\sigma)_* (q_\sigma)^* : \ \ H^6(Y_\sigma, \mathbb{Q})_{van} \ \to \ H^{20}(X_\sigma, \mathbb{Q})_{van}$$

is also an isomorphism. In particular, using hard Lefschetz, this means that the composition

$$H^2(Y_\sigma,\mathbb{Q})_{van} \xrightarrow{\cdot g^2} H^6(Y_\sigma,\mathbb{Q})_{van} \xrightarrow{(p_\sigma)_*(q_\sigma)^*} H^{20}(X_\sigma,\mathbb{Q})_{van} \xrightarrow{(q_\sigma)_*(p_\sigma)^*} H^2(Y_\sigma,\mathbb{Q})_{van}$$

is non-zero (and actually an isomorphism). Hence, the assignment

$$<\alpha,\beta>_{\mathrm{vv}}:=<\alpha,(q_\sigma)_*(p_\sigma)^*(p_\sigma)_*(q_\sigma)^*(g^2\cdot\beta)>_{Y_\sigma}$$

defines a polarization on $H^2(Y_\sigma,\mathbb{Q})_{van}$. Here, $<\alpha,\beta>_{Y_\sigma}$ is the Beauville–Bogomolov form. However, as explained in [18, Proof of Lemma 2.2], for very general σ the Hodge structure on $H^2(Y_\sigma,\mathbb{Q})_{van}$ is simple, and admits a unique polarization up to a coefficient. That is, there exists a non-zero number $\mu\in\mathbb{Q}$ such that

$$<\alpha,\beta>_{\mathrm{vv}}=\mu<\alpha,\beta>_{Y_\sigma}\ .$$

The Beauville–Bogomolov form being non-degenerate, this proves that

$$(q_\sigma)_*(p_\sigma)^*(p_\sigma)_*(q_\sigma)^*(g^2\cdot\beta)=\mu\beta\quad\forall\beta\ \in\ H^2(Y_\sigma,\mathbb{Q})_{van}\ .$$

Reasoning likewise starting from $H^{20}(X_\sigma,\mathbb{Q})_{van}$ (now using the cup product instead of the Beauville–Bogomolov form), we find that the other composition is also the identity.

Finally, the fact that the constant μ is the same for all fibres X_σ is because the map in cohomology $H^2(Y_\sigma,\mathbb{Q})_{van}\to H^{20}(X_\sigma,\mathbb{Q})_{van}$ is locally constant in the family. ☐

Let us define the relative correspondence

$$\Gamma:=\mu\Delta_{\mathcal{X}}-\Gamma_p\circ{}^t\Gamma_q\circ\Gamma_{g^2}\circ\Gamma_q\circ{}^t\Gamma_p\ \in\ A^{20}(\mathcal{X}\times_{B'}\mathcal{X})\ ,$$

where $\Gamma_{g^2}\in A^6(\mathcal{Y}\times_{B'}\mathcal{Y})$ is the correspondence acting fibrewise as intersection with two Plücker hyperplanes. Lemma 2.1 implies that

$$(\Gamma|_{X_\sigma\times X_\sigma})_*H^{20}(X_\sigma,\mathbb{Q})_{van}=0\quad\text{for very general }\sigma\in B'\ .$$

That is, the relative correspondence Γ satisfies the assumption of theorem 2.2. Thanks to theorem 2.2, we thus conclude that

$$(\Gamma|_{X_\sigma\times X_\sigma})_*A^{11}_{hom}(X_\sigma)=0\quad\forall\sigma\in B\ .$$

Unraveling the definition of Γ, this means in particular that there is a surjection

$$(p_\sigma)_*(q_\sigma)^*:\ \ A^4_{hom}(Y_\sigma)\ \twoheadrightarrow\ A^{11}_{hom}(X_\sigma)\quad\forall\sigma\in B'\ .$$

