A perfect Roman dominating function on a graph $G = (V,E)$ is a function $f : V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex $u$ with $f(u) = 0$ is adjacent to exactly one vertex $v$ for which $f(v) = 2$. The weight of a perfect Roman dominating function $f$ is the sum of the weights of the vertices. The perfect Roman domination number of $G$, denoted by $\gamma_{pR}(G)$, is the minimum weight of a perfect Roman dominating function in $G$. In this paper, we study the graphs for which adding any new edge decreases the perfect Roman domination number. We call these graphs $\gamma_{pR}$-edge critical. The purpose of this paper is to characterize the class of $\gamma_{pR}$-edge critical trees.

1. Introduction

Roman domination is a variation of domination introduced by ReVelle [12, 13]. Emperor Constantine had the requirement that an army or legion could be sent from its home to defend a neighboring location only if there was a second army which would stay and protect the home. Thus, there are two types of armies, stationary and traveling. A vertex with no army must have a neighboring vertex with a traveling army. Stationary armies then dominate their own vertices. A
A neighbor in low \([6]\). We consider finite, undirected, and simple graphs \(G\) functions introduced in \([7]\). We first present some necessary definitions and notions. For notation and graph theory terminology not given here, we follow [6]. We consider finite, undirected, and simple graphs \(G\) with vertex set \(V = V(G)\) and edge set \(E = E(G)\). The number of vertices of a graph \(G\) is called the order of \(G\) and is denoted by \(n = n(G)\). The open neighborhood of a vertex \(v \in V\) is \(N(v) = N_G(v) = \{u \in V \mid uv \in E\}\), and the degree of \(v\), denoted by \(\deg_G(v)\), is the cardinality of its open neighborhood. A leaf of a tree \(T\) is a vertex of degree one, while a support vertex of \(T\) is a vertex adjacent to a leaf. A strong support vertex is a support vertex adjacent to at least two leaves. In this paper, we denote the set of all support vertices of \(T\) by \(S(T)\) and the set of leaves by \(L(T)\). We denote \(\ell(T) = |L(T)|\) and \(s(T) = |S(T)|\). We also denote by \(L(x)\) the set of leaves adjacent to a support vertex \(x\), and denote \(\ell_x = |L(x)|\). If \(T\) is a rooted tree, then for any vertex \(v\) we denote by \(T_v\) the sub-rooted tree rooted at \(v\). A subset \(S \subseteq V\) is a dominating set of \(G\) if every vertex in \(V \setminus S\) has a neighbor in \(S\). The domination number \(\gamma(G)\) is the minimum cardinality of a dominating set of \(G\). A perfect dominating set is a set \(S \subseteq V\) such that for all \(v \in V, |N[v] \cap S| = 1\). The minimum size of a perfect dominating set for a graph \(G\) is the perfect domination number of \(G\), denoted by \(\gamma_p(G)\). Perfect dominating sets and several variations on perfect domination have received much attention in the literature; for example, see some discussion in [6] or the survey in [11].

For a graph \(G\), let \(f : V(G) \to \{0, 1, 2\}\) be a function, and let \((V_0, V_1, V_2)\) be the ordered partition of \(V(G)\) induced by \(f\), where \(V_i = \{v \in V(G) : f(v) = i\}\) for \(i = 0, 1, 2\). There is a one-by-one correspondence between the functions \(f : V(G) \to \{0, 1, 2\}\) and the ordered partitions \((V_0, V_1, V_2)\) of \(V(G)\). So, we will write \(f = (V_0, V_1, V_2)\). A function \(f : V(G) \to \{0, 1, 2\}\) is a Roman dominating function (or briefly, RDF) if every vertex \(u\) for which \(f(u) = 0\) is adjacent to at least one vertex \(v\) for which \(f(v) = 2\). The weight of an RDF \(f\) is \(w(f) = f(V(G)) = \sum_{u \in V(G)} f(u)\). The Roman domination number of a graph \(G\), denoted by \(\gamma_R(G)\), is the minimum weight of an RDF on \(G\). Roman dominating functions with several further conditions have been studied, for example, among other types, see [1–3, 10].

