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# SPACES OF DIMENSION THREE WITH CONGRUENCE

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## 1. Introduction.

It is well known that every Euclidean plane $(E, \mathcal{L}, \alpha, \equiv)$ is isomorphic to an affine plane AG(2,K) over a Pythagorean ordered commutative field $K$. There is a corrisponding theorem for hyperbolic panes (cf. [2]). For both proofs one first considers the group of motions, in particular the line reflections.

Reflections at all are a powerful tool in order to prove geometric theorems. For example they have been used by J. Hjelmslev, G. Thomsen, A. Schmidt, F. Bachmann, E. Sperner and many others as a basic concept for an axiomatization of plane absolute geometry. For characterizations of absolute planes one has to consider products of reflections. One central theorem of absolute planes is the well known three reflections theorem. Let $A, B, C$ be lines, either through a common point or perpendicular to a common line. Then the product of the line reflections about $A, B, C$ is a line reflection again.

In the famous paper [9], K. Sörensen introduces planes with congruence using only three clear axioms and shows that for the definition of a line reflection an order relation is not necessary. One needs only linear structure of $(E, \mathcal{L})$ and the congruence relation $\equiv$, but for the proof that the line reflection is a motion it seems that additional assumptions on the geometry are necessary. For example K. Sörensen assumes that for given lines $G_1, G_2$, there exist distinct lines $H_1, H_2$ through a common point $z$ which intersect $G_1, G_2$. But it suffices to assume an exchange plane with congruence not using this property.

In the plane with congruence $(E, \mathcal{L}, \alpha, \equiv)$ two perpendicular lines may have an empty intersection. In the case that in a plane any two perpendicular lines have an empty intersection, then the relation "perpendicular" together with the "identity relation" is transitive (cf. 3.1).

If there are two perpendicular lines with a non empty intersection, one can show that for any line $G$ and any point $x \in G$ there exists a perpendicular line through $x$. This is true for finite or affine planes, but not known in general. But for spaces with congruence of dimension at least three an answer is possible. We assume that there are two points with a midpoint, all known examples in which the line reflections are motions have dimension two.

We use the property that if two points have midpoint, then they have a midline in every plane. For a space of dimension greater than two one can show that for any line $G$ of a plane $E$ and any point $x \in G$ there is a unique perpendicular line through $x$ in $E$. First we have to prove that any line reflection is a motion. Therefore we need some steps. We show that the restriction of a line reflection to a plane is a motion (section 2). Then we consider a subspace $U$ of dimension three and construct some special lines in $U$ for which the reflections are motions. Finally we generalize this result to all lines of $U$. It follows that for any two points $b, z$ there exists a unique point $b' \in \overline{z, b} \setminus \{b\}$ with $(z, b) \equiv (z, b')$ and also point reflections are motions. Hence one can introduce a loop operation $+$ in a more general context then in [1].

This is the preparation of a talk given at Conference Combinatorics 2004 in Catania–Capomulini. For complete proofs of all propositions mentioned here, we refer to the papers [9], [4], [5].

## 2. Planes with congruence.

Let $(P, \mathcal{L})$ denote a *linear space* or *incidence space* with the point set $P$, the line set $\mathcal{L}$ and at least three points on every line, i.e., for any two points there is exactly one line containing them and for any line $L \in \mathcal{L}$ we have $|L| \geq 3$.

A *subspace* is a subset $U \subset P$ such that for all distinct points $x, y \in U$ the unique line passing through $x$ and $y$, denoted by $\overline{x, y}$, is contained in $U$. Let $\mathcal{U}$ denote the set of all subspaces. For every subset $X \subset P$ we define the following *closure operation*

$$(1) \qquad \overline{\phantom{X}} : \mathfrak{P}(P) \to \mathfrak{U},\ X \mapsto \overline{X} \qquad \text{by} \qquad \overline{X} := \bigcap_{\substack{U \in \mathfrak{U} \\ X \subset U}} U$$

For $U \in \mathfrak{U}$ we call $\dim U := \inf\{|X| - 1 : X \subset U \text{ and } \overline{X} = U\}$ the *dimension* of U. A subspace of dimension two is a *plane*. For a set $\{a, b, c, \ldots\}$ we write $\overline{a, b, c, \ldots}$ instead of $\overline{\{a, b, c, \ldots\}}$.