As we have seen, for any point $y \in Y_\sigma$ the fibre $(q_\sigma)^{-1}(y)$ is a 9-dimensional Grassmannian $\mathrm{Gr}(3, W_6)$ such that the 3-form σ vanishes on W_6. Such a Grassmannian is contained in the hypersurface X_σ, and so

$$(p_\sigma)_*(q_\sigma)^*(y) = \mathrm{Gr}(3, W_6) \quad \text{in } A^{11}(X_\sigma) \quad \forall y \in Y_\sigma .$$

The theorem is proven. □

Remark 2.3. The above argument actually shows that

$$A^{11}_{hom}(X_\sigma) \xrightarrow{(q_\sigma)_*(p_\sigma)^*} A^2_{hom}(Y_\sigma) \xrightarrow{\cdot g^2} A^4_{hom}(Y_\sigma) \xrightarrow{(p_\sigma)_*(q_\sigma)^*} A^{11}_{hom}(X_\sigma)$$

is a non-zero multiple of the identity, for any $\sigma \in B'$. This is very much reminiscent of cubic fourfolds and their Fano varieties of lines [1], [14]. Inspired by this analogy, it is tempting to ask the following: can one somehow prove that

$$\mathrm{Im}\big(A^{11}(X_\sigma) \to A^4(Y_\sigma)\big)$$

is the same as the subgroup of 0-cycles supported on a uniruled divisor ?

3. An injectivity result

Theorem 3.1. Let $\mathcal{L}$ be the Plücker polarization on $\mathrm{Gr}(3, V_{10})$, and let $X \in |\mathcal{L}|$ be a smooth hypersurface. Let $R^{11}(X) \subset A^{11}(X)_{\mathbb{Q}}$ be the subgroup containing intersections of two cycles of positive codimension, the Chern class $c_{11}(T_X)$ and the image of the restriction map $A^{11}(\mathrm{Gr}(3, V_{10})) \to A^{11}(X)$. The cycle class map induces an injection

$$R^{11}(X) \hookrightarrow H^{22}(X, \mathbb{Q}) .$$

In order to prove theorem 3.1, we first establish a "generalized Franchetta conjecture" type of statement (for more on the generalized Franchetta conjecture, cf. [11], [13], [6]):

Theorem 3.2. Let $\mathcal{X} \to B$ denote the universal family of Plücker hyperplanes in $\mathrm{Gr}(3, V_{10})$ (as in section 2). Let $\Psi \in A^{11}(\mathcal{X})$ be such that

$$\Psi|_{X_\sigma} = 0 \ \text{ in } H^{22}(X_\sigma) \quad \forall \sigma \in B .$$

Then

$$\Psi|_{X_\sigma} = 0 \ \text{ in } A^{11}(X_\sigma) \quad \forall \sigma \in B .$$

Proof. This is a two-step argument:

Claim 3.3. There is equality

$$\mathrm{Im}\big(A^{11}(\mathcal{X})\to A^{11}(X_\sigma)\big)=\mathrm{Im}\big(A^{11}(\mathrm{Gr}(3,V_{10}))\to A^{11}(X_\sigma)\big)\quad \forall \sigma\in B\ .$$

Claim 3.4. Restriction of the cycle class map induces an injection

$$\mathrm{Im}\big(A^{11}(\mathrm{Gr}(3,V_{10}))\to A^{11}(X_\sigma)\big)\ \hookrightarrow\ H^{22}(X_\sigma)\quad \forall \sigma\in B\ .$$

Clearly, the combination of these two claims proves theorem 3.2. To prove claim 3.3, let $\bar{B}:=\mathbb{P}H^0(\mathrm{Gr}(3,V_{10}),\mathcal{L})$ and let

$$\begin{array}{ccc} \bar{\mathcal{X}} & \xrightarrow{\pi} & \mathrm{Gr}(3,V_{10}) \\ \downarrow \phi & & \\ \bar{B} & & \end{array}$$

denote the universal hyperplane (including the singular hyperplanes). The morphism π is a projective bundle, and so any $\Psi\in A^{11}(\bar{\mathcal{X}})$ can be written

