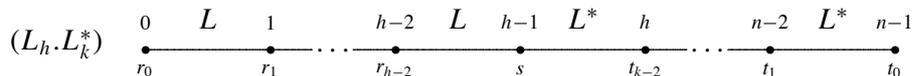


## FLAG-TRANSITIVE $L_h.L^*$ -GEOMETRIES

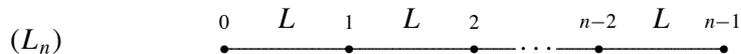
ANTONIO PASINI

### 1. Introduction.

The classification of finite flag-transitive linear spaces, obtained by Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [20] at the end of the eighties, gave new impulse to the program of classifying various classes of locally finite flag-transitive geometries belonging to diagrams obtained from a Coxeter diagram by putting a label  $L$  or  $L^*$  on some (possibly, all) of the single-bond strokes for projective planes. (I recall that the symbols  $L$  and  $L^*$ , when used as labels in a diagram, denote the class of linear spaces and, respectively, dual linear spaces). The reader may see Buekenhout and Pasini [23], Section 4 for a survey of results in this trend, updated to 1994. In this survey, we focus on the following diagram:



where  $k := n + 1 - h$ , the integers  $0, 1, \dots, n - 1$  are the types,  $r_0, r_1, \dots, r_{h-2}, s, t_{k-2}, \dots, t_1, t_0$  are finite orders,  $2 \leq h \leq n$  and  $n$ , which is the rank, is at least 3. Clearly,  $r_0 \leq r_1 \leq \dots \leq r_{h-2} \leq s$  and  $s \geq t_{k-2} \geq \dots \geq t_1 \geq t_0$ . In particular, when  $h = n$ , we have the following:



and when  $h = k = 2$  we have the following:

$$(L.L^*) \quad \begin{array}{ccccccc} & 0 & L & 1 & L^* & 2 & \\ & \bullet & \text{---} & \bullet & \text{---} & \bullet & \\ r & & & s & & & t \end{array}$$

It is well known that locally finite  $L_n$ -geometries are finite. Finite flag-transitive  $L_n$ -geometries have been classified by A. Delandtsheer soon after the publishing of [20].

**Theorem 1.1.** (Delandtsheer [31], [32]) *Let  $\Gamma$  be a finite flag-transitive geometry for diagram  $L_n$  with  $n \geq 3$ . Then  $\Gamma$  is one of the following:*

- (1) *An  $n$ -dimensional simplex or the  $\{n, n + 1, \dots, m\}$ -truncation of an  $m$ -dimensional simplex, obtained by dropping all elements of dimension  $\geq n$ .*
- (2) *An  $n$ -dimensional projective geometry or the  $\{n, n + 1, \dots, m\}$ -truncation of an  $m$ -dimensional projective geometry,  $m > n$ .*
- (3) *An  $n$ -dimensional affine geometry or the  $\{n, n + 1, \dots, m\}$ -truncation of an  $m$ -dimensional affine geometry.*
- (4) *One of the Steiner systems  $S(24, 8, 5)$ ,  $S(23, 7, 4)$  or  $S(22, 6, 3)$  for  $M_{24}$ ,  $M_{23}$  and  $M_{22}$  respectively, regarded as geometries of rank 5, 4 and, respectively, 3.*
- (5) *One of the Steiner systems  $S(12, 6, 5)$  or  $S(11, 5, 4)$  for  $M_{12}$  and  $M_{11}$ , regarded as geometries of rank 5 and, respectively, 4.*
- (6) *A 1-point extension of the point-line system of  $AG(d, q)$ , constructed as follows: The 0-elements are the  $q^d + 1$  points of  $PG(1, q^d)$ , the 1-elements are the unordered pairs of points of  $PG(1, q^d)$  and the 2-elements are the images by  $P\Gamma L_2(q^d)$  of a given copy of  $PG(1, q)$  naturally embedded in  $PG(1, q^d)$ . (Note that, when  $d = 2$ , this construction yields the classical Möbius planes. When  $q = 2$ , we obtain a truncated simplex).*
- (7) *A 1-point extension of a Netto triple system, constructed as follows: For  $q \equiv 7 \pmod{12}$ , the points of  $PG(1, q) = \{\infty\} \cup GF(q)$  are taken as 0-elements and the unordered pairs of points of  $PG(1, q)$  as 1-elements. The 2-elements are the images by  $P\Sigma L_2(q)$  of  $\{\infty\} \cup K$ , where  $K$  is the set of solutions of the equation  $x^3 = 1$  in  $GF(q)$ .*

In view of the above, we may keep our  $L_h.L_k^*$ -diagram distinct from either  $L_n$  and its dual  $L_n^*$ . So, we assume  $2 \leq h < n$  and  $r_0, t_0 < s$ . In particular, when  $n = 3$  then  $h = k = 2$  and  $r, t < s$ .

The case of  $n = 3$  is crucial for the project we are discussing. Indeed, once we had classified locally finite flag-transitive  $L.L^*$ -geometries, we could combine that classification with Theorem 1.1 to finish the job, hopefully without encountering too big obstacles. Some difficulties may arise at that latter

stage mainly because, whereas locally finite  $L.L^*$ -geometries are finite (Del Fra and Ghinelli [35]), locally finite  $L_h.L_k^*$ -geometries of rank  $n > 3$  are infinite in general. However, the rank 3 case is the most difficult to cope with. Indeed, a quick inspection of all possible combinations of two families of linear and dual linear spaces in an  $L.L^*$ -diagram is sufficient to realize that the classification we are dreaming of is probably too difficult to accomplish in that general setting. Many cases can be ruled out but too many remain (see Theorem 1.2). Furthermore, some combinations cover very wild areas, for which a classification seems hard to achieve. On the other hand, the existence of many interesting examples of  $L_h.L_k^*$ -geometries was too appealing for people neglected this topic at all. Moreover, characterizations had earlier been found for a few classes and exceptional examples of  $L_h.L_k^*$ -geometries (Hughes [41], Sprague [69], [70], Lefevre and Van Nypelseer [52], [73], for instance). So, authors interested in these geometries have focused on some special cases regarded as more interesting or promising, and classifications have been obtained for some of those cases. We will expose those results in sections 8, 9, and 10, fusing them in a few theorems tailored as if a complete classification were possible, sooner or later. Sections 2, 3, ..., 7 are devoted to examples.

In the rest of this introduction we recall some terminology for diagram geometries (next subsection) and the list of finite flag-transitive linear spaces (Subsection 1.2), also stating some notation for them. A preliminary reduction theorem for flag-transitive  $L.L^*$ -geometries is stated in Subsection 1.3.

### 1.1. Notation and terminology for diagram geometries

We follow [56] for basic notions of diagram geometry. In particular, as in [56], all geometries are residually connected and firm by definition. However, we make a few changes in the notation of [56]: We use the symbol  $\text{Res}$  to denote residues and, for a proper subset  $J$  of the type-set  $I$  of  $\Gamma$ , we write  $\text{Tr}_J(\Gamma)$  to denote the  $J$ -truncation of  $\Gamma$ , namely the subgeometry of  $\Gamma$  obtained by removing all elements of type  $j \in J$ . For distinct types  $i, j \in I$ , we put  $\mathcal{S}_{i,j}(\Gamma) := \text{Tr}_{I \setminus \{i,j\}}(\Gamma)$  and regard  $\mathcal{S}_{i,j}(\Gamma)$  as a point-line geometry, its  $i$ -elements being taken as points and the  $j$ -elements as lines. We call  $\mathcal{S}_{i,j}(\Gamma)$  the  $(i, j)$ -space of  $\Gamma$ . We write  $\text{Aut}(\Gamma)$  instead of  $\text{Aut}_s(\Gamma)$  for the full group of type-preserving automorphism of  $\Gamma$ , calling it the *automorphism group* of  $\Gamma$ .

Some of the geometries consider in this paper arise from buildings. As many authors do, we allow a building to be non-thick. The thin buildings are the Coxeter complexes.

Chamber systems will also be mentioned in this paper, at a few places. We refer to [56], Chapter 12, and Ceccherini and Pasini [28], Section 7, (also [61],

Section 2) for chamber systems and their relations with geometries.

## 1.2. Notation for linear spaces

According to Buekenhout et al. [20] (see also Beuekenhout, Delandtsheer and Doyen [19]), finite flag-transitive linear spaces can be divided in seven main classes:

- (1) Projective spaces, namely point-line systems of projective geometries. In particular, flag-transitive projective planes.
- (2) Affine spaces, namely point-line systems of affine geometries. In particular, flag-transitive affine planes.
- (3) Circular spaces, namely vertex-edge systems of complete graphs. We warn that this class intersects (2) non-trivially. Indeed, affine spaces over  $GF(2)$  are circular spaces.
- (4) Hermitian and Ree unitals.
- (5) Witt-Bose-Shrikande spaces (Witt spaces, for short).
- (6) Hering spaces. This class only contains two examples, both of which have orders (8,90).
- (7) 1-dimensional spaces. We warn that, in [20], this class includes non-desarguesian affine planes with 1-dimensional automorphism group, but we will do differently. In this paper, a 1-dimensional linear space is a non-affine flag-transitive linear space with  $q$  points ( $q$  a prime power) and automorphism group contained in  $A\Gamma L_1(q)$ .

When writing a diagram, we will use the following labels for the above classes and their duals:

class	symbol	dual class
Projective spaces	$PG$	$PG^*$
Affine spaces	$AG$	$AG^*$
Affine planes	$Af$	$Af^*$
Circular spaces	$c$	$c^*$
Unitals	$U$	$U^*$
Witt spaces	$W$	$W^*$
Hering spaces	$H$	$H^*$
1-dimensional spaces	$1D$	$1D^*$

Although projective planes are particular projective spaces, we will use the symbol  $PG$  only for projective spaces of dimension at least 3, keeping the usual convention of writing no label for the class of projective planes. On the other hand, we will regard affine planes as included in the class named  $AG$ . We will

use the label  $Af$  (or  $Af^*$ ) only when we want to recall that the (dual) affine spaces we are referring to are 2-dimensional.

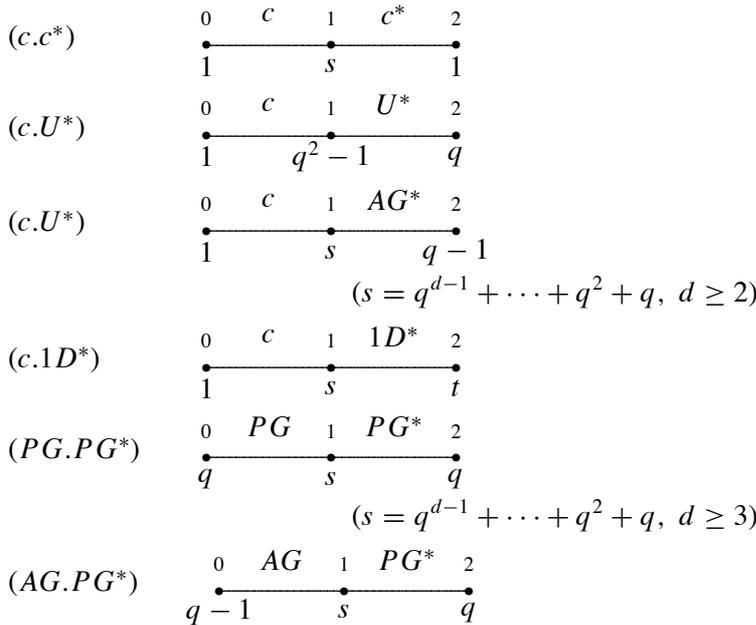
The symbol  $AG(d, q)$  usually denotes the  $d$ -dimensional affine geometry over  $GF(q)$ , which is a geometry of rank  $d$ . It is convenient to have a different symbol for the corresponding affine space, namely the point-line system of  $AG(d, q)$ . We will denote it by  $AS(d, q)$ . In view of this convention,  $PS$  and  $AS$  would be more appropriate than  $PG$  and  $AG$  as labels for strokes representing projective or affine spaces, but the latter are the symbols normally used in the literature, so we will keep them here.

We state one more definition: Given a  $d$ -dimensional affine space  $\Sigma$  of order  $q$ , we say that a flag-transitive subgroup  $G \leq \text{Aut}(\Sigma)$  is 1-dimensional if  $G \leq \text{AGL}_1(q^d)$ .

**1.3. A reduction theorem**

The possibilities for a diagram of a flag-transitive locally finite  $L.L^*$ -geometry have been thoroughly analyzed by C. Huybrechts [42], Theorems 5.5.9 and 6.7.1. We list them in the next theorem, also giving each diagram a conventional name for further reference. Names for diagrams dual of those listed here can be formed according to an obvious rule:  $U.c^*$  stands for the dual of  $c.U^*$ ,  $AG.c^*$  is the dual of  $c.AG^*$ , and so on.

**Theorem 1.2.** (Huybrechts) *Up to dualities, the following are the only possible diagrams for a flag-transitive locally finite  $L.L^*$ -geometry  $\Gamma$ :*



$$(s = q^{d-1} + \dots + q^2 + q, d \geq 3)$$

(AG.AG\*) 
$$\begin{array}{c} 0 \quad AG \quad 1 \quad AG^* \quad 2 \\ \bullet \text{---} \bullet \text{---} \bullet \\ q_1 - 1 \quad s \quad q_2 - 1 \end{array}$$
  

$$(s = \frac{q_1^{d_1} - q_1}{q_1 - 1} = \frac{q_2^{d_2} - q_2}{q_2 - 1})$$

(AG.1D\*) 
$$\begin{array}{c} 0 \quad AG \quad 1 \quad 1D^* \quad 2 \\ \bullet \text{---} \bullet \text{---} \bullet \\ q - 1 \quad s \quad t \end{array}$$
  

$$(s = q^{d-1} + \dots + q^2 + q, d \geq 2)$$

(H.H\*) 
$$\begin{array}{c} 0 \quad H \quad 1 \quad H^* \quad 2 \\ \bullet \text{---} \bullet \text{---} \bullet \\ 8 \quad 90 \quad 8 \end{array}$$

(U.U\*) 
$$\begin{array}{c} 0 \quad U \quad 1 \quad U^* \quad 2 \\ \bullet \text{---} \bullet \text{---} \bullet \\ q \quad q^2 - 1 \quad q \end{array}$$

(W.W\*) 
$$\begin{array}{c} 0 \quad W \quad 1 \quad W^* \quad 2 \\ \bullet \text{---} \bullet \text{---} \bullet \\ q/2 - 1 \quad q \quad q/2 - 1 \end{array} \quad (q = 2^h, h \geq 2)$$

(W.Af\*) 
$$\begin{array}{c} 0 \quad W \quad 1 \quad Af^* \quad 2 \\ \bullet \text{---} \bullet \text{---} \bullet \\ q/2 - 1 \quad q \quad q - 1 \end{array} \quad (q = 2^h, h \geq 2)$$

(W.1D\*) 
$$\begin{array}{c} 0 \quad W \quad 1 \quad 1D^* \quad 2 \\ \bullet \text{---} \bullet \text{---} \bullet \\ q/2 - 1 \quad q \quad r \end{array}$$
  

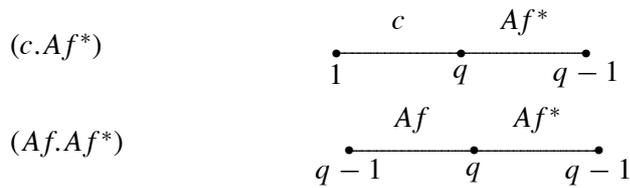
$$((q + 1)r + 1 \leq 2 = 2^{2h}, h \geq 2)$$

(1D.1D\*) 
$$\begin{array}{c} 0 \quad 1D \quad 1 \quad 1D^* \quad 2 \\ \bullet \text{---} \bullet \text{---} \bullet \\ r \quad s \quad t \end{array}$$

Moreover:

- (1) If  $\Gamma$  belongs to  $H.H^*$ ,  $W.W^*$  or  $U.U^*$ , then the  $\{1, 2\}$ -residues of  $\Gamma$  are dually isomorphic to the  $\{0, 1\}$ -residues.
- (2) In case  $c.AG^*$  with  $d > 2$ , either  $(q, d) = (4, 3)$  or the stabilizer in  $Aut(\Gamma)$  of a  $\{1, 2\}$ -residue induces a 1-dimensional group on that residue.
- (3) In case  $AG.AG^*$  with  $q_1 \neq q_2$ , the stabilizers in  $Aut(\Gamma)$  of the residues of type  $\{0, 1\}$  and  $\{1, 2\}$  induce 1-dimensional groups on those residues.
- (4) In case  $AG.1D^*$  the stabilizer in  $Aut(\Gamma)$  of a  $\{0, 1\}$ -residue induces a 1-dimensional group on that residue.