We introduce the concept of a space $(P, \mathfrak{L}, \equiv)$ with congruence (cf. [9]). We assume in this section a linear space $(P, \mathfrak{L})$ which satisfies the following exchange condition.

(EC) Let $S \subset P$ and let $x, y \in P$ with $x \in \overline{S \cup \{y\}} \setminus \overline{S}$. Then $y \in \overline{S \cup \{x\}}$

Let $\equiv$ be a *congruence relation* on $P \times P$, i.e. $\equiv$ is a equivalence relation with $(a, b) \equiv (b, a)$ and $(a, a) \equiv (b, c)$ if and only if $b = c$.

We use the notation $(x_1, x_2, x_3) \equiv (y_1, y_2, y_3)$ if and only if $(x_i, x_j) \equiv (y_i, y_j)$ for $i, j \in \{1, 2, 3\}$. $(P, \mathfrak{L}, \equiv)$ is a *space with congruence* if the axioms (W1), (W2) and (W3) are satisfied.

(W1) Let $a, b, c \in P$ be distinct and collinear, and let $a', b' \in P$ with $(a, b) \equiv (a', b')$. Then there exists exactly one $c' \in \overline{a', b'}$ with $(a, b, c) \equiv (a', b', c')$.

(W2) Let $a, b, x \in P$ be non-collinear and let $a', b', x' \in P$ with $(a, b, x) \equiv (a', b', x')$. For any $c \in \overline{a, b}$ and $c' \in \overline{a', b'}$ with $(a, b, c) \equiv (a', b', c')$ it holds $(x, c) \equiv (x', c')$.

(W3) For $a, b, x \in P$ non-collinear there exists exactly one $x' \in \overline{\{a, b, x\}} \setminus \{x\}$ with $(a, b, x) \equiv (a, b, x')$.

We call a bijective mapping $\phi : P \to P$ a *motion*, if $(x, y) \equiv (\phi(x), \phi(y))$ for all $x, y \in P$. We quote from [9]:

**Lemma 2.1.**
*(i) If $a, b, c$ are collinear points and $a', b', c' \in P$ with $(a, b, c) \equiv (a', b', c')$, then $a', b', c'$ are collinear.*
*(ii) Any motion $\phi$ is a collineation.*
*(iii) Two distinct points $a, b$ have at most one point $m \in \overline{a, b}$ with $(a, m) \equiv (b, m)$.*

An important proposition is (1.5) of [9]. There implicitly the exchange condition is used. We give here a proof using that in a plane the set of all midpoints of two points is contained in a line. For a subspace $U$ and points $a, b \in U$ we define $M_U(a, b) := \{x \in U : (a, x) \equiv (b, x)\}$. We call $M_U(a, b)$ a *midpoint*, a *midline*, or a *midplane* of $a, b$, respectively, if it is a point, a line, or a plane, respectively.

**Lemma 2.2.**
*(i) $M_U(a, b)$ is a subspace of $U$.*
*(ii) Let $a, b, x \in P$ be non-collinear points and let $x' \in \overline{a, b, x} \setminus \{x\}$ with $(a, b, x) \equiv (a, b, x')$. Then for any $c \in \overline{a, b, x}$ it holds $(x, c) \equiv (x', c)$ if and only if $c \in \overline{a, b}$.*

*Proof.* (i) Assume $a \neq b$ and let $x, y \in M_U(a, b)$ be distinct points, hence $(a, x, y) \equiv (b, x, y)$. By (W1) the points $a, b, x, y$ are not collinear. For $w \in \overline{x, y}$ by (W2) it follows $(a, w) \equiv (b, w)$, i.e., $w \in M_U(a, b)$.
(ii) For $E := \overline{a, b, x}$ we have $a, b \in M_E(x, x')$. Since $x \notin M_E(x, x')$ it follows $M_E(x, x') \neq E$. By (i), $M_E(x, x')$ is a subspace of $E$, and since $E$ is an exchange plane and $a, b \in M_E(x, x')$, it follows $M_E(x, x') = \overline{a, b}$. □