$$\Psi=\sum_\ell \pi^*(a_\ell)\cdot\phi^*(h^\ell)\quad \text{in } A^{11}(\bar{\mathcal{X}})\ ,$$

where $a_\ell\in A^{11-\ell}(\mathrm{Gr}(3,V_{10}))$ and $h:=c_1(\mathcal{O}_{\bar{B}}(1))\in A^1(\bar{B})$. For any $\sigma\in B$, the restriction of $\phi^*(h)$ to the fibre X_σ vanishes, and so

$$\Psi|_{X_\sigma}=a_0|_{X_\sigma}\quad \text{in } A^{11}(X_\sigma)\ ,$$

which establishes claim 3.3.

Let us prove claim 3.4. For any given $\sigma\in B$, let $\iota\colon X_\sigma\to \mathrm{Gr}(3,V_{10})$ denote the inclusion morphism. We know that

$$\iota_*\iota^*\colon\ \ A^j(\mathrm{Gr}(3,V_{10}))\ \to\ A^{j+1}(\mathrm{Gr}(3,V_{10}))$$

equals multiplication by the ample class $c_1(\mathcal{L})\in A^1(\mathrm{Gr}(3,V_{10}))$. Now let

$$b\in A^{11}(\mathrm{Gr}(3,V_{10}))$$

be such that the restriction $\iota^*(b)\in A^{11}(X_\sigma)$ is homologically trivial. Then we have that also

$$b\cdot c_1(\mathcal{L})=\iota_*\iota^*(b)=0\quad \text{in } H^{24}(\mathrm{Gr}(3,V_{10}))=A^{12}(\mathrm{Gr}(3,V_{10}))\ .$$

To conclude that $b=0$, it suffices to show that

$$\cdot c_1(\mathcal{L})\colon\ \ A^{11}(\mathrm{Gr}(3,V_{10}))\ \to\ A^{12}(\mathrm{Gr}(3,V_{10}))$$

is injective (and hence, by hard Lefschetz, an isomorphism). By hard Lefschetz, this is equivalent to showing that

$$\cdot c_1(\mathcal{L}):\ \ A^9(\mathrm{Gr}(3,V_{10}))\ \to\ A^{10}(\mathrm{Gr}(3,V_{10}))$$

is surjective (hence an isomorphism).

According to [5, Theorem 5.26], the Chow ring of the Grassmannian is of the form

$$A^*(\mathrm{Gr}(3,V_{10}))=\mathbb{Q}[c_1,c_2,c_3]/I\ ,$$

where $c_j\in A^j(\mathrm{Gr}(3,V_{10}))$ are Chern classes of the universal subbundle, and I is a certain complete intersection ideal generated by the 3 relations

$$\begin{aligned}&c_1^8+7c_1^6c_2+15c_1^4c_2^2+10c_1^2c_2^3+\cdots+3c_2c_3^2\ ,\\&c_1^9+8c_1^7c_2+21c_1^5c_2^2+20c_1^3c_2^3+\cdots+c_3^3\ ,\\&c_1^{10}+9c_1^8c_2+28c_1^6c_2^2+35c_1^4c_2^3+\cdots+4c_1c_3^3\ \ ,\end{aligned}$$

in degree $8,9,10$. With the aid of the relations in I, we find that

$$\begin{aligned}A^{10}(\mathrm{Gr}(3,V_{10}))=\mathbb{Q}[&c_1^{10},c_1^8c_2,c_1^6c_2^2,c_1^4c_2^3,c_1^7c_3,c_1^5c_2c_3,c_1^4c_3^2,c_1^3c_2^2c_3,\\&c_1^2c_2c_3^2,c_1c_2^3c_3]\end{aligned}$$

is 10-dimensional (the classes $c_1^2c_2^4,c_2^5$ are eliminated thanks to the relation in degree 8 containing c_2^4; the class $c_1c_3^3$ is eliminated thanks to the relation in degree 9; the class $c_2^2c_3^2$ is eliminated thanks to the relation in degree 10). Inspecting this description of $A^{10}(\mathrm{Gr}(3,V_{10}))$, we observe that the inclusion

$$c_1\cdot A^9(\mathrm{Gr}(3,V_{10}))\ \subset\ A^{10}(\mathrm{Gr}(3,V_{10}))$$

is an equality. Since c_1 is proportional to $c_1(\mathcal{L})$, this proves claim 3.4. □