**Remark 1.1.** Diagrams  $c.AG^*$  with  $q = 2$  and  $AG.AG^*$  with  $q_1 = q_2 = 2$  are special cases of  $c.c^*$ . Diagrams  $c.AG^*$  with  $s = q$  and  $AG.AG^*$  with  $d_1 = d_2 = 2$  (hence  $s = q_1 = q_2 = q$ , say) are usually written as follows:



**Remark 1.2.** The equation

$$(*) \quad q_1 + q_1^2 + \dots + q_1^{d_1-1} = q_2 + q_2^2 + \dots + q_2^{d_2-1}$$

to be satisfied in case  $AG.AG^*$  is the famous Goormaghtigh equation (see Makowski and Schinzel [53]). Trivially, (\*) holds for  $q_1 = q_2$  and  $d_1 = d_2$ . Apart from that, assuming  $q_1 < q_2$ , the only solution I know for (\*) is  $(q_1, d_1, q_2, d_2) = (2, 5, 5, 3)$ . Flag-transitive  $AG.AG^*$ -geometries with  $(q_1, d_1, q_2, d_2) = (2, 5, 5, 3)$  actually exist (see Example 4.1).

**Remark 1.3.** Comparing the classification of 2-transitive groups one can see that, in case of  $c.AG^*$  with a 1-dimensional group induced on  $\{1, 2\}$ -residues, 1-dimensional groups are also induced on  $\{0, 1\}$ -residues. This forces

$$(**) \quad 2 + q + q^2 + \dots + q^{d-1} = p^n$$

for a prime  $p$  and a positive integer  $n$ . This equation is also unsolved in general. When  $d$  is odd, then  $p = 2$  and we are back to (\*) of Remark 1.2. When  $d$  is even,  $p$  must be odd.

We warn that the  $c.AG^*$ -case with  $(q, d) = (4, 3)$  and non-1-dimensional groups induced on  $\{1, 2\}$ -residues is missing in [42], Theorem 6.7.1. That error has been corrected later [48].

**Remark 1.4.** Flag-transitive examples are known for all diagrams listed in Theorem 1.2 but  $U.U^*$ ,  $c.1D^*$ ,  $AG.1D^*$  and  $1D.1D^*$  (but non-flag-transitive examples for  $U.U^*$  are easy to construct, by the gluing procedure described in Section 4). For the diagrams for which examples are known, we indicate below where in this paper those examples are described or mentioned:

$c.c^*$  Subsections 2.1, 2.3, 2.4, 3.4, 3.8, 4.3, 6.2, 6.4 and Section 7.

$c.U^*$	Example 6.7.
$c.AG^*$	Examples 3.1, 3.2, 4.1 and Subsection 6.1.
$PG.PG^*$	Subsection 2.1.
$AG.PG^*$	Subsections 2.2 and 3.2.
$AG.AG^*$	Subsections 2.3, 2.4, 3.2, 4.3.
$H.H^*$	Subsection 4.4.
$W.W^*$	Subsection 5.1.
$W.Af^*$	Subsections 3.5 and 4.4.

#### 1.4. Property $T_{i,j}^k$ , IP and flatness

We finish this introduction recalling some terminology to be used in the sequel. Given a geometry  $\Gamma$  and three types  $i, j, k$ , we say that  $\Gamma$  satisfies the  $(i, j, k)$ -triangular property ( $T_{i,j}^k$ , for short) if any three mutually collinear points of the  $(i, j)$ -space  $\mathcal{S}_{i,j}(\Gamma)$  are incident with a common  $k$ -element.

Turning to  $L.L^*$ -geometries, we say that an  $L.L^*$ -geometry  $\Gamma$  is *flat* if  $\mathcal{S}_{0,2}(\Gamma)$  is a generalized digon. In the situation opposite to the above,  $\Gamma$  satisfies the *Intersection Property* (IP, for short) which in this context amounts to say that  $\mathcal{S}_{0,1}(\Gamma)$  is semi-linear. (See [56, Chapter 6] for formulations of IP in more general situations.)

We recall that the  $c.c^*$ -geometries that satisfy IP are called *semiplanes*. The semiplanes where any two 0-elements are incident to a common 1-element are called *biplanes*.

**Part I**

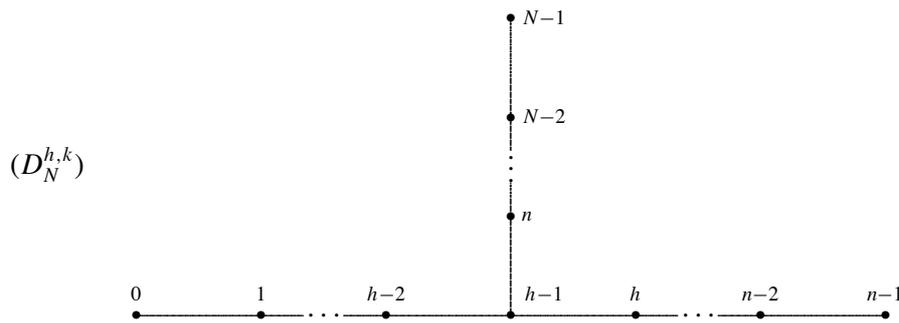
**A survey of examples**

In the next six sections we shall describe all examples of flag-transitive  $L_h.L_k^*$ -geometries known to us. As the reader will see, most of those geometries arise from general constructions, as truncations of a building of type  $D_N$ ,  $E_N$  or similar, possibly after having removed one or two hyperplanes from it (Section 2), or as affine expansions (Section 3) or gluings (Section 4). Different constructions are described in Section 5 and Section 7 (paragraph on projective semiplanes). Many non-flag-transitive examples can also be obtained by some of those constructions, but we are not going to insist on them. Exceptional examples will be described in Section 6. Most (but not all) of them are somehow related to Mathieu groups. Section 7 is only devoted to  $c.c^*$ -geometries.

Group-free characterizations are known for a few of the examples we are going to describe. When that is the case, we will mention that characterization just after the description of the example.

**2.  $L_h.L_k^*$ -geometries from buildings.**

Henceforth we denote by  $D_N^{h,k}$  the following Coxeter diagram of rank  $N$ , where  $h, k$  are positive integers with  $h + k \leq N + 1$ , and  $n = h + k - 1$ :

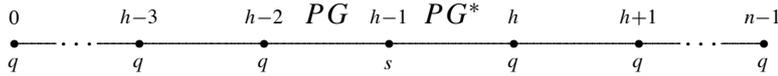


The diagram  $D_N^{h,k}$  is spherical for quite a few choices of  $h, k$  and  $N$ . For instance,  $D_N^{2,2}$ ,  $D_N^{N-2,2}$  and  $D_N^{2,N-2}$  are the same as the Lie diagram  $D_N$ . For  $N = 6, 7, 8$ , the diagrams  $D_N^{N-3,3}$ ,  $D_N^{3,N-3}$ ,  $D_N^{N-3,2}$ ,  $D_N^{2,N-3}$ ,  $D_N^{3,2}$  and  $D_N^{2,3}$  are the same as the Lie diagram  $E_N$ . When  $h = 1$  or  $k = 1$  or  $h + k = N + 1$ , then  $D_N^{h,k} = A_N$ . Diagram  $D_N^{h,k}$  is non-spherical for all other choices of the triple  $(h, k, N)$ .

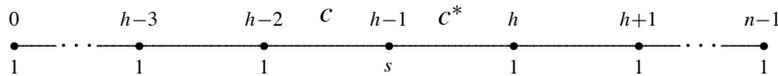
As in this paper we are only interested in locally finite flag-transitive geometries, we put a finite order  $q$  at all nodes of diagram  $D_N^{h,k}$ , calling  $q$  the order of  $D_N^{h,k}$ .

**2.1. Truncations**

Let  $\Delta$  be a building belonging to  $D_N^{h,k}$  with order  $q$ , possibly  $q = 1$ . Assuming that  $h, k \geq 2$  and  $h + k \leq N$ , we denote by  $\text{Tr}_\uparrow(\Delta)$  the  $\{h + k - 1, h + k, \dots, N - 1\}$ -truncation of  $\Delta$ . If  $q > 1$  then  $\text{Tr}_\uparrow(\Delta)$  belongs to the following diagram, which we will denote by  $\text{Tr}_\uparrow(D_N^{h,k})_q$ :



( $s = q^{d-1} + \dots + q^2 + q$  and  $d = N - h - k + 3$ ). In particular, for  $h = k = 2$ ,  $\text{Tr}_\uparrow(\Delta)$  is a  $PG.PG^*$ -geometry. If  $q = 1$ , then  $\text{Tr}_\uparrow(\Delta)$  belongs to the following diagram  $\text{Tr}_\uparrow(D_N^{h,k})_1$ , where  $s = N - h - k + 2$ :



In particular, when  $h = k = 2$ ,  $\text{Tr}_\uparrow(\Delta)$  is a  $c.c^*$ -geometry. In any case,  $\text{Tr}_\uparrow(\Delta)$  is flag-transitive. Furthermore, by the 2-simple connectedness of buildings and [28], Theorem 7.19,

**Proposition 2.1.**  $\text{Tr}_\uparrow(\Delta)$  is 2-simply connected.

By exploiting a lemma of Brouwer and Cohen [14], Lemma 5, on automorphisms of regular graphs, one can also prove the following:

**Proposition 2.2.** If  $D_N^{h,k}$  is spherical and  $q > 1$ , then  $\text{Tr}_\uparrow(\Delta)$  does not admit any proper 2-quotient.

Proper flag-transitive 2-quotients exist in all remaining cases.

**Proposition 2.3.** Let  $\Gamma$  be a geometry for diagram  $\text{Tr}_\uparrow(D_N^{h,k})_q$ , where  $h > 2$  or  $k > 2$ . Put  $m := N - h - k + 4$  and suppose that every  $\{h - 2, h - 1, h\}$ -residue  $\Phi$  of  $\Gamma$  is 2-covered by  $\text{Tr}_\uparrow(\Psi)$  for a  $D_m^{2,2}$ -building  $\Psi$  (when  $q > 1$ ,  $\Phi \cong \text{Tr}_\uparrow(\Psi)$  by Proposition 2.2). Then  $\Gamma$  is 2-covered by  $\text{Tr}_\uparrow(\Delta)$  for a  $D_N^{h,k}$ -building  $\Delta$ . Moreover, if  $q > 1$  and  $D_N^{h,k}$  is spherical, then  $\Gamma \cong \text{Tr}_\uparrow(\Delta)$ .

*Proof.* We can define a  $D_N^{h,k}$ -sheaf  $\mathbf{S}$  for the chamber system of the  $(h - 1)$ -truncation of  $\Gamma$ , as in [61, Theorem 5.1]. As in the proof of [61], Theorem 5.1, one can show that all rank 3 residues of the completion  $\mathcal{C}(\mathbf{S})$  of  $\mathbf{S}$  are 2-covered by buildings. Hence, by a celebrated theorem of Tits [72], the chamber system  $\mathcal{C}(\mathbf{S})$  is 2-covered by the chamber system  $\mathcal{C}(\Delta)$  of a  $D_N^{h,k}$ -building  $\Delta$ . However, the  $\{h + k - 1, h + k, \dots, N - 1\}$ -truncation of  $\mathcal{C}(\mathbf{S})$  is isomorphic to the chamber system of  $\Gamma$ . Hence  $\Gamma$  is 2-covered by  $\text{Tr}_\uparrow(\Delta)$ . The last claim follows from Proposition 2.2.  $\square$

Finally, let  $h = k = 2$ . It is well known that every  $D_N^{2,2}$  building satisfies the Intersection Property IP and both triangular properties  $T_{0,1}^2$  and  $T_{2,1}^0$ . Conversely,

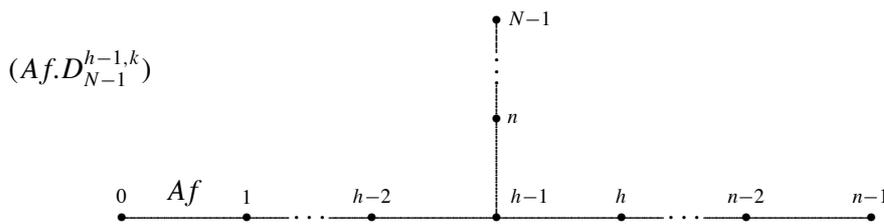
**Proposition 2.4.** (Baumeister and Pasini [11]) *Let  $\Gamma$  be a locally finite  $L.L^*$ -geometry satisfying IP,  $T_{0,1}^2$  and  $T_{2,1}^0$ . Then  $\Gamma$  is 2-covered by  $\text{Tr}_\uparrow(\Delta)$  for a building  $\Delta$  of type  $D_N^{2,2} = D_N$ . In particular, if  $\Gamma$  is thick, then  $\Gamma \cong \text{Tr}_\uparrow(\Delta)$ .*

**2.2. Removing a hyperplane**

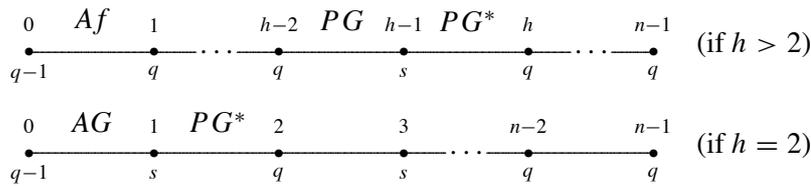
Let  $\Delta$  and  $q$  be as in the previous subsection, but assume  $q > 1$ . Let  $H$  be a geometric hyperplane of the  $(0, 1)$ -space  $\mathcal{S} := \mathcal{S}_{0,1}(\Delta)$  of  $\Delta$ , namely a proper subspace of  $\mathcal{S}$  meeting every line of  $\mathcal{S}$  non-trivially. (Proper subspaces of  $\mathcal{S}$  with these properties always exist when  $D_N^{h,k}$  is spherical). The complement  $\Delta \setminus H$  of  $H$  in  $\Delta$  is the substructure of  $\Delta$  as follows: The complement  $\mathcal{S} \setminus H$  of  $H$  in  $\mathcal{S}$  is taken as set of 0-elements for  $\Delta \setminus H$  and, for  $i > 0$ , the  $i$ -elements of  $\Delta \setminus H$  are the  $i$ -elements of  $\Delta$  that are incident to some element of  $\mathcal{S} \setminus H$ . Two elements  $x, y$  of  $\Delta \setminus H$  are declared to be incident when they are incident in  $\Delta$  and the flag  $\{x, y\}$  is incident to an element of  $\mathcal{S} \setminus H$ . Suppose the following:

- (\*)  $\mathcal{S}$  induces a connected point-line geometry on  $\mathcal{S} \setminus H$  and on  $(\mathcal{S} \setminus H) \cap \text{Res}_\Delta(F)$ , for every flag  $F$  of  $\Gamma$  with  $t(F) \cap \{0, 1\} = \emptyset$ .

(For instance, this happens when  $D_N^{h,k} = D_N$ ; see Shult [67], [68].) Then  $\Delta \setminus H$  is residually connected, hence it is a geometry. It belongs to the following diagram of rank  $N$ :



The  $\{h+k-1, h+k, \dots, N-1\}$ -truncation  $\text{Tr}_\uparrow(\Delta \setminus H)$  of  $\Delta \setminus H$  belongs to one of the following diagrams, where  $s = q^{d-1} + \dots + q^2 + q$ ,  $d = N - h - k + 3$ :



We will denote the above diagrams by  $\text{Tr}_\uparrow(Af.D_{N-1}^{h-1,k})$ . In particular, if  $h = k = 2$  then  $\text{Tr}_\uparrow(\Delta \setminus H)$  is an  $AG.PG^*$ -geometry. Note that  $\text{Tr}_\uparrow(\Delta \setminus H)$  is not flag-transitive, in general.