Recently, Henning, Klostermeyer and MacGillivray [7] introduced the concept of perfect Roman domination in graphs. As defined in [7], an RDF \(f = (V_0, V_1, V_2)\) is called a perfect Roman dominating function (or just PRDF) if every vertex \(u\) with \(f(u) = 0\) is adjacent to exactly one vertex \(v\) for which \(f(v) = 2\). The perfect Roman domination number \(\gamma_R^p(G)\) is the minimum weight of a
PRDF. Note that $\gamma_P^R(G)$ is defined for any graph $G$, since $(\emptyset,V(G),\emptyset)$ is a PRDF for $G$. We refer to a $\gamma_P^R(G)$-function as a PRDF of $G$ with minimum weight.

For many graph parameters, criticality is a fundamental question. The concept of criticality with respect to various operations on graphs has been studied for several domination parameters. Much has been written about graphs, where a parameter increases or decreases whenever an edge or vertex is removed or added. This concept has been considered for several domination parameters such as domination, 2-rainbow domination and Roman domination, by several authors and the concept is now well studied in domination theory. For references on the criticality concept on various domination parameters see, for example [4, 5, 8, 9]. In this paper, we consider this concept for perfect Roman domination number.

Our aim is to study the graphs for which adding any new edge decreases the perfect Roman domination number. We say that $G$ is perfect Roman domination edge critical, or just $\gamma_P^R$-edge critical, if for any $e \in E(\bar{G})$, we get $\gamma_P^R(G) < \gamma_P^R(G+e)$, where $\bar{G}$ is the complement of $G$. The purpose of this paper is to give a descriptive characterization of the class of $\gamma_P^R$-edge critical trees.

2. Main Results

We first present some properties of the $\gamma_P^R$-edge critical graphs.

Lemma 2.1. For every edge $e = xy$ in a graph $\bar{G}$, we get $\gamma_P^R(G) - 1 \leq \gamma_P^R(G+e)$.

Proof. Let $e = xy \in E(\bar{G})$ and $f = (V_0,V_1,V_2)$ be a $\gamma_P^R(G+e)$-function. If $V_2 \cap \{x,y\} = \emptyset$ or $\{x,y\} \subseteq V_2 \cup V_1$, then $f$ is a PRDF of graph $G$, as desired. Thus we may assume that $x \in V_2$ and $y \in V_0$. Now we define function $g$ by $g(y) = 1$ and $g(u) = f(u)$, if $u \in V - \{y\}$. Then function $g$ is a PRDF of graph $G$, and therefore $\gamma_P^R(G) \leq \gamma_P^R(G+e) + 1$. \hfill \Box

The next corollary is immediate from Lemma 2.1.

Corollary 2.2. For any edge $e \in E(\bar{G})$ in a $\gamma_P^R$-edge critical graph $G$, we have $\gamma_P^R(G+e) + 1 = \gamma_P^R(G)$.

Next, we give a characterization of $\gamma_P^R$-edge critical graphs.

Theorem 2.3. A graph $G$ is $\gamma_P^R$-edge critical if and only if for any two non-adjacent vertices $u,v$, there exists a $\gamma_P^R(G)$-function $f = (V_0,V_1,V_2)$ such that $\{f(u),f(v)\} = \{1,2\}$ and also if for $x \in \{u,v\}$, $f(x) = 1$, then for any vertex $y \in N(x)$, $f(y) \neq 2$. 
Proof. Let $G$ be a graph and $e = uv \in E(G)$. First, suppose that there exists a $\gamma_R^p$-function $f = (V_0, V_1, V_2)$ such that $\{f(u), f(v)\} = \{1, 2\}$ and also if for $x \in \{u, v\}$, $f(x) = 1$, then for any vertex $y \in N(x)$, $f(y) \neq 2$. Suppose that $f(u) = 1$ and $f(v) = 2$. We define $g : V(G + uv) \longrightarrow \{0, 1, 2\}$ by $g(u) = 0$ and $g(z) = f(z)$ if $z \neq u$. Then $g$ is a perfect Roman dominating function for $G + uv$, and so $\gamma_R^p(G + uv) \leq \gamma_R^p(G) - 1$. This implies that $G$ is $\gamma_R^p(G)$-edge critical graph.