For a line $L \in \mathcal{L}$, $x \in P \setminus L$ and $a, b \in L$ with $a \neq b$ there exists by (W3) the unique point $x' \in \overline{L \cup \{x\}} \setminus \{x\}$ with $(a, b, x) \equiv (a, b, x')$. By (W2), $x'$ is independent of the choice of $a, b \in L$, hence we may denote $x' = L(x)$.

The following mapping is called *line reflection*

$$\tilde{L} : P \to P; \qquad x \to \begin{cases} x & \text{if } x \in L, \\ L(x) & \text{if } x \notin L. \end{cases}$$

Clearly $\tilde{L}$ is an involutory bijection with $z = \tilde{L}(z)$ if and only if $z \in L$.

For the proof that the restriction of a line reflection to a plane is a motion, Sörensen uses an additional property mentioned in the introduction. If two perpendicular lines have an intersection, a distinct proof can be found in [4], which work also in general (cf. [5] (2.4), (2.5) and (2.6)). We obtain:

**Theorem 2.3.** *Let $L$ be a line of a plane $E$. Then $\tilde{L}|_E$ is an involutory motion with $z = \tilde{L}(z)$ if and only if $z \in L$.*

We define for lines $A, B \in \mathcal{L}$:

$$A \perp B \qquad \Longleftrightarrow \qquad \tilde{A}(B) = B \text{ and } A \neq B.$$

**Lemma 2.4.** *If $A \perp B$, then $B \perp A$.*

*Proof.* Let $a_0, a_1 \in A \setminus B$ be distinct points, $b \in B \setminus A$ and $b' = A(b) \in B$. Then $(a_0, a_1, b) \equiv (a_0, a_1, b')$. For $a_2 = B(a_1)$ we have $(b', b, a_1) \equiv (b', b, a_2)$, and by Lemma 2.2 (ii), $(b, a_2) \equiv (b, a_1) \equiv (b', a_1) \equiv (b', a_2)$ implies $a_2 \in A = \overline{a_1, a_0}$, hence by the involutoriety of $\tilde{B}$ we have $\tilde{B}(A) = A$. □

By [9], (1.15):

**Theorem 2.5.** *Let $A, B, C$ be lines of a plane $E$ through a point $a$. Then there exists a line $D \subset E$ through $a$ with $\tilde{A}\tilde{B}\tilde{C}|_E = \tilde{D}|_E$*

**Lemma 2.6.** *Let $E$ be a plane with distinct points $a, m, c \in E$:*

*(i) There exists at most one line $L$ in $E$. with $L(a) = c$*
*(ii) If $(a, m) \equiv (c, m)$, there exists a unique line $X \subset E$ with $m \in X$ and $X(a) = c$, i.e., $X = M_E(a, c)$ (cf.* [9] (1.7.2),(1.8),(4.2)*).*

**Lemma 2.7.** *The following propositions are equivalent (cf.* [9] (4.3)*)*

*(a)* *There are three collinear points* $a, b, b'$ *with* $(a, b) \equiv (a, b')$.
*(b)* *For any points* $x, y$ *there is a point* $y' \in \overline{x, y} \setminus \{y\}$ *with* $(x, y) \equiv (x, y')$.
*(c)* *There are two perpendicular lines* $A, B$ *with* $A \cap B \neq \emptyset$.
*(d)* *For a line* $L$ *of any plane* $E$ *and* $x \in L$*, there exists a line* $G \subset E$ *through* $x$ *with* $L \perp G$.

It is an open problem for a plane with congruence, if for $x \in L$ there exists only one perpendicular line $G \perp L$ through $x$. For spaces an answer is given in Corollary 3.5 (i)).

## 3. Spaces with congruence.

In this section let $(P, \mathcal{L}, \equiv)$ be a linear space with congruence and dim $P \geq 3$ satisfying the exchange condition .