**Proposition 2.5.** *Suppose that  $D_N^{h,k}$  is  $D_N$ , namely either*

- (1)  $N = h + k$  with  $k = 2$ , or
- (2)  $N = h + k$ ,  $k > 2$  but  $h = 2$ , or
- (3)  $N > h + k$  but  $h = k = 2$ .

*In case (1) the geometry  $\text{Tr}_\uparrow(\Delta \setminus H)$  is flag-transitive. In the remaining two cases,  $\text{Tr}_\uparrow(\Delta \setminus H)$  is flag-transitive if and only if  $H$  is the set of points of  $\mathcal{S}$  at non-maximal distance from a given element  $p$  of  $\Delta$  where  $p$  is a 0-element if  $N$  is even, an  $(N - 1)$ -element in case (2) with  $N$  odd, and a 2-element in case (3) with  $N$  odd.*

*Proof.* In case (1),  $\Delta \setminus H$  is an affine polar space. Affine polar spaces are well known to be flag-transitive. We refer to [62, Proposition 3.6] for cases (2) and (3). □

**Notation.** When  $H$  is the set of points of  $\mathcal{S}$  at non-maximal distance from an element  $p$  as in Proposition 2.5, then the elements of  $\Delta \setminus H$  are precisely the elements of  $\Delta$  that, compatibly with their type, have maximal distance from  $p$  in the incidence graph of  $\Delta$ . This subgeometry is also denoted by  $\text{Far}_\Delta(p)$  and it is called the subgeometry of  $\Delta$  far from  $p$ .

**Proposition 2.6.** *Let  $\Delta \setminus H$  be as in Proposition 2.5, with  $\Delta \setminus H = \text{Far}_\Delta(p)$  in cases (2) and (3). Then  $\text{Tr}_\uparrow(\Delta \setminus H)$  is 2-simply connected.*

*Proof.* By the 2-simple connectedness of buildings and affine geometries and [28], Theorem 7.19,  $\text{Tr}_\uparrow(\Delta \setminus H)$  is 2-simply connected precisely when  $\Delta \setminus H$  is simply connected. The latter is indeed 2-simply connected (see [56], Proposition 12.51, for case (1) of Proposition 2.5 and [60] for the other two cases). □

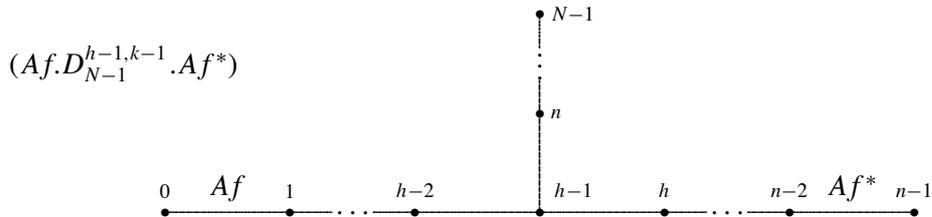
The following is straightforward:

**Proposition 2.7.** *Let  $\Delta \setminus H$  be as in Proposition 2.6. Then  $\text{Tr}_\uparrow(\Delta \setminus H)$  does not admit any flag-transitive proper 2-quotient.*

We are not aware of any example for a spherical  $D_N^{h,k}$  where (\*) fails to hold for some hyperplane  $H$  of  $\mathcal{S}$ . On the other hand, when  $D_N^{h,k}$  is non-spherical, I do not even know if  $\mathcal{S}$  admits any hyperplane at all. However, suppose that  $H$  is a hyperplane of  $\mathcal{S}$  such that (\*) fails to hold in  $\Delta \setminus H$ . Then we can do as follows: Let  $\mathcal{C}(\Delta \setminus H)$  be the graph induced by the chamber system  $\mathcal{C}(\Delta)$  of  $\Delta$  on the set of chambers of  $\Delta$  that are contained in  $\Delta \setminus H$ . As  $\Delta \setminus H$  is supposed not to be a geometry,  $\mathcal{C}(\Delta \setminus H)$  is disconnected. However, as the chambers and the cells of a connected component  $\mathcal{X}$  of  $\mathcal{C}(\Delta \setminus H)$  are chambers and residues of flags of  $\Delta$ , the incidence structure  $\Xi = \Gamma(\mathcal{X})$  associated to  $\mathcal{X}$  is a geometry and belongs to  $Af.D_{N-1}^{h-1,k}$ . Its truncation  $\text{Tr}_\uparrow(\Xi)$  belongs to  $\text{Tr}_\uparrow(Af.D_{N-1}^{h-1,k})$ .

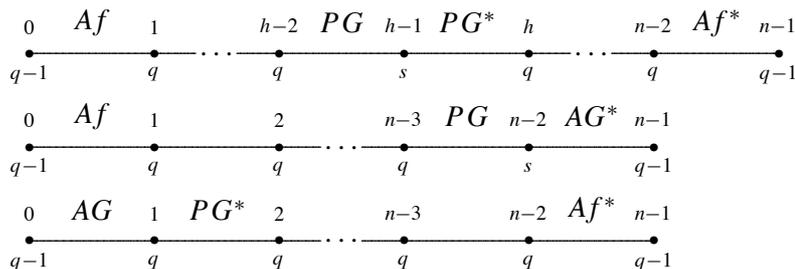
**2.3. Removing two hyperplanes**

Take  $D_N^{h,k}$ ,  $\Delta$  and  $q$  as in Subsection 2.2, with the same restrictions on the triple  $(h, k, N)$ . With  $\mathcal{S}$  and  $H$  as in that subsection, let  $H^*$  be a geometric hyperplane of  $\mathcal{S}^* := \mathcal{S}_{n-1,n-2}(\Delta)$  and put  $\mathcal{H} := \{H, H^*\}$  and  $\Delta \setminus \mathcal{H} := (\Delta \setminus H) \cap (\Delta \setminus H^*)$ . The structure  $\Delta \setminus \mathcal{H}$  might be disconnected even if  $\Delta \setminus H$  and  $\Delta \setminus H^*$  were residually connected. However, we can apply a trick similar to that described at the end of Subsection 2.2: Denoting by  $\mathcal{C}(\Delta \setminus \mathcal{H})$  the graph induced by  $\mathcal{C}(\Delta)$  on the set of chambers of  $\Delta$  contained in  $\Delta \setminus \mathcal{H}$ , the incidence structure  $\Xi = \Gamma(\mathcal{X})$  associated to a connected component  $\mathcal{X}$  of  $\mathcal{C}(\Delta \setminus \mathcal{H})$  is a geometry and belongs to the following diagram:



When  $(h, k) \neq (2, 2)$ , the diagram of  $\text{Tr}_\uparrow(\Xi)$  is as follows, according to whether

$h, k > 2, h > 2 = k$  or  $h = 2 < k$ :



We will denote these diagram by  $\text{Tr}_\uparrow(Af.D_{N-2}^{h-1,k-1}.Af^*)$ . When  $h = k = 2$ ,  $\text{Tr}_\uparrow(\Xi)$  is an  $AG.AG^*$ -geometry (a  $c.c^*$ -geometry when  $q = 2$ ). A special case of the latter situation is considered in the next proposition:

**Proposition 2.8.** *Let  $h = k = 2$  (so,  $D_N^{h,k} = D_N$ ) and, given a  $\{0, 2\}$ -flag  $\{p, p^*\}$  of  $\Delta$ , consider the following hyperplanes  $H$  and  $H^*$  of  $\mathcal{S}$  and  $\mathcal{S}^*$ :*

- (1) *if  $N$  is even,  $H$  is the set of 0-elements at maximal distance from  $p$  and  $H^*$  is the set of 2-elements at maximal distance from  $p^*$ ;*
- (2) *if  $N$  is odd,  $H$  is the set of 0-elements at maximal distance from  $p^*$  and  $H^*$  is the set of 2-elements at maximal distance from  $p$ .*

*Put  $\mathcal{H} = \{H, H^*\}$ . Then  $\Delta \setminus \mathcal{H}$  is residually connected and  $\text{Tr}_\uparrow(\Delta \setminus \mathcal{H})$  is flag-transitive. If  $q > 2$ , then  $\text{Tr}_\uparrow(\Delta \setminus \mathcal{H})$  is 2-simply connected. If  $q = 2$ , then the universal 2-cover of  $\text{Tr}_\uparrow(\Delta \setminus \mathcal{H})$  is  $\text{Tr}_\uparrow(\mathcal{C})$ , where  $\mathcal{C}$  is the Coxeter complex of type  $D_M^{h,k} = D_M$ ,  $M = 2^N$ .*

*Proof.* The residual connectedness of  $\Delta \setminus \mathcal{H}$  follows from Blok and Brouwer [13]. The flag-transitivity of  $\text{Tr}_\uparrow(\Delta \setminus \mathcal{H})$  is straightforward. As for the rest, see Baumeister, Meixner and Pasini [7] and Baumeister and Stroth [12] (also Pasini [60]). □

Flag-transitive proper 2-quotients of  $\text{Tr}_\uparrow(\Delta \setminus \mathcal{H})$  exist even in the hypotheses of the above proposition. Minimal ones are obtained by factorizing by the elementwise stabilizer of  $\text{Res}_\Delta(p, p^*)$  in  $\text{Aut}(\Delta)$ .

**Notation.** For  $H, H^*, p, p^*$  and  $\mathcal{H}$  as in Proposition 2.8,  $\Delta \setminus \mathcal{H}$  is the geometry formed the elements of  $\Delta$  at maximal distance from the flag  $\{p, p^*\}$ . This geometry is also denoted by  $\text{Far}_\Delta(p, p^*)$ .

**2.4. Biaffine geometries and their quotients**

Biaffine geometries arise from the same construction of the previous subsection, but with  $N = n = h + k - 1$ . So,  $D_N^{h,k}$  is the Coxeter diagram  $A_n$

and  $\Delta = PG(n, q)$ . The hyperplanes of  $\mathcal{S}$  are the  $(n - 1)$ -elements of  $\Delta$  and every hyperplane of  $\mathcal{S}^*$  can be regarded as the set  $H(p)$  of  $(n - 1)$ -elements incident to a given point  $p$  of  $\mathcal{S}$ . Let  $\mathcal{H} = \{H, H^*\}$  where  $H$  is a hyperplane of  $\mathcal{S}$  and  $H^* = H(p)$ . Then  $\Delta \setminus \mathcal{H}$  is a flag-transitive geometry belonging to the following diagram, usually denoted by  $Af.A_{n-2}.Af^*$ :



In particular, when  $n = 3$ ,  $\Delta \setminus \mathcal{H}$  is an  $Af.Af^*$ -geometry. We call  $\Delta \setminus \mathcal{H}$  a *biaffine geometry* of order  $q$  and rank  $n$ . If  $p \in H$  (equivalently,  $H \in H^*$ ) we say that  $\Delta \setminus \mathcal{H}$  is of *incident type*. Otherwise,  $\Delta \setminus \mathcal{H}$  is of *non-incident type*. The elements of  $\Delta \setminus \mathcal{H}$  are precisely the elements of  $\Delta$  that, compatibly with their type, have maximal distance from both  $p$  and  $H$ . Accordingly, we will denote  $\Delta \setminus \mathcal{H}$  by  $\text{Far}_\Delta(p, H)$ .

Note that, when  $p \in H$  (case of incident type) the geometry  $\text{Far}_\Delta(p, H)$  can also be obtained as an affine expansion of the dual of  $AG(n - 1, q)$  (see Subsection 3.3).

Flag-transitive proper 2-quotients of  $\text{Far}_\Delta(p, H)$  also exist, except when  $q = 2$  and  $p \notin H$ . Minimal ones are obtained by factorizing by the group  $Z$  of homologies (if  $p \notin H$ ) or elations (when  $p \in H$ ) with  $p$  as the center and  $H$  as the axis. When  $p \in H$  then  $\text{Far}_\Delta(p, H)/Z$  is isomorphic to a (twisted) gluing of two copies of  $AG(n - 1, q)$  (Buekenhout, Huybrechts and Pasini [21], 6.1; also Del Fra, Pasini and Shpectorov [37]). In particular, when  $p \in H$  and  $n = 3$  then  $\text{Far}_\Delta(p, H)/Z$  is the canonical gluing of two copies of  $AG(2, q)$  (see Subsection 4.3). On the other hand, let  $p \notin H$ . Then  $\text{Far}_\Delta(p, H)/Z$  can be recovered as follows from  $\Delta_0 := H \cong PG(n - 1, q)$  (Del Fra, Pasini and Shpectorov [37]): for  $i = 0, 1, \dots, n - 1$ , the  $i$ -elements are the pairs  $(X, Y)$  where  $X$  and  $Y$  are subspaces of  $\Delta_0$  of respective dimension  $i - 1$  and  $i$  and  $X \subset Y$ , with the convention that  $\dim(\emptyset) = -1$ ; two elements  $(X, Y)$  and  $(X', Y')$  of type  $i$  and  $j$  respectively, with  $i < j$ , are incident if and only if  $X \subset X', Y \subset Y'$  but  $Y \not\subset X'$ . In particular, a 0-element  $(\emptyset, x)$  and an  $(n - 1)$ -elements  $(X, \Delta_0)$  are incident if and only if  $x \notin X$ .

**Proposition 2.9.** *Let  $\Gamma$  be a geometry of rank  $n \geq 3$  belonging to diagram  $Af.A_{n-2}.Af^*$ . If  $n > 3$  then  $\Gamma$  is a (possibly non-proper) quotient of a biaffine geometry. In any case,  $\Gamma$  is biaffine if and only if it satisfies the Intersection Property IP (which, in this contexts, amount to say that the  $(0, 1)$ -space  $\mathcal{S}_{0,1}(\Gamma)$  is semilinear).*

The first claim of the above proposition has been proved by Del Fra, Pasini and Shpectorov [37], the second one by Lefevre and Van Nypelseer [52], [73]. Not so much is known on  $Af.Af^*$ -geometries without IP. The reader is referred to Del Fra and Pasini [36] for some partial results on that case. The next corollary is a trivial consequence of Proposition 2.9:

**Corollary 2.10.** *All biaffine geometries are 2-simply connected.*

### 3. $L_h.L_k^*$ -geometries obtained as affine expansions.

We will firstly define affine expansions in general. Next, we will show how they can be exploited to produce  $L_h.L_k^*$ -geometries.

#### 3.1. Definitions

We define affine expansions in a setting as general as we need in this paper (but see Buekenhout, Huybrechts and Pasini [21], Section 4, for a more abstract approach.) In the sequel  $\mathcal{D}$  is a diagram of rank  $N \geq 2$ , 0 is a distinguished type and  $\Sigma$  is a geometry belonging to  $\mathcal{D}$ . We assume that  $\Sigma$  satisfies the Intersection Property IP with respect to 0. We firstly suppose that  $\mathcal{D}$  is a string with 0 as its leftmost node, next we will consider a more general situation.

**Case 1.** Assume that  $\mathcal{D}$  is a string, with types 0, 1, ...,  $N - 1$  labelling the nodes of  $\mathcal{D}$  from left to right. Thus, 0 corresponds to the leftmost node of  $\mathcal{D}$  and 1 to the node next to it. We denote the set of 0-elements of  $\Sigma$  by  $P$  and, for an element  $x$  of  $\Sigma$ , we denote by  $P(x)$  the set of 0-elements incident to  $x$ . For a vector space  $V$ , suppose that an injective mapping  $e$  is given from  $P$  to the set of points of the projective geometry  $PG(V)$  of linear subspaces of  $V$ , in such a way that:

- (E1)  $e(P)$  spans  $PG(V)$ ;
- (E2) for a given  $d > 0$  and every 1-element  $x$  of  $\Sigma$ ,  $e(P(x))$  is the full point-set of a  $d$ -dimensional projective subspace  $e(x)$  of  $PG(V)$ .

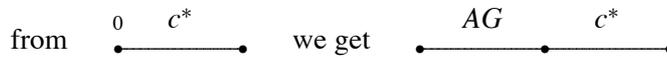
In particular, if  $d = 1$  then  $e$  is a full projective embedding of the point-line geometry  $\mathcal{S}_{0,1}(\Sigma)$ . Accordingly, we call  $e$  a *full  $d$ -projective embedding* of  $\Sigma$ . We can extend  $e$  to a mapping from the whole of  $\Sigma$  to the set of projective subspaces of  $PG(V)$  by sending every element  $x$  to the span  $e(x) := \langle e(P(x)) \rangle$  of  $e(P(x))$  in  $PG(V)$ . When  $N > 2$  we also assume this:

- (E3) For  $p \in P$  and an element  $x$  of  $\Sigma$  of type  $t(x) > 0$ , we have  $e(p) \in e(x)$  only if  $p \in P(x)$ .