For the converse, suppose that $G$ is $\gamma_R^p(G)$-edge critical graph. Then by Corollary 2.2, we have $\gamma_R^p(G + uv) = \gamma_R^p(G) - 1$. Let $g = (V_0, V_1, V_2)$ be a $\gamma_R^p(G + uv)$-function. If $\{g(u), g(v)\} \neq \{0, 2\}$, then $g$ is a perfect Roman dominating function for $G$, which implies that $\gamma_R^p(G) \leq \gamma_R^p(G + uv) = \gamma_R^p(G) - 1$, a contradiction. Thus, $\{g(u), g(v)\} = \{0, 2\}$. Let $g(u) = 0$, then for any vertex $w \in N_{G+uv}(u) - \{v\}$, $g(w) \neq 2$, since $g$ is a perfect Roman dominating function. We define $h : V(G) \longrightarrow \{0, 1, 2\}$ by $h(u) = 1$ and $h(z) = g(z)$ if $z \neq u$. Then $h$ is a perfect Roman dominating function for $G$ with weight $\gamma_R^p(G + uv) + 1$ and also any vertex $w \in N_G(u)$, $h(w) \neq 2$. On other hand, since $\gamma_R^p(G) = \gamma_R^p(G + uv) + 1$, it follows that $h$ is a $\gamma_R^p(G)$-function, and the result follows. □

Next, we give a characterization of the class of $\gamma_R^p$-edge critical trees. Let $T_1$ be a tree obtained from two path $P_3$ by joining central vertices which is depicted in Fig. 1(a), and $T_2$ be a tree obtained from a path $P_3$ with central vertex $u$ and a path $P_4$ with support vertex $v$ by joining $u$ to $v$ illustrated in Fig. 1(b).

![Trees T1 and T2](image)

Figure 1: The trees $T_1$ and $T_2$.

Theorem 2.4. A tree $T$ is $\gamma_R^p$-edge critical if and only if $T \in \{T_1, T_2\}$.

Proof. Let $T$ be a $\gamma_R^p$-edge critical tree. If $diam(T) \in \{2, 3\}$, $T$ is star or a double star. It is straightforward to see that $T$ is not $\gamma_R^p$-edge critical. Thus we assume that $diam(T) \geq 4$. We root $T$ at a leaf $x_0$ of a diametrical path $x_0, x_1 \ldots x_d$ from $x_0$ to a leaf $x_d$ farthest from $x_0$. Without loss of generality, we may assume that for $i \in \{2, d - 2\}$, $\deg(x_{i-1}) \geq \deg(u)$, where $u$ is any child support vertex of $x_i$. We proceed with the following claims:
Claim 1. $T$ has no strong support vertex with exactly one adjacent non-leaf vertex and degree at least 4.

Proof Assume, towards a contradiction, that $u$ is a strong support vertex of $T$ with degree at least 4 such that $\deg(u) = \ell_u + 1$. Let $w, z$ be two leaves adjacent to $u$ and $x$ be non-leaf neighbors of $u$. It follows from Theorem 2.3, that there exists a $\gamma^2_R(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(w), f(z)\} = \{1,2\}$ and $f(u) \neq 2$. Clearly, for each leaf $v$ adjacent to $u$, $f(v) \geq 1$, $f(u) = 0$ and since $f$ is a $\gamma^2_R(T)$-function, $(N(u) - \{w, z\}) \cap V_2 = \emptyset$. Then, $g : V(T) \rightarrow \{0,1,2\}$ defined by $g(u) = 2$, $g(v) = 0$ if $v \in L(u)$, $g(x) = \max\{f(x), 1\}$ and $g(v) = f(v)$ if $v \not\in L(u) \cup \{u, x\}$, is a perfect Roman dominating function for $T$ with weight less than $\gamma^2_R(T)$, a contradiction.

By Claim 1, we get $\max\{\deg(x_1), \deg(x_{d-1})\} \leq 3$.

Claim 2. $T$ has no two strong support vertices $u$ and $v$ such that $|N(u) \cup N(v)| = \ell_u + \ell_v + 1$.