Let $A \in \mathcal{L}$ be any line. For any two points $x, y \in P$ there is a three dimensional subspace $U$ of $P$ with $A \cup \{x, y\} \subset U$. To show that $\tilde{A}$ is a motion it suffices to show that the restriction of $\tilde{A}$ to $U$ is a motion. Therefore in the following we consider a three dimensional subspace $U$ of $P$ and a line $A \subset U$.

**Remarks.** We recall that if there are two points in $P$ with a midpoint, then by Lemma 2.7 for any distinct $a, m \in A$ there exists a point $c \in A$ with $(a, m) \equiv (c, m)$.
Further we remark that for affine spaces $(P, \mathcal{L})$ with dim $P \geq 3$ one can show that perpendicular lines have a non empty intersection. With Lemma 2.7 it follows that for affine spaces $(P, \mathcal{L})$ with congruence and with dim $P \geq 3$ on every line there exist points $a, c$ with a midpoint.
By [9] (2.3) finite planes with congruence are affine planes. Further finite affine spaces with congruence have dimension two. Therefore finite spaces with congruence are affine planes.

One can also show that in the case that any perpendicular lines have an empty intersection, the relation “perpendicular” together with the “identity” is transitive:

**Theorem 3.1.** *Assume that for any two perpendicular lines* $A, B$ *we have* $A \cap B = \emptyset$. *Then for distinct lines* $G, H, L$ *of a plane* $E$ *with* $L \perp G, H$ *we have* $G \perp H$.

*Proof.* Let $a \in L$. Since $\tilde{G}|_E, \tilde{H}|_E$, are motions by Theorem 2.3,

$$(G(a), H(a)) \equiv (GHG(a), G(a)).$$

Because $L \perp G, H$, clearly $a, H(a), GHG(a) \in L$, and by assumption and Lemma 2.7, there exits only one point $x \in L$ with $(x, G(a)) \equiv (H(a), G(a))$, i.e., $GHG(a) = H(a)$.

Now it is easy to show that $\tilde{G}\tilde{H}\tilde{G}|_E = \widetilde{G(H)}|_E$, and since $a \notin H$, $\tilde{G}(H) = H$ is by Lemma 2.6 (i) the unique midline of $a$ and $H(a)$ in $E$. Hence $G \perp H$. □

We know already that the restriction of a line reflection to a plane is a motion. To show that the line reflections are motion at all, we start with the proof for some special lines. One can show that the reflection on the intersection line of two mid planes is a motion. Then one shows that this is the case for every line. First we consider the intersection of mid planes:

**Lemma 3.2.**

*(i) Let $a, c \in U$ be distinct points, then $dim M_U(a, c) \leq 2$. If $a, c$ have a midpoint $m \in \overline{a, c}$, then $M_U(a, c)$ is a plane of $U$.*

*(ii) Let $A, B \subset U$ be perpendicular lines, let $a \in A \setminus B$ and $b \in B \setminus A$. If the point $m = A \cap B$ exists, then $X := M_U(a, B(a)) \cap M_U(b, A(b))$ is a line with $m \in X$.*

*Proof.* (i). Since $a, c \notin M_U(a, c)$, $dim M_U(a, c) \leq 2$. If $m$ is the midpoint of $a, c$, then by Lemma 2.6 (ii) in every plane $E$ of $U$ containing $\overline{a, c}$ there exists a line $L \subset M_U(a, c)$, hence $\dim M_U(a, c) = 2$.
(ii). The point $m$ is the midpoint of $a, B(a)$ and $b, A(b)$. By (i), $M_U(b, A(b))$ is a plane with $A \subset M_U(b, A(b))$ and by Lemma 2.6 (ii) there is a line $X \subset M_U(b, A(b))$ with $m \in X$ and $X(a) = B(a)$, hence $X = M_U(a, B(a)) \cap M_U(b, A(b))$. □

Using the method of [4] one can show (cf. [5]):