By (E3) on  $e$  and IP on  $\Sigma$  we immediately obtain the following:

- (E4) For two elements  $x, y$  of  $\Sigma$  of type  $t(x), t(y) > 0$ , we have  $e(x) \subseteq e(y)$  if and only if  $x$  and  $y$  are incident and  $t(x) \leq t(y)$ . In particular,  $e(x) = e(y)$  only if  $x = y$ .

We can now define the *affine expansion* of  $\Sigma$  by  $e$ . We shall denote it by  $Af_e(\Sigma)$  (also by  $Af(\Sigma)$ , when the embedding  $e$  is implicit in the definition of  $\Sigma$ ). We take  $\{0, 1, \dots, N\}$  as type-set for  $Af_e(\Sigma)$ . The 0-elements of  $Af_e(\Sigma)$  are the points of the affine geometry  $AG(V)$ . Regarding  $PG(V)$  as the geometry at infinity of  $AG(V)$ , the 1-elements of  $Af_e(\Sigma)$  are the lines  $L$  of  $AG(V)$  with point at infinity  $L^\infty \in e(P)$ . For  $i > 1$ , the  $i$ -elements of  $Af_e(\Sigma)$  are the affine subspaces  $X$  of  $AG(V)$  with space at infinity  $X^\infty = e(x)$  for an  $(i - 1)$ -element  $x$  of  $\Sigma$ . The incidence relation is the natural one, namely inclusion. Exploiting (E1), (E3) and (E4), it is easy to prove that  $Af_e(\Sigma)$  is indeed a geometry (namely, it is residually connected) and that the residues of its 0-elements are isomorphic to  $\Sigma$ . In view of (E2), the residues of the 2-elements of  $Af_e(\Sigma)$  are  $(d + 1)$ -dimensional affine spaces. Thus, a diagram for  $Af_e(\Sigma)$  can be obtained by attaching a stroke labelled by  $AG$  to  $\mathcal{D}$ . For instance,



Let  $G_e$  be the setwise stabilizer in  $\Gamma L(V)$  of the image  $e(\Sigma)$  of  $\Sigma$  and  $T$  be the translation group of  $AG(V)$ . Then the subgroup  $TG_e$  of  $A\Gamma L(V)$  is contained in  $\text{Aut}(Af_e(\Sigma))$ . If  $G_e$  acts flag-transitively on  $e(\Sigma)$ , then  $TG_e$  is flag-transitive on  $Af_e(\Sigma)$ .

**Case 2.** Suppose that  $\mathcal{D}$  has the shape of a tree. Note that  $\mathcal{D}$  might still be a string, but now we do not assume that 0 is its leftmost node. Let  $\text{Sh}_0(\Sigma)$  be the 0-shadow geometry of  $\Sigma$  (called 0-Grassmann geometry in [56]). We recall that the elements of  $\text{Sh}_0(\Sigma)$  are the elements of  $\Sigma$  and the flags of  $\Sigma$  of 0-reduced type. In particular,  $\text{Sh}_0(\Sigma)$  and  $\Sigma$  have the same 0-elements, but the 1-elements of  $\text{Sh}_0(\Sigma)$  are the flags of  $\Sigma$  of type  $\mathcal{D}(0)$ , where  $\mathcal{D}(0)$  is the set of types joined to 0 in  $\mathcal{D}$ . We refer to [56, Chapter 5] for more details. We only recall that  $\text{Sh}_0(\Sigma)$  inherits IP from  $\Sigma$  and belongs to a string diagram with 0 as the leftmost node.

Suppose that a full  $d$ -projective embedding  $e$  satisfying (E3) exists for  $\text{Sh}_0(\Sigma)$ , as in Case 1. Then we can consider the affine expansion  $Af_e(\text{Sh}_0(\Sigma))$ . Suppose furthermore that  $\Sigma$  is thick at all types but possibly those that are end-nodes of the tree  $\mathcal{D}$ . Then, as  $\mathcal{D}$  is a tree, we can distinguish the elements of  $\text{Sh}_0(\Sigma)$  that are elements of  $\Sigma$  from the other ones, thus recovering  $\Sigma$  from  $\text{Sh}_0(\Sigma)$  (see [57]). In the same way, a geometry  $\Xi$  of rank  $n + 1$  such that

$\text{Sh}_0(\Xi) = \text{Af}_e(\text{Sh}_0(\Sigma))$  can be recovered inside  $\text{Af}_e(\text{Sh}_0(\Sigma))$ . We call  $\Xi$  the *affine expansion* of  $\Sigma$  by  $e$  and we denote it by  $\text{Af}_e(\Sigma)$ . To obtain a diagram for  $\text{Af}_e(\Sigma)$  we must only attach a stroke labelled by  $AG$  to  $\mathcal{D}$  at the node 0. For instance,



The group induced on  $e(\text{Sh}_0(\Sigma))$  by the stabilizer of  $e(\text{Sh}_0(\Sigma))$  in  $\Gamma L(V)$ , regarded as a subgroup of  $\text{Aut}(\text{Sh}_0(\Sigma))$  (as is possible, in view of (E4)) induces on  $\Sigma$  a subgroup of  $\text{Aut}(\Sigma)$ . If that subgroup is flag-transitive, then  $\text{Af}_e(\Sigma)$  is flag-transitive.

**3.2. Affine expansions of  $D_N^{h,k}$ -buildings**

Let  $D_N^{h,k}$ ,  $\Delta$  and  $q$  be as in Subsection 2.2, with the same restrictions on the triple  $(h, k, N)$  (in particular,  $N \geq h+k$ ), but allowing  $h = 1$ . Thus,  $D_N^{h,k}$  might be any of the spherical diagrams  $A_N, D_N$  or  $E_N$ , but 0 is neither the leftmost nor the rightmost node when  $D_N^{h,k} = A_N$ . Let  $e$  be a full 1-projective embedding  $e$  of  $\text{Sh}_0(\Delta)$  satisfying (E3). The affine expansion  $\text{Af}_e(\Delta)$  belongs to diagram  $\text{Af}.D_N^{h,k}$ . Diagrams for  $\text{Tr}_\uparrow(\text{Af}_e(\Delta))$  are as for  $\text{Tr}_\uparrow(\Delta \setminus H)$  in Subsection 2.2. In particular, if  $h = 1$  and  $k = 2$  then  $\text{Tr}_\uparrow(\text{Af}_e(\Delta))$  is an  $AG.PG^*$ -geometry.

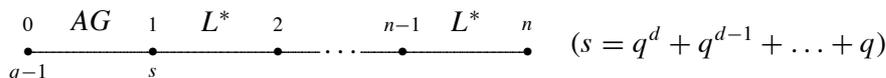
In a few cases,  $\text{Af}_e(\Delta)$  is a (possibly non-proper) quotient of the geometry  $\widehat{\Delta} \setminus H$  obtained by removing a hyperplane  $H$  from a  $D_{N+1}^{h+1,k}$ -building  $\widehat{\Delta}$  (see Subsection 2.2). For instance, this happens when  $k = 2$  and  $N = h + k$ , or  $h = 1$  and either  $N = h + k$  or  $k = 2$ . However, the diagram  $D_{N+1}^{h+1,k}$  is non-spherical for most choices of  $h, k$  and  $N$ . In those cases, it is very unlikely that  $\text{Af}_e(\Delta)$  can be obtained from a  $D_{N+1}^{h+1,k}$ -building as said above.

A variation of the previous construction can also be considered. Let  $\Sigma$  belong to diagram  $\text{Af}.D_{N-1}^{h-1,k}$  of Subsection 2.2, but allow  $k = 1$  and take  $n - 1$  instead of 0 as distinguished type for the expansion. Suppose that  $\text{Sh}_{n-1}(\Sigma)$  admits a full 1-projective embedding  $e$  satisfying (E3). (For instance, if  $\Sigma = \Delta \setminus H$  as in Subsection 2.2,  $e$  might be induced by an embedding of  $\text{Sh}_{n-1}(\Delta)$ .) Then  $\text{Af}_e(\Sigma)$  belongs to diagram  $\text{Af}.D_{N-1}^{h-1,k}.Af^*$ . Diagrams for  $\text{Tr}_\uparrow(\text{Af}_e(\Sigma))$  are as for the geometry  $\text{Tr}_\uparrow(\Xi)$  of Subsection 2.3. In particular, if  $h = 1$  and  $k = 2$  then  $\text{Tr}_\uparrow(\text{Af}_e(\Sigma))$  is an  $AG.AG^*$ -geometry.

**3.3. Affine expansions of  $L_n^*$ -geometries. Generalities**

In this and the next four subsections  $\Sigma$  is the dual of an  $L_n$ -geometry and  $e : \Sigma \rightarrow PG(m, q)$  a full  $d$ -projective embedding of  $\Sigma$ . Accordingly, the affine

expansion  $Af_e(\Sigma)$  is an  $L_2.L_n^*$ -geometry with  $\{0, 1\}$ -residues isomorphic to the affine space  $AS(d + 1, q)$  and  $\{1, 2, \dots, n\}$ -residues isomorphic to  $\Sigma$ :



For instance, let  $\Sigma$  be the dual of  $AG(n, q)$  and  $e$  be its natural embedding in  $PG(n, q)$ . Then  $Af_e(\Sigma)$  is isomorphic to a biaffine geometry of incident type and rank  $n + 1$ . Examples less trivial than this will be considered in the next subsections.

**3.4. Affine expansions of dimensional dual hyperovals**

Let  $\Delta$  be a circular space, with point-set  $\mathcal{L}$  and line-set  $P$ , and  $\Sigma = (P, \mathcal{L})$  be its dual. Let  $e$  be a full  $d$ -embedding of  $\Sigma$  in  $PG(m, q)$ . As the 1-elements of  $\Sigma$  are sent by  $e$  to  $d$ -dimensional subspaces of  $PG(m, q)$ , condition (E2) forces  $|P(x)| = q^d + \dots + q + 1$  for every 1-element  $x \in \mathcal{L}$  of  $\Sigma$ . Hence  $|\mathcal{L}| = q^d + \dots + q + 2$  and  $e(\mathcal{L})$  is a  $d$ -dimensional dual hyperoval in the sense of Huybrechts and Pasini [48], namely:

- (1) any two members of  $e(\mathcal{S})$  meet in exactly one point,
- (2) every point of  $PG(m, q)$  belongs to either 2 or 0 members of  $e(\mathcal{S})$ .

We have  $2d \leq m$ , by (1). In particular, if  $m = 2$  then  $d = 1$  and  $e(\mathcal{L})$  is a dual hyperoval of  $PG(2, q)$  in the usual sense. In that case,  $q$  is even. We refer the interested reader to the following papers for more information on dimensional dual hyperovals: Cooperstein and Thas [29], Del Fra [34], Huybrechts [45], Pasini and Yoshiara [65], [66], Taniguchi [71], Yoshiara [74], [75], Buratti and Del Fra [24], Del Fra and Yoshiara [38].

The expansion  $Af_e(\Sigma)$  belongs to diagram  $AG.c^*$ , with  $\{0, 1\}$ -residues isomorphic to  $AG(d + 1, q)$ . In particular, when  $m = 2$  the dual of  $Af_e(\Sigma)$  is a special Laguerre plane. When  $q = 2$  then  $Af_e(\Sigma)$  is a semiplane. The following is sufficient for  $Af_e(\Sigma)$  to be flag-transitive:

- (3) the stabilizer  $G_e$  of  $e(\mathcal{L})$  in  $P\Gamma L_{m+1}(q)$  acts two-transitively on  $e(\mathcal{L})$ .

Condition (3) is fairly easy to satisfy when  $q = 2$  (see Yoshiara [74] for a large family of examples; also Pasini and Yoshiara [65]). When  $q > 2$ , only two examples are known where (3) holds. We discuss them in the next two paragraphs.

**Example 3.1.** [The special Laguerre plane of order 4]. Let  $\mathcal{L}$  be the dual hyperoval of  $PG(2, 4)$ ,  $P$  be the set of points contained in lines of  $\mathcal{L}$  and put

$\Sigma := (P, \mathcal{L})$ . Then  $\mathcal{L}$  satisfies (3) and  $\text{Af}(\Sigma)$  is the dual of the classical special Laguerre plane of order 4, hence it is an  $\text{Af.c}^*$ -geometry with orders (3, 4, 1):

$$(\text{Af.c}^*) \quad \begin{array}{ccccccc} & 0 & & \text{Af} & & 1 & & \text{c}^* & & 2 \\ & \bullet & & \text{---} & & \bullet & & \text{---} & & \bullet \\ 3 & & & & & 4 & & & & 1 \end{array}$$

Clearly,  $G := \text{Aut}(\text{Af}(\Sigma)) = 2^6 : 3 \cdot \text{Sym}(6)$ . The geometry  $\text{Af}(\Sigma)$  admits two flag-transitive proper quotients  $\text{Af}(\Sigma)/X$  and  $\text{Af}(\Sigma)/Y$ , where  $Y < X < G$ ,  $|X| = 2$  with  $N_G(X)/X = 2^5 : \text{Sym}(5)$  and  $|Y| = 2^2$  with  $N_G(Y)/Y = 2^4 : (Z_3 \times \text{Sym}(5))$  (see Baumeister et al. [6]). We will denote  $\text{Af}(\Sigma)$ ,  $\text{Af}(\Sigma)/X$  and  $\text{Af}(\Sigma)/Y$  by  $\Gamma_L(2^6 3 S_6)$ ,  $\Gamma_L(2^5 S_5)$  and  $\Gamma_L(2^4 3 S_5)$ , respectively. (The index  $L$  is a reminder for ‘Laguerre’ and we write  $S_5$  and  $S_6$  instead of  $\text{Sym}(6)$  and  $\text{Sym}(5)$  to make our notation less clumsy.)

**Example 3.2.** [An exceptional family of 22 planes of  $PG(5, 4)$ ]. It is well known (Conway et al. [30], page 39) that  $PG(5, 4)$  contains a 2-dimensional dual hyperoval  $\mathcal{L}$  with the following properties: The 22 planes of  $\mathcal{L}$  are totally singular for a given non-singular hermitian form; the stabilizer of  $\mathcal{L}$  in  $\Gamma L_6(4)$  is contained in  $\Gamma U_6(2^2)$ , is isomorphic to the non-split extension  $3 \cdot \text{Aut}(M_{22})$  and acts on  $\mathcal{L}$  just as  $\text{Aut}(M_{22})$  does on the 22 points of the Steiner system  $S(22, 6, 3)$ . The kernel of that action is the center of  $GL_6(4)$ .

Denoting by  $P$  the set of points of  $PG(4, 4)$  contained in members of  $\mathcal{L}$ , we can consider the affine expansion  $\text{Af}(\Sigma)$  of the dual circular space  $\Sigma = (P, \mathcal{L})$ . The geometry  $\text{Af}(\Sigma)$  is flag-transitive with  $\text{Aut}(\text{Af}(\Sigma)) = 2^{12} : 3 \cdot \text{Aut}(M_{22})$ , but  $2^{12} : 3 \cdot M_{22}$  also acts flag-transitively on it. We shall denote this geometry by  $\Gamma(2^{12} 3 M_{22})$ . It has diagram and orders as follows:

$$(\text{AG.c}^*) \quad \begin{array}{ccccccc} & 0 & & \text{AG} & & 1 & & \text{c}^* & & 2 \\ & \bullet & & \text{---} & & \bullet & & \text{---} & & \bullet \\ 3 & & & & & 20 & & & & 1 \end{array}$$

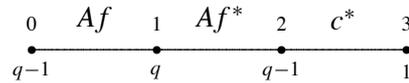
The geometry  $\Gamma(2^{12} 3 M_{22})$  is not simply connected. Its universal cover, say  $\tilde{\Gamma}$ , is a double cover and can be constructed inside the affine polar space obtained by removing a singular hyperplane from the non-singular hermitian variety  $\mathcal{H}(7, 2^2)$  of  $PG(7, 4)$ . (We refer to Hybrechts and Pasini [48] for the details of that construction). We have  $\text{Aut}(\tilde{\Gamma}) = 2_+^{1+12} : 3 \cdot \text{Aut}(M_{22})$  with center  $Z = Z_2$ , and  $\tilde{\Gamma}/Z = \Gamma(2^{12} 3 M_{22}) = \tilde{\Gamma}/Z$ . (We warn that this group was negligently described as  $2^{13} : 3 \cdot \text{Aut}(M_{22})$  in [48]). Henceforth we shall denote  $\tilde{\Gamma}$  by  $\Gamma(2^{13} 3 M_{22})$ .