Proof Assume that $u$ and $v$ are two strong support vertex of tree $T$ such that $|N(u) \cup N(v)| = \ell_u + \ell_v + 1$ and $N(u) \cap N(v) = \{w\}$. By Claim 1, $\ell_u = \ell_v = 2$. Let $L(u) = \{u_1, u_2\}$ and $L(v) = \{v_1, v_2\}$. It follows from Theorem 2.3, that there is a $\gamma^2_R(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(u_1), f(v_1)\} = \{1,2\}$. Without loss of generality, we may assume that $u_1 \in V_1$ and $v_1 \in V_2$. Then, $f(u) \neq 2, f(v_2) = 1$ and $f(v) = 0$. Since $f$ is a $\gamma^2_R(T)$-function, $f(w) \geq 2$ and so $f(u) + f(u_2) \geq 2$. Then, $g : V(T) \rightarrow \{0,1,2\}$ defined by $g(u) = 2$, $g(v) = 2$, $g(u_1) = g(u_2) = g(v_1) = g(v_2) = 0$, $g(w) = \max\{1, f(w)\}$ and $g(z) = f(z)$ if $z \not\in N[u] \cup N[v]$, is a perfect Roman dominating function for $T$ with weight less than $\gamma^2_R(T)$, a contradiction.

Claim 2, implying that if $x_1$ be a strong support vertex, then each vertex of $N(x_2) - \{x_3\}$ is a weak support vertex or a leaf.

We now assume that $diam(T) = 4$. Without loss of generality, we may assume that $\deg(x_1) \geq \deg(x_3)$. By Claim 1, $\deg(x_3) \leq \deg(x_1) \leq 3$. We first assume that $\deg(x_1) = 3$. Then Claim 2, implying that every neighbor of $x_2$ is a leaf or a weak support vertex and so $\deg(x_3) = 2$. Let $L(x_1) = \{x_0, x_1\}$. It follows from Theorem 2.3, that there is a $\gamma^2_R(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_0), f(x_1)\} = \{1,2\}$. Without loss of generality, we may assume that $x_0 \in V_1$ and $x_1 \in V_2$. Then $f(x_1) = 0$, $f(x_2) \neq 2$ and $f(x_3) + f(x_4) = 2$. Then, $g : V(G) \rightarrow \{0,1,2\}$ defined by $g(x_1) = 2$, $g(x_1) = g(x_0) = g(x_2) = 0$ and $g(z) = 1$ if $z \not\in \{x_0,x_1,x_2,x_1\}$, is a perfect Roman dominating function for $T$ with weight less than $\gamma^2_R(T)$, a contradiction. Thus, we assume that $\deg(x_3) = \deg(x_1) = 2$ and also any neighbor of $x_2$ is a leaf or weak support vertex. If $\deg(x_2) = 2$, then $T = P_3$ and clearly $T$ is not $\gamma^2_R$-edge critical. Hence, we assume that $\deg(x_2) \geq 3$. It follows from Theorem 2.3, that there is a $\gamma^2_R(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_0), f(x_4)\} = \{1,2\}$. Without loss of generality, we may assume
that $x_0 \in V_1$ and $x_4 \in V_2$. Then $f(x_3) = 0, f(x_2) \neq 2$ and $f(x_1) = 1$. Hence if $u \in L(x_2)$, then $f(u) = 1$ and also if $v \in N(x_2)$ is a support vertex with leaf neighbors $v'$, then $f(v) + f(v') = 2$. Therefore, $w(f) = 2 \deg(x_2) - \ell_{x_2} + f(x_2) \geq 2 \deg(x_2) - \ell_{x_2}$. Thus, we may assume that $f(x_2) = 1$, that is, reassigning to each neighbor of $x_1$ the weight 0 and to each other weight unchanged produces a new PRDF with weight less than $\gamma^R_T$, a contradiction. Hence $\gamma^R_T = 5$.

**Claim 3.** $\text{diam}(T) = 5$.

**Proof** Assume that $\text{diam}(T) \geq 6$. Then $x_2 \notin N[x_{d-2}]$ and so by Theorem 2.3, there is a $\gamma^R_T$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_2), f(x_{d-2})\} = \{1, 2\}$. Without loss of generality, we may assume that $x_2 \in V_1$ and $x_{d-2} \in V_2$. By Theorem 2.3, for any $u \in N(x_2)$, $f(u) \neq 2$ and so $f(x_3) \neq 2$. Since $f(x_2) = 1$, we have $f(N[x_1]) \geq 3$. Then, reassigning to each neighbor of $x_1$ the weight 0 and to $x_4$ the weight 2 produces a PRDF with weight less than $\gamma^R_T$, a contradiction. Hence $\text{diam}(T) = 5$.