It remains to prove theorem 3.1:

Proof. (of theorem 3.1) Clearly, the Chern class is universally defined: for any $\sigma\in B$, we have

$$c_{11}(T_{X_\sigma})=c_{11}(T_{\mathcal{X}/B})|_{X_\sigma}\ .$$

Also, the image

$$\mathrm{Im}\big(A^{11}(\mathrm{Gr}(3,V_{10}))\to A^{11}(X_\sigma)\big)$$

consists of universally defined cycles. (For a given $a\in A^{11}(\mathrm{Gr}(3,V_{10}))$, the relative cycle

$$(a\times B)|_{\mathcal{X}}\ \in\ A^{11}(\mathcal{X})$$

does the job.)

Likewise, for any $j<10$ the fact that $A^j_{hom}(X_\sigma)=0$, combined with weak Lefschetz in cohomology, implies that

$$A^j(X_\sigma)=\operatorname{Im}\bigl(A^j(\operatorname{Gr}(3,V_{10}))\to A^j(X_\sigma)\bigr)\ ,$$

and so $A^j(X_\sigma)$ consists of universally defined cycles for $j<10$. In particular, all intersections

$$A^j(X_\sigma)\cdot A^{11-j}(X_\sigma)\ \subset\ A^{11}(X_\sigma)\ ,\quad 1<j<10$$

consist of universally defined cycles.

It remains to make sense of intersections

$$A^{10}(X_\sigma)\cdot A^1(X_\sigma)\ \subset\ A^{11}(X_\sigma)\ .$$

To this end, we note that $A^1(X_\sigma)$ is 1-dimensional, generated by the restriction g of the Plücker line bundle $\mathcal{L}$. Let $\iota\colon X_\sigma\to \operatorname{Gr}(3,V_{10})$ denote the inclusion. The normal bundle formula implies that

$$a\cdot g=\iota^*\iota_*(a)\ \ \text{in}\, A^{11}(X_\sigma)\ \ \forall\, a\in\, A^{10}(X_\sigma)\ .$$

It follows that

$$A^{10}(X_\sigma)\cdot A^1(X_\sigma)\ \subset\ \operatorname{Im}\bigl(A^{11}(\operatorname{Gr}(3,V_{10}))\xrightarrow{\iota^*}A^{11}(X_\sigma)\bigr)$$

also consists of universally defined cycles.

In conclusion, we have shown that $R^{11}(X_\sigma)$ consists of universally defined cycles, and so theorem 3.1 is a corollary of theorem 3.2. □

Remark 3.5. There are more cycle classes that can be put in the subgroup $R^{11}(X)$ of theorem 3.1. For instance, let Y_σ be the hyperkähler fourfold associated to $X=X_\sigma$, and assume Y_σ is smooth. Then (as we have seen above) the class

$$(p_\sigma)_*(q_\sigma)^*c_4(T_{Y_\sigma})\ \in\ A^{11}(X)$$

is universally defined, hence it can be added to the subgroup $R^{11}(X)$ of theorem 3.1.

Remark 3.6. Theorem 3.1 is an indication that perhaps the hypersurfaces $X\subset \operatorname{Gr}(3,V_{10})$ have a *multiplicative Chow–Künneth decomposition*, in the sense of [14, Chapter 8]. Unfortunately, establishing this seems difficult; one would need something like theorem 3.2 for

$$A^{40}(\mathcal{X}\times_B\mathcal{X}\times_B\mathcal{X})\ .$$

Acknowledgements. Thanks to my mythical colleague Gilberto Kiwi for inspiring conversations, and thanks to a zealous referee for constructive comments.

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