**3.5. Affine expansions of dual Witt spaces**

Given a hyperoval  $O$  of the projective plane  $PG(2, q)$ ,  $q$  even, let  $\Sigma$  be the dual Witt space formed by the points and the lines of  $PG(2, q)$  exterior to  $O$ . Then  $Af(\Sigma)$  is an  $Af.W^*$ -geometry. If  $O$  is classical, then its stabilizer  $G_O$  in  $P\Gamma L_3(q)$  acts flag-transitively on  $\Sigma$  and  $Af(\Sigma)$  is flag-transitive. Moreover, denoting by  $p_0$  the nucleus of  $O$  and regarding the linear subspace  $T_0$  of  $V(3, q)$  corresponding to  $p_0$  as a group of translations of  $AG(3, q)$ ,  $T_0$  defines a flag-transitive quotient of  $Af(\Sigma)$ , which is flat. When  $q = 4$ , then  $Af(\Sigma)$  is an  $Af.c^*$ -geometry, isomorphic to the dual of the special Laguerre plane of order 4 (Example 3.1).

**3.6. Affine expansions of dual Möbius planes**

Let  $\Delta$  be an embeddable Möbius plane of order  $q$ , realized as an  $L_3$ -subgeometry of  $PG(3, q)$ , and  $\Sigma$  be its dual. Namely, the 2-elements of  $\Sigma$  (points of  $\Delta$ ) form an ovoid  $O$  of  $PG(3, q)$ , the 1-elements of  $\Sigma$  are the pairs of points of  $O$  and the 0-elements of  $\Sigma$  (circles of  $\Delta$ ) are the planes of  $PG(3, q)$  that meet  $O$  in a conic. By applying a correlation of  $PG(3, q)$  (a polarity, for instance), we obtain a full 1-embedding  $e$  of  $\Sigma$  in  $PG(3, q)$ . The expansion  $Af_e(\Sigma)$  has diagram and orders as follows:



The geometry  $\Gamma = Af_e(\Sigma)$  is flag-transitive if and only if  $O$  is classical. If that is the case, then  $Aut(\Gamma) = V : \Gamma L_2(q^2)$  where  $V$  is the additive group of  $V(4, q)$  and  $\Gamma L_2(q^2)$  is the stabilizer of  $O$  in  $\Gamma L_4(q)$ , acting on  $V$  according to the adjoint action of  $\Gamma L_4(q)$ . The  $\{0, 1, 2\}$ -residues of  $\Gamma$  are biaffine geometries of incident type. When  $q = 2$ ,  $Af_e(\Sigma)$  is a truncation of the Coxeter complex of type  $D_5$ . The case of  $q = 3$  is also interesting. We will turn back to it at the end of the next subsection.

**3.7. A series for  $3^6 2M_{12}$ ,  $3^5 M_{11}$  and  $3^4 \Gamma L_2(9)$**

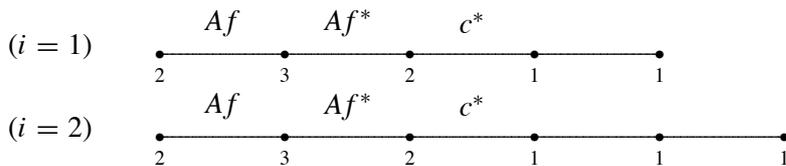
For  $i = 0, 1, 2$ , let  $\Delta_i$  be the the Steiner system  $S(10 + i, 4 + i, 3 + i)$ , regarded as an  $L_{3+i}$ -geometry. So,  $\Delta_0$  is the Möbius plane of order 3,  $\Delta_1$  is the Steiner system for  $M_{11}$  and  $\Delta_2$  that for  $M_{12}$ . As recalled in the previous subsection,  $\Delta_0$  is (laxly) embeddable in  $PG(3, 3)$ . The Steiner systems  $\Delta_1$  and  $\Delta_2$  also admit a lax embedding in  $PG(4, 3)$  and  $PG(5, 3)$  respectively. To show this, we need to recall some properties of the 6-dimensional ternary Golay code  $C_6(3)$  and its dual  $C_6^*(3)$ .

Regarding  $C_6^*(3)$  as a vector space  $V = V(6, 3)$  equipped with a suitable weight function, let  $S$  be the set of 1-dimensional linear subspaces of  $V$  spanned

by vectors that have weight 1 as words of  $\mathcal{C}_6^*(3)$ . Then  $S$  is a set of 12 points of  $PG(5, 3)$  with the following properties: Any 5 of them span a hyperplane of  $PG(5, 3)$  and every hyperplane of  $PG(5, 3)$  meets  $S$  in 6, 3 or 0 points (see [63]). The dual  $V^*$  of  $V$  can be regarded as a copy of  $\mathcal{C}_6(3)$ . More explicitly, let  $B = \{b_p\}_{p \in S}$  be the natural basis of  $\widehat{V} := V(12, 3)$  and pick a representative  $v_p \in V$  for every point  $p \in S$ . Then the codewords of  $\mathcal{C}_6(3)$  are the vectors  $\sum_{p \in S} \langle v_p, v^* \rangle b_p$  for  $v^* \in V^*$  (where the symbol  $\langle \cdot, \cdot \rangle$  stands for scalar product) and we may regard  $V^*$  as a 6-dimensional subspace of  $\widehat{V}$ . A vector  $\sum_{p \in S} \lambda_p b_p$  of  $\widehat{V}$  belongs to  $V^*$  if and only if  $\sum_{p \in S} \lambda_p v_p = 0$  (see [63]). The natural isomorphism  $\widehat{V}/V^* \cong V$  obtained in this way makes it clear that  $\mathcal{C}_6(3)$  is just  $V^*$ , equipped with the weight function relative to  $B$ .

Turning to  $\Delta_2$ , and with  $S$  as above, we can take  $S$  as the set of 0-elements of  $\Delta_2$ . The lines, planes, 3-spaces and hyperplanes of  $PG(5, 3)$  that meet  $S$  in 2, 3, 4 and, respectively, 6 points will be taken as elements of type 1, 2, 3 and 4, respectively. Thus, we obtain a lax 1-embedding  $\delta_2 : \Delta_2 \rightarrow PG(5, 3)$ . Clearly,  $\delta_2$  induces lax 1-embeddings  $\delta_1 : \Delta_1 \rightarrow PG(4, 3)$  and  $\delta_0 : \Delta_0 \rightarrow PG(3, 3)$ , and  $\delta_0$  is the same as considered in Subsection 3.6.

For  $i = 0, 1, 2$ , let  $V_i = V(4 + i, 3)$  be the underlying vector space of the projective space  $PG(V_i) = PG(3 + i, 3)$  in which  $\Delta_i$  is embedded by  $\delta_i$  and let  $V_i^*$  be its dual. (In particular,  $V_2$  and  $V_2^*$  are the spaces previously called  $V$  and  $V^*$ .) let  $\Sigma_i$  be the dual of  $\Delta_i$ . The embedding  $\delta_i : \Delta_i \rightarrow PG(V_i)$  induces a full 1-embedding  $e_i$  of  $\Sigma_i$  in  $PG(V_i^*)$  and we can consider the affine expansion  $Af_{e_i}(\Sigma_i)$ . We have described the diagram of  $Af_{e_0}(\Sigma_0)$  in Example 3.6. For  $i = 1, 2$ , the expansion  $Af_{e_i}(\Sigma_i)$  has diagram and orders as follows:



Clearly,  $Af_{e_0}(\Sigma_0)$  is a residue of  $Af_{e_1}(\Sigma_1)$  and the latter is a residue of  $Af_{e_2}(\Sigma_2)$ . We recall that  $\text{Aut}(Af_{e_0}(\Sigma_0)) = 3^4 : \Gamma L_2(9)$ . Both  $Af_{e_1}(\Sigma_1)$  and  $Af_{e_2}(\Sigma_2)$  are flag-transitive, too, with  $\text{Aut}(Af_{e_1}(\Sigma_1)) = 3^5 : M_{11}$  and  $\text{Aut}(Af_{e_2}(\Sigma_2)) = 3^6 : 2M_{12}$ . In view of this, we will denote  $Af_{e_1}(\Sigma_1)$  and  $Af_{e_2}(\Sigma_2)$  by  $\Gamma(3^5 M_{11})$  and  $\Gamma(3^6 2M_{12})$ , respectively. We also denote  $Af_{e_0}(\Sigma_0)$  by  $\Gamma(3^4 \Gamma L_2(9))$ .

**3.8. Intersecting an affine expansion with its dual**

Suppose that  $\Sigma$  is a geometry with string diagram of rank  $n$  and let  $e$  be a full embedding of  $\Sigma$  in  $\Pi_0 = PG(n, q)$ , satisfying (E3). So,  $e(x)$  is  $i$ -dimensional for every  $i$ -element  $x$  of  $\Sigma$ . Let  $\delta$  be a correlation of  $\Pi :=$

$PG(n + 1, q)$  (e.g., a polarity). Then, regarding  $\Pi_0$  as a hyperplane of  $\Pi$ , the composition  $\delta e$  is a lax embedding of the dual of  $\Sigma$  in the star  $\text{Res}_\Pi(p_0)$  of the point  $p_0 := \delta(\Pi_0)$ . We may also assume that the affine geometry  $AG(n + 1, q)$  in which  $\text{Af}_e(\Sigma)$  is realized, is just the complement of  $\Pi_0$  in  $\Pi$ . Accordingly,  $\delta$  induces a dual isomorphism from  $\text{Af}_e(\Sigma)$  to a subgeometry  $\text{Af}_{\delta e}^*(\Sigma)$  of the dual affine space  $\Pi \setminus \text{Res}_\Pi(p_0)$ . So, both  $\text{Af}_e(\Sigma)$  and its dual  $\text{Af}_{\delta e}^*(\Sigma)$  live inside  $\Pi$ . The intersection  $\text{Af}_{\delta e}^*(\Sigma) \cap \text{Af}_e(\Sigma)$ , regarded as an induced substructure of  $\Pi$ , is not a geometry in general. However, for certain choices of  $\Sigma$ ,  $e$  and  $\delta$  we do get a geometry. For instance, all biaffine geometries arise in this way. This construction has also been exploited by Del Fra [33] to create two infinite families of semiplanes. However, apart from biaffine geometries, only one flag-transitive geometry is known that can be obtained by this construction. It is a semiplane and belongs to one of the two families of Del Fra [33], but Janko and Van Trung [50] are those who discovered it first.

**Example 3.3.** [The Janko-Van Trung semiplane] Let  $\Sigma = (P, \mathcal{L})$  be the dual of the circular space with 6 points embedded in  $\Pi_0 := PG(2, 4)$  in such a way that  $e(\mathcal{L})$  is a dual hyperoval. Choose the correlation  $\delta$  of  $\Pi = PG(3, 4)$  in such a way that the point  $p_0 = \delta(\Pi_0)$  is exterior to the plane  $\Pi_0$  of  $\Pi$  and  $\{\delta e(l) \cap \Pi_0\}_{l \in \mathcal{L}}$  is the hyperoval formed by the six points of  $\Pi_0$  exterior to the dual hyperoval  $e(\mathcal{L})$ . Then  $\Gamma := \text{Af}_{\delta e}^*(\Sigma) \cap \text{Af}_e(\Sigma)$  is a flag-transitive semiplane of order 4 and  $\text{Aut}(\Gamma) = 3 \cdot \text{Sym}(6)$  is the stabilizer of  $e(\mathcal{L})$  in  $\Gamma L_3(4)$ , the latter being regarded as the stabilizer in  $P\Gamma L_4(4)$  of the antiflag  $\{p_0, \Pi_0\}$ . The multiplier 3 of  $\text{Sym}(6)$  in  $3 \cdot \text{Sym}(6)$  is the center  $Z$  of  $GL_3(4)$  and defines a quotient of  $\Gamma$ . Note that  $\Gamma/Z$  is flat and  $\text{Aut}(\Gamma/Z) = \text{Sym}(6)$ .

#### 4. Gluings.

Gluings can be defined in a very general way (Buekenhout, Huybrechts and Pasini [21], Section 3), but we will only consider gluings of linear spaces here. We firstly recall a few basics on parallelisms of linear spaces, next we define gluings and, finally, we will survey examples of flag-transitive  $L.L^*$ -geometries obtained as gluings.

##### 4.1. Parallelisms in linear spaces

Given a linear space  $\Sigma = (P, \mathcal{L})$  with point-set  $P$  and line-set  $\mathcal{L}$ , a *parallelism* of  $\Sigma$  is a partition  $\pi$  of  $\mathcal{L}$  in spreads. We denote by  $\Sigma/\pi$  the set of classes of  $\pi$  and, for a line  $l \in \mathcal{L}$ ,  $l^\pi$  is the class of  $\pi$  containing  $l$ . We denote by  $\text{Aut}(\Sigma, \pi)$  the group of all automorphisms of  $\Sigma$  that permute the classes of  $\pi$ , by  $K_\pi$  the kernel of the action of  $\text{Aut}(\Sigma, \pi)$  on  $\Sigma/\pi$  and, for an element

$g \in \text{Aut}(\Sigma, \pi)$  (a subgroup  $X \leq \text{Aut}(\Sigma, \pi)$ ) we denote by  $g^\pi$  (resp.  $X^\pi$ ) the permutation (the group) induced on  $\Sigma/\pi$  by  $g$  (resp.  $X$ ). If  $\text{Aut}(\Sigma, \pi)$  acts flag-transitively on  $\Sigma$ , then we say that  $(\Sigma, \pi)$  is *flag-transitive*, also that  $\pi$  is *flag-transitive*, for short. A sufficient condition for  $(\Sigma, \pi)$  to be flag-transitive is the following:  $K_\pi$  and  $(\text{Aut}(\Sigma, \pi))^\pi$  are transitive on  $P$  and  $\Sigma/\pi$  respectively.

Obviously, the natural parallelism of an affine space is flag-transitive. Each of the two Hering spaces admit a flag-transitive parallelism. Classical Witt spaces also admit flag-transitive parallelisms. Indeed, Let  $W(O)$  be the Witt space of lines and points of  $PG(2, 2^h)$  ( $h \geq 2$ ) exterior to a given classical hyperoval  $O = C \cup \{p\}$ , for a conic  $C$  with nucleus  $p$ . The lines of  $PG(2, 2^h)$  through  $p$  form a partition of the set of points exterior to  $O$ , which are the lines of  $W(O)$ . That partition is in fact a flag-transitive parallelism of  $W(O)$ . We call it the *natural* parallelism of  $W(O)$ . Note that, when  $h > 2$ , every classical hyperoval of  $PG(2, 2^h)$  can be regarded in a unique way as a conic plus its nucleus. In that case  $W(O)$  admits a unique natural parallelism. When  $h = 2$ , every hyperoval  $O$  of  $PG(2, 4)$  is classical and  $C := O \setminus \{p\}$  is a conic for every  $p \in O$ . In that case,  $W(O)$  admits six different flag-transitive parallelisms, but they are pairwise isomorphic. However, when  $h = 2$ ,  $W(O)$  is in fact isomorphic to the circular space  $\mathcal{C}_6$  with six points.

The circular spaces  $\mathcal{C}_{12}$  and  $\mathcal{C}_{28}$  with respectively 12 and 28 points also admit a flag-transitive parallelism. In either of them the flag-transitive parallelism is unique. Its automorphism group is isomorphic to  $L_2(11)$  and  $R(3) = {}^2G_2(3)$ , respectively (see Cameron and Korchmaros [26]).

We recall that the  $n$ -dimensional affine space  $AS(n, 2)$  over  $GF(2)$  is a circular space  $\mathcal{C}_v$  with  $v = 2^n$  points. When  $n > 2$ ,  $\mathcal{C}_v$  admits many parallelisms different from the natural parallelism of  $AS(n, 2)$ , but the latter is the unique flag-transitive parallelism of  $\mathcal{C}_v$  (Cameron and Korchmaros [26]). The circular spaces  $\mathcal{C}_6$ ,  $\mathcal{C}_{12}$  and  $\mathcal{C}_{28}$  and  $\mathcal{C}_v$  with  $v = 2^n$  are the only circular spaces admitting a flag-transitive parallelism (Cameron and Korchmaros [26]).