**Lemma 3.3.** *Let $A, B \subset U$ be lines with $A \perp B$ and for $a \in A \setminus B$ and $b \in B \setminus A$ let $X := M_U(a, B(a)) \cap M_U(b, A(b))$ be a line. Then:*

*$\tilde{X}|_U$ is a motion.*

A generalization of this Lemma to all lines shows (cf. [5] (3.4)):

**Theorem 3.4.** *Let $(P, \mathcal{L}, \equiv)$ be a space with congruence and* dim $P \geq 3$. *If there are two points in $P$ with a midpoint, then any line reflection is a motion.*

Now using line reflections one proves that in a plane for every line we have a unique perpendicular line through every point of this line:

**Corollary 3.5.** *Let $A, B \in \mathcal{L}$ be lines in the three-dimensional subspace $U$ with $A \perp B$ and $m = A \cap B$. Then:*

*(i) $B$ is the unique line through $m$ in $\overline{A \cup B}$ with $A \perp B$.*
*(ii) For $a \in A \setminus \{m\}$ there exists exactly one point $b \in A \setminus \{a\}$ with $(a, m) \equiv (b, m)$.*
*(iii) There exists only one line $X \subset U$ with $X \perp A, B$ and it holds $X \perp C$ for any line $C \subset \overline{A \cup B}$ with $m \in C$.*
*(iv) Let $z, a, b$ be distinct points, let $a' \in \overline{z, a} \setminus \{a\}$ and $b' \in \overline{z, b} \setminus \{b\}$ with $(z, a) \equiv (z, a')$ and $(z, b) \equiv (z, b')$, then $(a, b) \equiv (a', b')$.*

In the following we assume, that there are two perpendicular lines with a non-empty intersection. Then by Lemma 2.7 and Corollary 3.5 for any two points $a, b$ there is a unique point $b' \in \overline{a, b} \setminus \{b\}$ with $(a, b) \equiv (a, b')$.
Now we consider point reflections. For distinct points $a, x \in P$ we denote with $a(x)$ the unique point $a(x) \in \overline{a, x} \setminus \{x\}$ with $(a, x) \equiv (a, a(x))$. We call the following mapping *point reflection*:

$$\tilde{a} : P \to P; \qquad x \to \begin{cases} x & \text{if } x = a, \\ a(x) & \text{if } x \neq a. \end{cases}$$

Now Lemma 3.5 (iv) imply:

**Theoem 3.6.** *Every point reflection $\tilde{a}$ is an involutory motion with $x = \tilde{a}(x)$ if and only if $x = a$.*

With this result one can introduce a binary operation $+$ in a more general context then in [1]. Assume that any two distinct points $a, a'$ have a midpoint $m \in \overline{a, a'}$ with $(a, m) \equiv (a', m)$. For a fixed point $0 \in P$ we denote for any point $a \in P \setminus \{0\}$ the unique midpoint of 0 and $a$ by $a/2$. We denote $0 = 0/2$. Then for the point reflection corresponding to $a/2$ we have

$$\widetilde{a/2}(0) = a \;\text{ and }\; \widetilde{a/2}(a) = 0\,.$$

We define for points $a, b$ the addition $+$ on the point set $P$ by (cf. [1], [4])

$$a + b := \widetilde{a/2} \circ \tilde{0}(b) = \widetilde{a/2}\ \tilde{0}(b)\ .$$

**Theorem 3.7.** ([3], [1], [4]) *Let $(P, \mathcal{L}, \equiv)$ be a space with congruence of* $\dim P \geq 3$ *such that any two distinct points have a midpoint. Then fixing a*

*point* $0 \in P$*, an addition* $+$ *can be defined on* $P$ *such that* $(P, +)$ *is a Bruck loop (for a definition cf.* [8] *) with the neutral element* $0$*. The point* $-a := \widetilde{0}(a)$ *is the inverse of* $a \in P$*.*
$(P, +)$ *is associative if and only if for three points* $a, b, c \in P$ *the product* $\tilde{a}\,\tilde{b}\,\tilde{c}$ *is a point reflection, too.*

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