Then Claim 1, implying that $\deg(x_1) \leq 3$ and $\deg(x_4) \leq 3$. Without loss of generality, we may assume that $\deg(x_4) \leq \deg(x_1)$. We consider the following cases:

**Case 1.** $\deg(x_1) = \deg(x_4) = 3$.

Suppose that $N(x_1) = \{x_0, x'_1, x_2\}$ and also $N(x_4) = \{x_5, x'_4, x_3\}$. We first assume that $5 \leq \max\{\deg(x_2), \deg(x_3)\}$. Without loss of generality, we assume that $\deg(x_2) \geq 5$. By Claim 2, each neighbor of $x_2$ other of $x_1$ and $x_3$ is a leaf or a weak support vertex. Let $K_i$ be the set of weak support neighbors of $x_i$ for $i \in \{2, 3\}$. Then $\ell_{x_2} + |K_2| \geq 3$. It follows from Theorem 2.3, that there is a $\gamma^R_T$-function $f = (V_0, V_1, V_2)$ such that $f(x_0) = 1, f(x'_1) = 2$. Then clearly $f(x_1) = 0, f(x_2) \neq 2$, for $u \in L(x_2), f(u) \geq 1$ and for any vertex $v \in K_2, f(v) + f(v') = 2$, where $v'$ is leaf neighbor of $v$. Hence $\gamma^R_T = w(f) \geq w(f|_{T_3}) + \ell_{x_2} + 2|K_2| + 3$. We define function $g : V(T) \rightarrow \{0, 1, 2\}$ defined by $g(x_2) = g(x_1) = 2, g(x_0) = g(x'_1) = 0$, for each vertex $u \in L(x_2)$, $g(u) = 0$, for $u \in K_2, g(u) = 0$ and $g(u') = 1$, $g(x_3) = \max\{1, f(x_3)\}$ and for $w \in V(T_3) - \{x_3\}, g(w) = f(w)$. Then $g$ is a perfect Roman dominating function with weight at most $5 + |K_2| + w(f|_{T_3})$ and so $w(g) \leq 5 + |K_2| + w(f|_{T_3}) < w(f|_{T_3}) + \ell_{x_2} + 2|K_2| + 3 \leq \gamma^R_T$, a contradiction. Thus, we assume that $max\{\deg(x_2), \deg(x_3)\} \leq 4$.

Then by Theorem 2.3, there is a $\gamma^R_T$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_1), f(x_4)\} = \{1, 2\}$. Without loss of generality, we may assume that $x_1 \in V_2$ and $x_4 \in V_1$. Clearly, $f(x_0) = f(x'_1) = 0, f(x_5) = f(x'_4) = 1$ and we can assume that $f(x_3) = 0$ and for each $v \in L(x_3), f(v) = 1$. Let $N(x_3) \cap V_2 = \{u\}$. If $u \in K_3$ and $L(u) = \{u'\}$, then $f(u') = 0$. Then, reassigning to $x_5$ and $x'_4$ the weight 0, to $x_4$ the weight 2 and to each $u \in K_3$ and $u'$ the weight 1, and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma^R_T$, a contradiction. Thus, we assume that $u \notin K_3$ and so $u = x_2$. Then for each
vertex \( v \in L(x_2) \), \( f(v) = 0 \) and for each vertex \( w \in K_2 \), \( f(w) = 0 \) and \( f(w') = 1 \), that \( \{w'\} = L(w) \). Hence \( w(f) = \ell_x + 2|K_3| + |K_2| + 7 \). Then, reassigning to \( x_2 \) the weight 0, to \( x_4 \) the weight 2, to \( x_5 \) and \( x'_4 \) the weight 0, to each \( u \in K_2 \) the weight 1, to each \( u \in L(x_2) \) the weight 1 and leaving all other weights unchanged produces a new PRDF \( g \) with weight \( \ell_x + 2|K_3| + 2|K_2| + \ell_{x_2} + 4 \) and so \( w(g) = \ell_x + 2|K_3| + 2|K_2| + \ell_{x_2} + 4 < \ell_x + 2|K_3| + |K_2| + 7 = w(f) = \gamma^P_R(T) \), a contradiction.