As noticed by Buekenhout, Huybrechts and Pasini [21], many projective spaces of odd dimension and all hermitian and Ree unitals admit a parallelism but, if  $\Sigma$  is such a space, every flag-transitive subgroup of  $\text{Aut}(\Sigma)$  acts primitively on the line-set of  $\Sigma$ , hence none of the parallelisms of  $\Sigma$  can be flag-transitive. I do not know if any 1-dimensional linear space exists that admits a flag-transitive parallelism.

#### 4.2. Gluing two linear spaces

Given two linear spaces  $\Sigma_1 = (P_1, \mathcal{L}_1)$  and  $\Sigma_2 = (P_2, \mathcal{L}_2)$ , equipped with parallelisms  $\pi_1$  and  $\pi_2$ , suppose that  $|\Sigma_1/\pi_1| = |\Sigma_2/\pi_2|$  and let  $\alpha$  be a bijection from  $\Sigma_1/\pi_1$  to  $\Sigma_2/\pi_2$ . The *gluing* of  $(\Sigma_1, \pi_1)$  with  $(\Sigma_2, \pi_2)$  by  $\alpha$  is the rank 3

geometry  $\text{Gl}_\alpha(\Sigma_1, \Sigma_2)$  defined as follows:  $P_1$  is the set of 0-elements and  $P_2$  is the set of 2-elements. The 1-elements are the pairs  $(l_1, l_2)$  with  $l_1 \in \mathcal{L}_1, l_2 \in \mathcal{L}_2$  and  $\alpha(l_1^{\pi_1}) = l_2^{\pi_2}$ . All 0-elements are declared to be incident to all 2-elements (so, this geometry is flat). A 0-element (a 2-element)  $p$  and a 1-element  $(l_1, l_2)$  are incident precisely when  $p \in l_1$  (resp.  $p \in l_2$ ). The residues of the 2-elements of  $\text{Gl}_\alpha(\Sigma_1, \Sigma_2)$  are isomorphic to  $\Sigma_1$  and the residues of the 0-elements are isomorphic to the dual of  $\Sigma_2$ . So,  $\text{Gl}_\alpha(\Sigma_1, \Sigma_2)$  is an  $L.L^*$ -geometry.

For  $i = 1, 2$ , put  $G_i := \text{Aut}(\Sigma_i, \pi_i)$ . Then  $\text{Aut}(\text{Gl}_\alpha(\Sigma_1, \Sigma_2))$  is the subgroup of  $G_1 \times G_2$  formed by the pairs  $(g_1, g_2)$  such that  $\alpha g_1^{\pi_1} \alpha^{-1} = g_2^{\pi_2}$ . Put  $L_1 := G_1^{\pi_1} \cap \alpha^{-1} G_2^{\pi_2} \alpha$  and  $L_2 := \alpha G_1^{\pi_1} \alpha^{-1} \cap G_2^{\pi_2}$ . Clearly,  $\alpha$  induces an isomorphism from the action of  $L_1$  on  $\Sigma_1/\pi_1$  to the action of  $L_2$  on  $\Sigma_2/\pi_2$ , and  $\text{Aut}(\text{Gl}_\alpha(\Sigma_1, \Sigma_2))$  is a (possibly non-split) extension of  $K_{\pi_1} \times K_{\pi_2}$  by a copy  $L$  of  $L_1 \cong L_2$ . The geometry  $\text{Gl}_\alpha(\Sigma_1, \Sigma_2)$  is flag-transitive if and only if the permutation group  $L_i$  acts transitively on the set  $\Sigma_i/\pi_i$  and the extension  $K_{\pi_i} \cdot L_i \leq G_i$  acts flag-transitively on  $(\Sigma_i, \pi_i)$ , for  $i = 1, 2$ . In particular, if  $K_{\pi_i}$  is transitive on  $P_i$  for  $i = 1, 2$  and  $L_i$  is transitive on  $\Sigma_i/\pi_i$ , then  $\text{Gl}_\alpha(\Sigma_1, \Sigma_2)$  is flag-transitive.

When  $(\Sigma_1, \pi_1) = (\Sigma_2, \pi_2) = (\Sigma, \pi)$ , we write  $\text{Gl}_\alpha(\Sigma)$  for short instead of  $\text{Gl}_\alpha(\Sigma, \Sigma)$ . The bijection  $\alpha$  is now a permutation of  $\Sigma/\pi$  and, putting  $G := \text{Aut}(\Sigma, \pi)$ , we can consider the double coset  $G^\pi \alpha G^\pi$  in the group of all permutations of  $\Sigma/\pi$ . The following has been proved by Buekenhout, Huybrechts and Pasini [21], Theorem 3.9:

**Proposition 4.1.** *Two gluings  $\text{Gl}_\alpha(\Sigma)$  and  $\text{Gl}_\beta(\Sigma)$  are isomorphic if and only if  $G^\pi \alpha G^\pi = G^\pi \beta G^\pi$ .*

We say that  $\text{Gl}_\alpha(\Sigma)$  is a *canonical gluing* if  $\alpha \in G^\pi$  (in particular, if  $\alpha$  is the identity permutation). By Proposition 4.1, canonical gluings are mutually isomorphic. This allows us to freely use the determinate article *the* when speaking of the canonical gluing.

**4.3. Gluing affine spaces with affine or circular spaces**

We firstly consider gluings of two copies of the  $n$ -dimensional affine space  $AS(n, q)$  over  $GF(q)$ .

**Proposition 4.2.** (Baumeister and Stroth [12]) *Let  $\Sigma := AS(n, q)$  and  $\pi$  be its parallelism (the natural one, when  $q = 2$ ). Given a permutation  $\alpha$  of  $\Sigma/\pi$ , put  $L := G^\pi \cap \alpha G^\pi \alpha^{-1}$ , where  $G := \text{Aut}(\Sigma, \pi)$ . Suppose that the gluing  $\text{Gl}_\alpha(\Sigma)$  is flag-transitive non-canonical. Then one of the following holds:*

- (1)  $n = 2, q \in \{5, 7, 11, 19, 23, 29, 59\}$  and  $L$  is as follows:  
     if  $q \in \{5, 7, 23\}$ , then  $L \cong \text{Sym}(4)$ ;

- if  $q \in \{19, 29, 59\}$ , then  $L \cong \text{Alt}(5)$ ;  
 if  $q = 11$ , then  $L \cong \text{Sym}(4)$ ,  $\text{Alt}(4)$  or  $\text{Alt}(5)$ .  
 (2)  $q = 3$ ,  $n = 4$  and  $L \cong L_2(13)$ .  
 (3)  $q = 3$ ,  $n = 6$ ,  $O_2(L) = 2^4$  and  $Z_5 \leq L/O_2(L) \leq \text{Frob}(5 : 2)$ .  
 (4)  $L$  is the normalizer of a Singer cycle in  $\Gamma L_n(q)$  and either  $n > 2$  or  $n = 2$  with  $q = 7$  or  $q > 9$ .  
 (5)  $L$  is embedded in  $\Gamma L_1(q^n)$  and, for  $r = |\text{Aut}(GF(q))|$ , the numbers  $nr$  and  $q^{n-1} + \dots + q + 1$  are non-coprime.

In cases (1), (2) and (3),  $\text{Gl}_\alpha(\Sigma)$  is simply connected. In the remaining two cases, if  $n = 2$  and  $q$  is an odd prime, then  $\text{Gl}_\alpha(\Sigma)$  is simply connected.

Not so much is known on the universal cover of  $\text{Gl}_\alpha(\Sigma)$  in cases (4) and (5) with  $n > 2$  or  $q$  even or non-prime. Actually, only the case of  $q = 2$  has been investigated: infinitely many non-canonical gluings of two copies of  $AS(n, 2)$  exist that admit a proper cover (Pasini and Yoshiara [65], Proposition 3.5), but it is likely that infinitely many simply connected examples also exist (Pasini and Yoshiara [66], Section 4).

**Proposition 4.3.** *Let  $\Gamma$  be the canonical gluing of two copies of  $AS(n, q)$  and define  $\widehat{\Gamma}$  as follows: if  $n = 2$ , then  $\widehat{\Gamma}$  is the biaffine geometry of rank 3, order  $q$  and incident type; if  $n > 2$ , then  $\widehat{\Gamma} = \text{Tr}_\uparrow(\text{Far}_\Delta(p, p^*))$ , where  $\Delta$  is the building of type  $D_{n+1}^{2,2} = D_{n+1}$  over  $GF(q)$  and  $\{p, p^*\}$  is a  $\{0, 2\}$ -flag of  $\Delta$ , as in Proposition 2.8. Then  $\Gamma$  is a quotient of  $\widehat{\Gamma}$ . If  $q > 2$ , then  $\widehat{\Gamma}$  is the universal cover of  $\Gamma$ . If  $q = 2$ , then the universal cover of  $\Gamma$  is  $\text{Tr}_\uparrow(\mathcal{C})$ , where  $\mathcal{C}$  is the Coxeter complex of type  $D_N^{2,2} = D_N$ ,  $N = 2^n$ .*

*Proof.* See [59] for the first claim. The rest follows from that claim and either Proposition 2.8 (when  $n > 2$ ) or Proposition 2.9 (when  $n = 2$ ). As for the case of  $n = 2$  with  $q = 2$ , note that the biaffine geometry of incident type, order 2 and rank 3 is isomorphic to the truncation  $\text{Tr}_\uparrow(\mathcal{C})$  of the Coxeter complex  $\mathcal{C}$  of type  $D_4^{2,2} = D_4$ .  $\square$

Flag-transitive  $Af.Af^*$ -geometries can also be obtained by gluing two copies of a non-desarguesian affine plane or even two non-isomorphic affine planes, but no thoroughful investigation of these gluings can be found in the literature. The case of two affine spaces of different orders and dimensions is discussed below.

**Example 4.1.** Let  $\Sigma_1 = AS(n_1, q_1)$  and  $\Sigma_2 = AS(n_2, q_2)$  with  $q_1 < q_2$ . A gluing of  $\Sigma_1$  with  $\Sigma_2$  exists if and only if  $(q_1, n_1, q_2, n_2)$  is a solution of the Goormaghtigh equation (\*) of Remark 1.2. Take  $(q_1, n_1, q_2, n_2) = (2, 5, 5, 3)$ . Only two flag-transitive gluings exist of  $AS(5, 2)$  with  $AS(3, 5)$  (Huybrechts

and Pasini [48], Section 4.2). They are simply connected. Their automorphism groups are extensions of  $2^5 \times 5^3$  by  $Z_{31}$ .

We now turn to gluings where at least one circular space is involved. We firstly recall the following:

**Proposition 4.4.** (Baumeister and Pasini [10]). *Let  $\Gamma$  be a flat flag-transitive  $c.c^*$ -geometry. Then  $\Gamma$  is either the flat quotient of the Janko-Van Trung semiplane (Example 3.3), or a gluing of two copies of  $AS(n, 2)$ .*

The flat quotient of the Janko-Van Trung semiplane is not a gluing. So, all flag-transitive gluings of two circular spaces are in fact gluings of two affine spaces. Let now  $\Gamma$  be a flag-transitive gluing of a circular space  $\mathcal{C}_v$  with  $v$  points with an affine space  $\Sigma$  of order  $q > 2$  and dimension  $n$ . Then

$$v = 2 + q + q^2 + \dots + q^{n-1}.$$

If  $v$  is a power of 2, we are back to Example 4.1. Suppose  $v = 6, 12$  or  $28$ . No solution  $(v, q, n)$  exists for the above equation with  $v = 12$  or  $28$ . Hence  $v = 6, q = 4$  and  $n = 2$ . A flag-transitive gluing of  $\mathcal{C}_6$  with  $AS(2, 4)$  actually exists: it is isomorphic to the flat quotient  $\Gamma_L(2^4 3 S_5)$  of the geometry  $\Gamma_L(2^6 3 S_6)$  of Example 3.1.

#### 4.4. Gluings involving Witt or Hering spaces

Let  $\mathcal{W}$  be the classical Witt space of order  $(2^{h-1} - 1, 2^h)$ . We can glue  $\mathcal{W}$  with the affine plane  $\Sigma := AG(2, 2^h)$ . If the gluing bijection  $\alpha$  is well chosen, then  $\text{Gl}_\alpha(\Sigma, \mathcal{W})$  is flag-transitive. However, it is not difficult to check that, in that case,  $\text{Gl}_\alpha(\Sigma, \mathcal{W})$  is in fact isomorphic to the flat quotient of the affine expansion of the dual of  $\mathcal{W}$ , discussed in Subsection 3.5.

We can also glue  $\mathcal{W}$  with itself, but a straightforward computation shows that there is no way to choose  $\alpha$  in such a way that  $\text{Gl}_\alpha(\mathcal{W})$  is flag-transitive. On the other hand, let  $\mathcal{H}$  be one of the two Hering spaces. Comparing the information given on Hering spaces by Huybrechts [42, 2.6] one can see that we can only glue  $\mathcal{H}$  with itself or with  $AS(3, 9)$ . The latter gluing is not flag-transitive, but the canonical gluing of  $\mathcal{H}$  with itself is flag-transitive. Thus, we obtain two flag-transitive  $H.H^*$ -geometries, one for each Hering space.

## 5. More constructions.

### 5.1. A family of flag-transitive $W.W^*$ -geometries

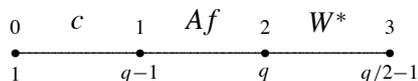
An infinite family of flag-transitive  $W.W^*$ -geometries has been discovered by C. Huybrechts [44]. Two equivalent constructions are given for those geometries in [44]. We will describe only one of them.

Given a non-singular quadric  $\mathcal{Q}$  of  $PG(4, q)$ ,  $q$  even, let  $p$  be a point of  $PG(4, q)$  exterior to  $\mathcal{Q}$  but different from the nucleus of  $\mathcal{Q}$ . Let  $\Gamma$  be the induced subgeometry of  $PG(4, q)$  defined as follows: The secant lines of  $\mathcal{Q}$  passing through  $p$  are the 0-elements, the 1-elements are the planes through  $p$  that meet  $\mathcal{Q}$  in a non-singular conic with nucleus different from  $p$ , and the 2-elements are the hyperplanes through  $p$  intersecting  $\mathcal{Q}$  in an elliptic quadric. Then  $\Gamma$  is a  $W.W^*$ -geometry with  $\{0, 1\}$ -residues ( $\{1, 2\}$ -residues) isomorphic to (the dual of) a Witt space. The stabilizer  $G$  of  $p$  and  $\mathcal{Q}$  in  $P\Gamma L_5(q)$  acts flag-transitively on  $\Gamma$ . It contains a normal subgroup of order  $q/2$  every subgroup of which defines a quotient of  $\Gamma$ .

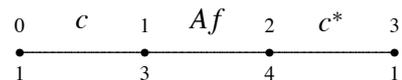
We guess that  $\Gamma$  is simply connected, but nothing is said about this in [44]. The reader might also wonder what happens if, in the above construction, we take exterior instead of secant lines and hyperplanes meeting  $\mathcal{Q}$  in a hyperbolic quadric rather than an elliptic one: the geometry obtained in that way is isomorphic to  $\Gamma$  (Huybrechts [44]).

**5.2. A family of flag-transitive  $c.Af.W^*$ -geometries**

For  $q = 2^h$  with  $h \geq 2$ , let  $\mathcal{Q}$  be a non-singular quadric of  $PG(4, q)$  and  $\mathcal{H}$  be the family of 3-spaces of  $PG(4, q)$  that meet  $\mathcal{Q}$  in an elliptic quadric. We can form a geometry of rank 4 as follows:  $\mathcal{H}$  is the set of 3-elements, the 0-elements are the points of  $\mathcal{Q}$ , the 1-elements are the non-collinear pairs of points of  $\mathcal{Q}$  and the 2-elements are the planes of  $PG(4, q)$  that meet  $\mathcal{Q}$  in a non-singular conic. The incidence relation is inclusion. This geometry is flag-transitive and has diagram and orders as follows:



The  $\{0, 1, 2\}$ -residues are Möbius planes and the  $\{1, 2, 3\}$ -residues are isomorphic to affine expansions of dual Witt spaces, as in Subsection 3.5. In particular, when  $q = 4$  we obtain the following:



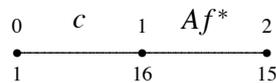
In this case, the  $\{1, 2, 3\}$ -residues are dual special Laguerre planes of order 4, as in Example 3.1.

**6. Exceptional and sporadic examples.**

A few exceptional geometries have already been met in Section 4, namely the  $AG.c^*$ -geometry  $\Gamma(2^{13}3M_{22})$  and its quotient  $\Gamma(2^{12}3M_{22})$  (Example 3.2) and the geometries  $\Gamma(3^5M_{11})$  and  $\Gamma(3^62M_{12})$  (Subsection 3.7). The following examples may also be added to that list, even if no sporadic group is involved in them: the dual Laguerre plane  $\Gamma_L(2^63S_6)$  and its quotients  $\Gamma_L(2^5S_5)$  and  $\Gamma_L(2^43S_5)$  (Example 3.1), the Janko-Van Trung semiplane and its flat quotient (Example 3.3), and the exceptional gluings mentioned in Proposition 4.2 ((1),(2),(3)) and in Example 4.1. More examples are gathered in this section.

**6.1. Flag-transitive  $c^i.Af^*$ -geometries**

**Example 6.1.** [A  $c.Af^*$ -geometry for  $L_2(17)$ ] Denoting by  $V$  the natural 18-dimensional  $GF(2)$ -module for the linear group  $L = L_2(17)$ ,  $V$  contains two non-isomorphic 9-dimensional submodules  $V_1$  and  $V_2$ , permuted by an outer automorphism of  $L_2(17)$  and intersecting in the 1-dimensional submodule  $Z := C_V(L)$  (see Ivanov and Praeger [49], (1.4) ). Let  $U$  be  $V_1$  or  $V_2$ . A flag-transitive  $c.Af^*$ -geometry  $\Gamma$  with orders as follows can be constructed over  $U$ .

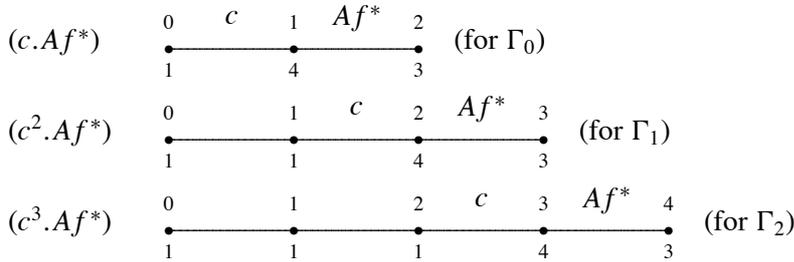


The vectors of  $U$ , regarded as functions from  $PG(1, 17)$  to  $GF(2)$ , are taken as 2-elements of  $\Gamma$  and the pairs  $(i, \varepsilon) \in PG(1, 17) \times GF(2)$  are the 0-elements. A pair  $(i, \varepsilon)$  and a vector  $v : PG(1, 17) \rightarrow GF(2)$  are declared to be incident precisely when  $v(i) = \varepsilon$ . Consider the 0-elements  $p_\infty = (\infty, 0)$  and  $p_0 = (0, 0)$ . There exist  $2^7$  2-elements incident with both  $p_\infty$  and  $p_0$ . Denoting by  $L_{\{\infty,0\}}$  the stabilizer of  $\{\infty, 0\}$  in  $L$ , we have  $L_{\{\infty,0\}} = D_{16}$ . Pick an element  $a \in L_{\{\infty,0\}}$  of order 8. Then  $|C_U(a)| = 2^5$  and the set, say  $l_0$ , of vectors of  $C_U(a)$  incident to both  $p_\infty$  and  $p_0$  is a subgroup of  $C_U(a)$  of index 2. The 1-elements of  $\Gamma$  are the images of  $l_0$  by elements of the group  $UL$ , in its affine action on the vector space  $U$ . A 1-element  $l$  and a 2-element  $v$  (a 0-element  $p$ ) are declared to be incident when  $v \in l$  (when  $p$  is incident to all 2-elements  $v \in l$ ). The geometry  $\Gamma$  is flag-transitive, with  $\text{Aut}(\Gamma) = UL = 2^9 : L_2(17)$ . The subgroup  $Z < U$  defines a flag-transitive quotient  $\Gamma/Z$  of  $\Gamma$ . We will denote  $\Gamma$  and  $\Gamma/Z$  by  $\Gamma(2^9L_2(17))$  and  $\Gamma(2^8L_2(17))$ , respectively.

Note that IP fails to hold in  $\Gamma(2^9L_2(17))$  and  $\Gamma(2^8L_2(17))$  is non-flat. We refer to Baumeister et al. [6] for more information on these geometries.

**Example 6.2.** [Three geometries for  $M_{23}, M_{22}L_3(4) \cdot 2$ ] For  $i = 0, 1, 2$ , let  $\Sigma_i$  be the Steiner system  $S(22 + i, 6 + i, 3 + i)$ , regarded as an  $L_{3+i}$ -geometry. So,

0, 1, ..., 2 + i are the types, the 0-elements are the points of  $S(22+i, 6+i, 3+i)$ , the 1-elements are the duads, and so on. Pick a 0-element  $p$  of  $\Sigma_i$  and let  $\Gamma_i$  be the induced subgeometry of  $\Sigma_i$  formed by the elements that are not incident to  $p$ . (In particular, the 0-elements of  $\Gamma_i$  are the points of  $\Sigma_i$  different from  $p$ .) Then  $\Gamma_i$  has diagram and orders as follows:



For  $i = 0, 1$ , the residues of the 0-elements of  $\Gamma_{i+1}$  are isomorphic to  $\Gamma_i$ . The automorphism group of  $\Gamma_i$  is the stabilizer of  $p$  in  $\text{Aut}(\Sigma_i)$  and acts flag-transitively on  $\Gamma_i$ . Explicitly,  $\text{Aut}(\Gamma_0) = P\Sigma L_3(4)$ ,  $\text{Aut}(\Gamma_1) = M_{22}$  and  $\text{Aut}(\Gamma_2) = M_{23}$ . We will use the following notation:  $\Gamma_{\text{far}}(L_3(4))$  for  $\Gamma_0$ ,  $\Gamma_{\text{far}}(M_{22})$  for  $\Gamma_1$  and  $\Gamma_{\text{far}}(M_{23})$  for  $\Gamma_2$ , where the subscript ‘far’ should remind us that the elements of  $\Gamma_i$  are those that are non-incident to  $p$  (namely, far from  $p$ ).

**Proposition 6.1.** (Sprague [69], [70]). *All but two of the  $c.Af^*$ -geometries satisfying IP are special Laguerre planes.  $\Gamma_{\text{far}}(L_3(4))$  and the biaffine geometry of order 2, rank 3 and non-incident type are the two exceptions and are characterized by the fact that  $T_{0,1}^2$  does not hold in them.*

**Proposition 6.2.** ([58]). *The geometries  $\Gamma_{\text{far}}(M_{22})$  and  $\Gamma_{\text{far}}(M_{23})$  are the unique  $c^i.Af^*$ -geometries of rank  $n = i + 2 \geq 4$  with  $c.Af^*$ -residues isomorphic to  $\Gamma_{\text{far}}(L_3(4))$ .*

**Example 6.3.** [Geometries of type  $c.Af^*$ ,  $c^2.Af^*$  and  $c^3.Af^*$  for  $2^4 : \text{Sym}(6)$ ,  $2^4 : \text{Alt}(7)$  and  $2^4 : \text{Alt}(8)$ ] With  $\Sigma_i = S(22 + i, 6 + i, 3 + i)$  as in Example 6.2, pick a block ( $(i + 2)$ -element)  $A$  of  $\Sigma_i$ , namely a hexad when  $i = 0$ , a heptad if  $i = 1$ , an octad if  $i = 2$ . Define a geometry  $\Delta_i$  as follows: The 0-elements of  $\Delta_i$  are the points of  $A$  and the  $(i + 2)$ -elements are the points of  $\Sigma_i$  exterior to  $A$ . For  $1 \leq j \leq i$ , the  $j$ -elements are the subsets of  $A$  of size  $j + 1$ . The  $(i + 1)$ -elements of  $\Delta_i$  are the blocks  $X$  of  $\Sigma_i$  with  $|X \cap A| = i + 2$ . All  $(i + 2)$ -elements of  $\Delta_i$  are declared to be incident to all  $j$ -elements for any  $j \leq i$  (in particular,  $\Delta_0$  is flat). The incidence relation between  $(i + 1)$ -elements and the remaining elements of  $\Delta_i$ , or between elements of type less

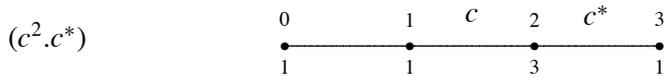
then  $i + 1$ , is inherited from  $\Sigma_i$ . Then  $\Delta_0, \Delta_1$  and  $\Delta_2$  belong to  $c.Af^*, c^2.Af^*$  and  $c^3.Af^*$  with orders  $(1, 4, 3), (1, 1, 4, 3)$  and  $(1, 1, 1, 4, 3)$  respectively, just as the geometries  $\Gamma_{\text{far}}(L_3(4)), \Gamma_{\text{far}}(M_{22})$  and  $\Gamma_{\text{far}}(M_{23})$  of Example 6.2.

For  $i = 0, 1$ , the residues of the 0-elements of  $\Delta_{i+1}$  are isomorphic to  $\Delta_i$ . The automorphism group of  $\Delta_i$  is flag-transitive and isomorphic to the stabilizer of  $A$  in  $\text{Aut}(\Sigma_i)$ . That is,  $\text{Aut}(\Delta_0) = 2^4 : \text{Sym}(6), \text{Aut}(\Delta_1) = 2^4 : \text{Alt}(7)$  and  $\text{Aut}(\Delta_2) = 2^4 : \text{Alt}(8)$ . We will denote  $\Delta_0, \Delta_1$  and  $\Delta_2$  by  $\Gamma_{\text{cl}}(2^4S_6), \Gamma_{\text{cl}}(2^4A_7)$  and  $\Gamma_{\text{cl}}(2^4A_8)$  respectively, where the subscript ‘cl’ stands for ‘close’ and should remind us that  $\Delta_i$  is formed by the elements of  $\Sigma_i$  different from  $A$  but as close as possible to  $A$ .

**Proposition 6.3.** ([58]) *The geometries  $\Gamma_{\text{cl}}(2^4A_8)$  and  $\Gamma_{\text{cl}}(2^4A_7)$  are the unique  $c^i.Af^*$ -geometries of rank  $n = i + 2 \geq 4$  with  $c.Af^*$ -residues isomorphic to  $\Gamma_{\text{cl}}(2^4S_6)$ .*

**6.2. Exceptional  $c^i.c^{*j}$ -geometries**

**Example 6.4.** [A  $c^2.c^*$ -geometry of order 3 for  $M_{11}$  and a semiplane for  $L_2(11)$ ] (Buekenhout [18], (27), Meixner [54]). It is known that  $M_{12}$  admits two conjugacy classes of subgroups isomorphic to  $M_{11}$ . One class contains the stabilizers of the points of the Steiner system  $\Sigma = S(12, 6, 5)$ . If  $G$  is a member of the other class, then  $G$  is 3-transitive on the point-set  $A$  of  $\Sigma$  and has an orbit  $\mathcal{O}$  of length 22 on the set of blocks of  $\Sigma$ . Regarding  $\Sigma$  as a geometry of rank 5, let  $\Gamma$  be the induced subgeometry of  $\Sigma$  obtained by removing the blocks (4-elements) that do not belong to  $\mathcal{O}$  and all 3-elements of  $\Sigma$  (namely, 4-subsets of  $A$ ). Then  $\Gamma$  is a flag-transitive geometry with  $\text{Aut}(\Gamma) = G \cong M_{11}$  and diagram and orders as follows:



We will denote this geometry by  $\Gamma_{12}(M_{11})$ , where the index 12 reminds us of the number of its 0-elements. The geometry  $\Gamma_{12}(M_{11})$  is simply connected (Meixner [54]). The residues of the 0-elements of  $\Gamma_{12}(M_{11})$  are isomorphic to a well known biplane for  $L_2(11)$ . We refer to Buekenhout [17] (also Baumeister and Buekenhout [5], 3,(3)(i)) for alternative descriptions of this biplane.

**Example 6.5.** [Some  $c^i.c^*$ -geometries of order 7 related to  $M_{12}, M_{11}$  and  $M_{10}$ ] (Meixner [54], Ceccherini and Pasini [28], 3.3). The construction we describe here is essentially taken from [28], but for a few corrections. (Indeed, the elements of type 2 of the geometry we are going to describe are defined in

[28] in an erroneous way). We firstly state some terminology and recall a few properties of the 6-dimensional ternary Golay code  $\mathcal{C}_3(6)$ . Let  $V = V(12, 3)$  be the vector space of all functions from  $I = \{1, 2, \dots, 12\}$  to  $GF(3)$ . For a vector  $v \in V$ , the *support* of  $v$  is the set  $\sigma(v) := \{i \in I | v(i) \neq 0\}$ . The size  $\lambda(v) = |\sigma(v)|$  of  $\sigma(v)$  is the *weight* of  $v$ . It is well known [30] that the non-trivial words of  $\mathcal{C}_6(3) (\subset V)$  have weight 6, 9 or 12. In particular,  $\mathcal{C}_3(6)$  admits 264 words of weight 6, 440 words of weight 9 and 24 words of weight 12. Moreover, if  $u$  and  $x$  are words of weight 9 and 6, then  $3 \leq |\sigma(u) \cap \sigma(x)| \leq 6$ . In the following table, we list all possibilities for  $\lambda(u+x)$ , for  $u$  and  $x$  as above. In the last column of the table we write the number of words  $x$  of weight 6 that, for a given  $u$  of weight 9, have  $|\sigma(x) \cap \sigma(u)|$  and  $\lambda(x+u)$  as in the considered row of the table.

	$ \sigma(x) \cap \sigma(u) $	$\lambda(x+u)$	number of words
(1)	6	6	24
(2)	5	6	54
(3)	5	9	54
(4)	4	9	108
(5)	3	9	12

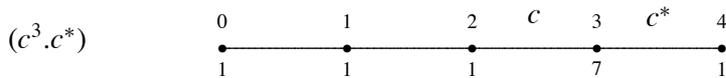
For  $1 \leq i \leq 6$ , let  $W_i(u)$  be the set of words  $x$  of weight 6 as in case (i) of the table. Note that the function sending every word to its opposite stabilizes  $W_1(u)$  and  $W_4(u)$  and permutes  $W_2(u)$  with  $W_3(u)$  and  $W_5(u)$  with  $W_6(u)$ . We are now ready to define our geometry.

Let  $\Gamma$  be the geometry of rank 5 defined as follows: The 4-elements of  $\Gamma$  are the vectors of  $V$  of weight 1. The 3-elements are the pairs  $(v_1, v_2)$  where  $v_1$  and  $v_2$  are vectors of weight 4 and 2 respectively and  $v_1 + v_2$  is a word of  $\mathcal{C}_6(3)$  of weight 6. The 2-elements are the words of  $\mathcal{C}_6(3)$  of weight 9. The 1- and 0-elements are the unordered pairs of elements of  $I$  and the elements of  $I$ , respectively.

Let  $v$  be a 4-element. We say that a 3-element  $(v_1, v_2)$  (a 2-element  $u$ ) is incident with  $v$  when  $\lambda(v_2 - v) = 1$  (respectively,  $\lambda(u - v) = 8$ ). A 1-element  $J$  (a 0-element  $i$ ) is incident to  $v$  when  $\sigma(v) \not\subset J$  (resp.  $\sigma(v) \neq i$ ). Let  $(v_1, v_2)$  be a 3-element. A 2-element  $u$  is incident to  $(v_1, v_2)$  when  $v_1 + v_2 \in W_6(u)$  and  $\lambda(u - v_2) = 7$ . A 1-element  $J$  (a 0-element  $i$ ) is said to be incident to  $(v_1, v_2)$  when  $J \subset \sigma(v_1)$  (resp.  $i \in \sigma(v_1)$ ). A 2-element  $u$  and a 1-element  $J$  (a 0-element  $i$ ) are incident if and only if  $J \cap \sigma(u) = \emptyset$  (resp.  $i \notin \sigma(u)$ ). A 1-element  $J$  and a 0-element  $i$  are incident if and only if  $i \in J$ .