**Case 2.** \( \deg(x_1) = 3 \) and \( \deg(x_4) = 2 \).

Without loss of generality, we may assume that any child of \( x_3 \) is a weak support or a leaf. We first assume that \( \deg(x_2) \geq 4 \). Then any neighbors of \( x_2 \) other of \( x_3 \) is a weak support or a leaf. By Theorem 2.3, there is a \( \gamma^P_R(T) \)-function \( f = (V_0, V_1, V_2) \) such that \( \{f(x_1), f(x_3)\} = \{1, 2\} \). We first assume that \( x_1 \in V_2 \) and \( x_3 \in V_1 \) and so by theorem 2.3, \( f(x_2) \neq 2 \). Clearly \( f(x_0) = f(x'_1) = 0 \), \( f(x_5) + f(x_4) = 2 \) and we can assume that for each \( v \in L(x_3) \), \( f(v) = 1 \). Then reassigning to \( x_3 \) and \( x_2 \) the weight 0, to \( x_4 \) the weight 2 and to each \( u \in K_3 \) and \( u' \in N(u) - \{x_3\} \) the weight 1, and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma^P_R(T) \), a contradiction. Thus, we assume that \( f(x_1) = 1 \), \( f(x_3) = 2 \) and \( f(x_2) \neq 2 \). Let \( T' = T - x_3 \). Then, reassigning to \( x_2 \) the weight 2, to \( u \in N(x_2) - \{x_3\} \) the weight 0 and to each other vertex of tree \( T' \) the weight 1, and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma^P_R(T) \), a contradiction. Thus, we assume that \( \deg(x_2) \leq 3 \).

Assume that \( \deg(x_2) = 3 \) and \( N(x_2) = \{x_1, x_3, x'_2\} \). Then Claim 2 implying that \( x'_2 \) is a leaf or a weak support vertex. We first assume that \( x_2 \) is a support vertex. By Theorem 2.3, there is a \( \gamma^P_R(T) \)-function \( f = (V_0, V_1, V_2) \) such that \( \{f(x_2), f(x_4)\} = \{1, 2\} \). We first assume that \( f(x_2) = 2 \) and \( f(x_4) = 1 \) and so \( f(x_3) \neq 2 \). Clearly, \( f(x'_2) = 0 \), \( f(x_5) = 1 \) and \( f(x_0) + f(x_1) + f(x'_1) = 2 \). We can assume that \( f(x_0) = f(x'_1) = 0 \) and \( f(x_1) = 2 \). If \( f(x_3) = 1 \), then reassigning to \( x_2 \) the weight 0, reassigning to \( x'_2 \) the weight 1 and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma^P_R(T) \), a contradiction. Thus we may assume that \( f(x_3) = 0 \). Then for \( u \in N(x_3) - \{x_2\} \), \( f(u) \neq 2 \), since \( f(x_2) = 2 \) and \( f \) is a \( \gamma^P_R(T) \)-function. Then reassigning to \( x_2 \) and \( x_5 \) the weight 0, reassigning to \( x'_2 \) the weight 1, reassigning to \( x_4 \) the weight 2 and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma^P_R(T) \), a contradiction. Thus we assume that \( f(x_2) = 1 \) and \( f(x_4) = 2 \) and so \( f(x_1) \neq 2 \). Then \( f(x_1) + f(x'_1) + f(x_0) = 3 \) and so reassigning to \( x_1 \) the weight 2, reassigning to \( x'_1 \) and \( x_0 \) the weight 0 and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma^P_R(T) \), a contradiction. Thus we assume that \( x'_2 \) is a weak support vertex. Let \( x'_2 \) is leaf adjacent to \( x'_2 \). Then by Theorem 2.3, there is a \( \gamma^P_R(T) \)-function \( f = (V_0, V_1, V_2) \) such that \( \{f(x_0), f(x_4)\} = \{1, 2\} \). We first assume that \( f(x_0) = 2 \) and \( f(x_4) = 1 \). Clearly
Assume that \( \deg f(x_1) = 2 \) and \( \deg f(x_3) = 3 \). Without loss of generality, we may assume that \( f(x_2) = 2 \) and \( f(x_4) = 1 \). Then, reassigning to \( x_1 \) the weight 2, reassigning to \( x_2 \) and \( x_3 \) the weight 1 and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma_R^f(T) \), a contradiction. Hence, we assume that \( f(x_0) = 1 \) and \( f(x_4) = 2 \). Clearly, \( f(x_5) = 0 \) and \( f(x_3) \neq 2 \). Then, reassigning to \( x_1 \) the weight 2, reassigning to \( x_2 \), and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma_R^f(T) \), a contradiction. Hence, we assume that \( f(x_2) = 0 \) and \( f(x_3) = 2 \). If \( f(x_3) \neq 2 \), then reassigning to \( x_2 \) the weight 2, reassigning to \( x_0 \) and \( x_2 \) the weight 0 and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma_R^f(T) \), a contradiction. Now assume that \( f(x_3) = 2 \). Then, reassigning to \( x_1 \) and \( x_0 \) the weight 1, reassigning to \( x_2 \) the weight 0, and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma_R^f(T) \), a contradiction. Hence, \( \deg f(x_3) \geq 3 \) and similarly \( \deg f(x_5) \geq 3 \).

Assume that \( \deg f(x_2) \geq 4 \). We can assume that any vertex in \( N(x_2) - \{x_3\} \) is a leaf or a weak support vertex. We first assume that \( x_2 \) is a strong support vertex. Let \( \{x'_2, x''_2\} \subseteq L(x_2) \). It follows from Theorem 2.3, that there is a \( \gamma_R^f(T) \)-function \( f = (V_0, V_1, V_2) \) such that \( \{f(x_0), f(x_2)\} = \{1, 2\} \). We first assume that \( f(x_0) = 2 \) and \( f(x_2) = 1 \). Clearly \( f(x_1) = 0 \) and \( f(x_3) \neq 2 \). Then, reassigning to \( x_1 \) the weight 2, reassigning to \( x_2 \) and \( x_3 \) the weight 0 and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma_R^f(T) \), a contradiction. Hence, we assume that \( f(x_1) = 0 \) and \( f(x_2) = 2 \). If \( f(x_3) \neq 2 \), then reassigning to \( x_1 \) the weight 2, reassigning to \( x_0 \) and \( x_2 \) the weight 0 and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma_R^f(T) \), a contradiction. Now assume that \( f(x_3) = 2 \). Then, reassigning to \( x_2 \) the weight 2, reassigning to \( u \in N(x_2) - \{x_3\} \) the weight 0, reassigning to each leaf of tree \( T - T_3 \), the weight 1, reassigning to vertex \( x_3 \) the weight \( \max \{1, f(x_3)\} \) and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma_R^f(T) \), a contradiction.
2, where \( v' \) is the leaf adjacent to \( v \). Hence \( w(f) \geq 2|K_2| + 2|K_3| + \ell_x + \ell_{x_3} \). Then, function \( g = (N(x_2), V(T) - N[x_2], x_2) \) is a perfect Roman dominating function with weight \( \ell_x + 2|K_3| + |K_2| + 2 \) and so \( w(g) = \ell_x + 2|K_3| + |K_2| + 2 < 2|K_2| + 2|K_3| + \ell_x + \ell_{x_3} = w(f) = \gamma_R^p(T) \), a contradiction. Next, assume that \( f(x_3) \neq 0 \). Let \( T' = T - T_{x_3} \). Then reassigning to \( x_2 \) the weight 2, reassigning to \( u \in N(x_2) - \{x_3\} \) the weight 0, reassigning to each leaf of tree \( T' \) the weight 1 and leaving all other weights unchanged produces a new PRDF with weight less than \( \gamma_R^p(T) \), a contradiction. Hence, \( \deg(x_2) = 3 \) and similarly \( \deg(x_3) = 3 \).

Let \( N(x_2) = \{x_2', x_1, x_3\} \) and \( N(x_3) = \{x_3', x_4, x_2\} \). We first assume that \( x_2' \) is a leaf. If \( x_3 \) be a support vertex, then by Theorem 2.3, there is a \( \gamma_R^p(T) \)-function \( f = (V_0, V_1, V_2) \) such that \( \{f(x_2'), f(x_3')\} = \{1, 2\} \). Without loss of generality, we may assume that \( f(x_2') = 2 \) and \( f(x_3') = 1 \). Clearly, \( f(x_2) = 0, f(x_3) \neq 2, f(x_0) + f(x_1) = 2, f(x_4) + f(x_5) = 2 \) and so \( w(f) \geq 7 > \gamma_R^p(T) = 6 \), a contradiction. Hence, \( x_3' \) is a weak support vertex and so \( T = T_1 \). Now, assume that \( x_2' \) is a support vertex. Then \( T \in \{T_1, T_2\} \). The converse part is obvious.

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