The structure  $\Gamma$  defined as above is a flag-transitive geometry with

$\text{Aut}(\Gamma) = 2:M_{12}$  (central non-split extension) and diagram and orders as follows:



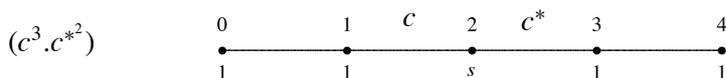
Moreover,  $\Gamma$  is simply connected (Ceccherini and Pasini [28]). The center  $Z = Z_2$  of  $\text{Aut}(\Gamma)$  defines a flag-transitive quotient  $\Gamma/Z$  of  $\Gamma$ , with  $\text{Aut}(\Gamma/Z) = M_{12}$ . We will denote  $\Gamma$  by  $\Gamma_{\text{Gol}}(2M_{12})$  and  $\Gamma/Z$  by  $\Gamma_{12}(M_{12})$ . (Needless to say, the subscript ‘Gol’ stands for ‘Golay’.) Note that  $\Gamma_{12}(M_{12})$  can also be recovered from the Steiner system  $\Sigma = S(12, 6, 5)$ . The points of  $\Sigma$  are taken both as 0- and as 4-elements, the elements of  $\Gamma_{12}(M_{12})$  of type 1 and 2 are the duads and the complements of triads of  $\Sigma$ ; the 3-elements are hexads partitioned in a tetrad and a duad. The incidence relation of  $\Gamma_{12}(M_{12})$  is implicit in the description of  $\Gamma_{\text{Gol}}(2M_{12})$ . In fact, the projection of  $\Gamma_{\text{Gol}}(2M_{12})$  onto  $\Gamma_{12}(M_{12})$  is induced by the function mapping every vector  $v \in V$  onto  $\sigma(u)$ .

The residue of a 0-element of  $\Gamma_{\text{Gol}}(2M_{12})$  is a simply connected flag-transitive  $c^2.c^*$ -geometry with orders  $(1, 1, 7, 1)$  and automorphism group isomorphic to  $Z_2 \times M_{11}$ . We will denote it by  $\Gamma_{\text{Gol}}(2M_{11})$ . The factor  $Z_2$  of the product  $Z_2 \times M_{11}$  defines a flag-transitive quotient of  $\Gamma_{\text{Gol}}(2M_{11})$ , which we will denote by  $\Gamma_{11}(M_{11})$ . Clearly,  $\Gamma_{11}(M_{11})$  is isomorphic to the residues of the 0-elements of  $\Gamma_{12}(M_{12})$ .

$\Gamma_{11}(M_{11})$  can be recovered from  $S(11, 5, 4)$  just like  $\Gamma_{12}(M_{12})$  from  $S(12, 6, 5)$ . We refer to Meixner [54] for that description of  $\Gamma_{11}(M_{11})$ . A description of  $\Gamma_{\text{Gol}}(2M_{11})$  by means of the 5-dimensional ternary Golay code for  $M_{11}$  is implicit in the above description of  $\Gamma_{\text{Gol}}(2M_{12})$ .

The residues of the 0-elements of  $\Gamma_{\text{Gol}}(2M_{11})$  are  $c.c^*$ -geometries with orders  $(1, 7, 1)$  and automorphism group isomorphic to  $Z_2 \times M_{10}$ . (We warn that these  $c.c^*$ -geometries are not semiplanes.) The residues of the 0-elements of  $\Gamma_{11}(M_{11})$  are  $c.c^*$ -geometries for  $M_{10} = L_2(9)2_3$ .

**Example 6.6.** [Geometries of type  $c^2.c^{*2}$  for  $M_{12}$  and  $2:M_{12}$ ] (Meixner [54], Leemans [51]). The geometry  $\Gamma_{12}(M_{11})$  is isomorphic to the residues of the 4-elements of a  $c^2.c^{*2}$ -geometry  $\Delta_1$  for  $M_{12}$  (Meixner [54], Leemans [51]):



where  $s = 3$ . The geometry  $\Delta_1$  is not simply connected. Its universal 2-cover  $\tilde{\Delta}_1$  is a double cover, with automorphism group isomorphic to the central non-split extension  $2:M_{12}$  (see Meixner [54], where a group-theoretic construction

of  $\tilde{\Delta}_1$  is given, by generators and relations). The outer automorphisms of  $M_{12}$  induce non-type-preserving automorphisms on  $\Delta_1$ . The same holds for  $\tilde{\Delta}_1$ .

$\Gamma_{\text{Gol}}(2M_{11})$  is isomorphic to the residues of the 4-elements of a 2-simply connected  $c^2.c^*2$ -geometry  $\tilde{\Delta}_2$  with  $s = 7$  and automorphism group isomorphic to  $2M_{12}$ , constructed group-theoretically by Meixner [54]. The center  $Z = Z_2$  of  $\text{Aut}(\tilde{\Delta}_2)$  defines a flag-transitive quotient  $\Delta_2 = \tilde{\Delta}_2/Z$ , with  $\text{Aut}(\Delta_2) = M_{12}$ . Clearly, the residues of the 4-elements of  $\Delta_2$  are isomorphic to  $\Gamma_{12}(M_{12})$ .

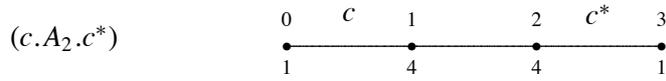
Geometric constructions of  $\Delta_1$  and  $\Delta_2$  are given by Leemans [51], starting from a partition of the Steiner system  $S(24, 8, 5)$  in two disjoint dodecads. We refer to [51] for details. No combinatorial construction of  $\tilde{\Delta}_1$  or  $\tilde{\Delta}_2$  can be found in the literature.

### 6.3. Geometries for $J_2$ and $HS$

**Example 6.7.** [A  $c.U^*$ -geometry for  $J_2$ ] (Buekenhout [18], (104), Buekenhout and Huybrechts [22]). For  $k \in \{100, 280\}$ , the Hall-Janko group  $J_2$  admits a unique conjugacy class  $\mathcal{C}_k$  of subgroups of index  $k$  (see Conway et al. [30]; the members of  $\mathcal{C}_{100}$  are isomorphic to  $U_3(3)$  and those of  $\mathcal{C}_{280}$  are non-split extensions  $3:PG L_2(9)$ ). Put  $S_0 := \mathcal{C}_{100}$ . If  $X \in \mathcal{C}_{280}$ , then  $X$  has two orbits  $O_{10}(X)$  and  $O_{90}(X)$  on  $S_0$ , of length 10 and 90 respectively. Put  $S_2 := \{O_{10}(X)\}_{X \in \mathcal{C}_{280}}$  and let  $S_1$  be the set of pairs of elements of  $S_0$ . Then the triple  $\Gamma = (S_0, S_1, S_2)$  with symmetrized inclusion as incidence relation is a flag-transitive  $c.U^*$ -geometry with orders  $(1, 8, 3)$  and  $\{1, 2\}$ -residues dually isomorphic to the hermitian unital of  $PG(2, 9)$ . We have  $\text{Aut}(\Gamma) = J_2$ , but  $J_2$  also acts flag-transitively on  $\Gamma$ . We will denote  $\Gamma$  by  $\Gamma(J_2)$ .

**Example 6.8.** [A  $c.A_2.c^*$ -geometry for  $HS$ ] (Buekenhout [16] and [18], (49)) Let  $\mathcal{H} = (V, \sim)$  be the graph with 100 vertices and valency 22 on which the Higman-Sims group  $HS$  acts as a rank 3 group (see [30]). A geometry  $\Gamma$  of rank 4 can be defined as follows: For  $x \in V$ , the pairs  $(x, 1)$  and  $(x, 2)$  are taken as elements of type 0 and 3 respectively; for  $i = 1, 2$ , the  $i$ -elements of  $\Gamma$  are the pairs  $\{(x, i), (y, i)\}$  with  $x \neq y$  and  $\{x, y\}$  a non-edge of  $\mathcal{H}$ . A 0-element  $(x, 1)$  and a 3-element  $(y, 2)$  are declared to be incident when  $\{x, y\}$  is an edge of  $\mathcal{H}$ . For  $\{i, j\} = \{1, 2\}$ , the elements of type 1 or 2 incident to  $(x, i)$  are the  $i$ -elements  $\{(x, i), (y, i)\}$  and the  $j$ -elements  $\{(y, j), (z, j)\}$  such that both  $\{x, y\}$  and  $\{x, z\}$  are edges of  $\mathcal{H}$ . Finally, a 1-element  $\{(x_1, 1), (y_1, 1)\}$  and a 2-element  $\{(x_2, 2), (y_2, 2)\}$  are declared to be incident if each of the pairs  $\{x_1, x_2\}$ ,  $\{y_1, y_2\}$ ,  $\{x_1, y_2\}$  and  $\{y_1, x_2\}$  is an edge of  $\mathcal{H}$ . The geometry  $\Gamma$  defined

in this way is flag-transitive, with  $\text{Aut}(\Gamma) = \text{Aut}(HS)$  (but  $HS$  is also flag-transitive on it). It has diagram and orders as follows:



We will denote this geometry by  $\Gamma(HS)$ .

**Proposition 6.4.** (Hughes [41]) *The geometry  $\Gamma(HS)$  and the two biaffine geometries of rank 4 and order 2 are the unique  $c.A_2.c^*$ -geometries that satisfy IP (which, in this context, amounts to say that  $\mathcal{S}_{0,1}(\Gamma)$  is semi-linear).*

**6.4. Exceptional  $c.c^*$ -geometries**

A few flag-transitive  $c.c^*$ -geometries which may be regarded as exceptional in some respect have already been mentioned in this paper: the Janko-Van Trung geometry and its flat quotient (Example 3.3), the biplane for  $L_2(11)$  (see Example 6.4), a  $c.c^*$ -geometry for  $Z_2 \times M_{10}$  with orders  $(1, 7, 1)$  and its 2-fold quotient for  $M_{10}$  (see Example 6.5). We list in the following table the remaining exceptional examples we are aware of. All of them are semiplanes. We are not going to describe their constructions; we refer to Baumeister and Buekenhout [5] for them (also Baumeister [3]). We will only mention the significant order (namely the order at the mid-node of the diagram) and the automorphism group, and we give one or two additional references.

	order	group	more references
(1)	10	$M_{12}$	Brouwer, Cohen and Neumaier [15], 11.4.F
(2)	9	$M_{12}$	Buekenhout [17]
(3)	13	$M_{22}$	Baumeister [2]
(4)	13	$2 \cdot M_{22}$	Baumeister [2]
(5)	8	$L_3(4)$	Baumeister [3], Grams and Meixner [39]
(6)	8	$2 \cdot L_3(4)$	Baumeister [3], Grams and Meixner [39]
(7)	5	$U_3(3)$	Neumaier [55]

All the above semiplanes but (3) and (5) are simply connected (Baumeister and Pasechnik [8]). (3) and (5) are quotients of (4) and (6) respectively. (5) and (6) are subgeometries of (3) and (4).

## 7. Non-sporadic flag-transitive $c.c^*$ -geometries.

Only four infinite families of  $c.c^*$ -geometries are known were infinitely many (possibly, all) of the members are flag-transitive.

**Truncated Coxeter complexes and quotients.** As noticed in Subsection 2.1, if  $\Delta$  is the Coxeter complex of type  $D_N^{2,2} = D_N$ , then  $\text{Tr}_\uparrow(\Delta)$  is a 2-simply connected flag-transitive semiplane with orders  $(1, N - 2, 1)$ . Although  $\Delta$  does not admit any proper quotient,  $\text{Tr}_\uparrow(\Delta)$  admits many flag-transitive quotients.

**Affine expansions of dimensional dual hyperovals of  $PG(n, 2)$ .** We have mentioned these semiplanes in Subsection 3.4. Some but not all of them are quotients of truncated Coxeter complexes of type  $D_N$  (see Pasini and Yoshiara [65], [66]).

**Projective semiplanes.** This family of semiplanes has been discovered by Hughes [40]. Given an involution  $\sigma$  of  $PG(2, q)$ , take as 0- and 2-elements the orbits of  $\sigma$  of size 2 on the set of points and, respectively, lines of  $PG(2, q)$ , declaring a 0-element  $\{p_1, p_2\}$  and a 2-element  $\{l_1, l_2\}$  to be incident when each of the points  $p_1, p_2$  belongs to one of the lines  $l_1, l_2$ . The 1-elements are the pairs of 0-elements incident to the same 2-element. The geometry, say  $\Gamma(\sigma)$ , defined in this way is a semiplane. We call it a *projective semiplane*; more explicitly, an *elation*, *homology* or *Baer semiplane* according to whether  $\sigma$  is an elation (in which case  $q$  is even) a homology ( $q$  odd) or a Baer involution ( $q$  is a square). The order of  $\Gamma(\sigma)$  at the mid-node of the diagram is  $q/2$  when  $\sigma$  is an elation,  $(q - 1)/2$  when  $\sigma$  is a homology and  $(q - \sqrt{q})/2$  when  $\sigma$  is a Baer involution.

Elation and homology semiplanes are simply connected (Baumeister and Pasechnik [9]). On the other hand, a Baer semiplane of order  $(q - \sqrt{q})/2$  admits a flag-transitive  $(\sqrt{q} - 1)$ -fold cover (Baumeister [3], Example (6); also [1], pp. 83-86). I don't know if that cover is simply connected.

All elation semiplanes can also be obtained as affine expansions from suitable dual hyperovals (see Pasini and Yoshiara [65]).

**Non-canonical gluings of two copies of  $AS(n, 2)$ .** These geometries correspond to cases (4) and (5) of Proposition 4.2, with  $q = 2$  in either case.

**Part II**  
**Classifications**

In the next three sections we gather all classification theorems for classes of flag-transitive locally finite  $L_h.L_k^*$ -geometries that can be found in the literature. We will consider geometries of rank 3 first, fusing in one long statement all results that have been obtained for them. Next we will turn to geometries of larger rank. In most cases, the theorems we can state contain gaps, namely cases for which no classification has been found so far, or even no example is known. We will point out them by writing *open case* at the beginning of the item where those cases are mentioned.

**8. The rank 3 case.**

**Theorem 8.1.** *Let  $\Gamma$  be a flag-transitive locally finite  $L.L^*$ -geometry. Then, possibly up to a permutation of the types 0 and 2, one of the following occurs:*

- (1)  $\Gamma$  belongs to diagram  $PG.PG^*$  and it is isomorphic to  $\text{Tr}_\uparrow(\Delta)$  for a building  $\Delta$  of type  $D_N^{2,2} = D_N$ ,  $N \geq 4$  (notation as in Section 2)
- (2)  $\Gamma$  belongs to diagram  $AG.PG^*$  and it is isomorphic to  $\text{Tr}_\uparrow(\Delta \setminus H)$  for a thick building  $\Delta$  of type  $D_N^{2,2} = D_N$ ,  $N \geq 4$ , and a hyperplane  $H$  of  $\mathcal{S}_{0,1}(\Delta)$  as in Subsection 2.2. Moreover, if  $N > 4$  then  $\Delta \setminus H = \text{Far}_\Delta(p)$  for an element  $p$  of  $\Delta$  of type 0 (if  $N$  is even) or 2 (if  $N$  is odd).
- (3)  $\Gamma$  is the  $c.U^*$ -geometry  $\Gamma(J_2)$  of Example 6.7.
- (4)  $\Gamma$  is one the following  $c.AG^*$ -geometries:

	name	orders	reference	notes
(4.1)	$\Gamma_L(2^6 3 S_6)$	1, 4, 3	Example 3.1	
(4.2)	$\Gamma_L(2^5 S_5)$	1, 4, 3	Example 3.1	quotient of (4.1)
(4.3)	$\Gamma_L(2^4 3 S_5)$	1, 4, 3	Example 3.1	quotient of (4.2)
(4.4)	$\Gamma(2^9 L_2(17))$	1, 16, 15	Example 6.1	
(4.5)	$\Gamma(2^8 L_2(17))$	1, 16, 15	Example 6.1	quotient of (4.4)
(4.6)	$\Gamma_{\text{far}}(L_3(4))$	1, 4, 3	Example 6.2	
(4.7)	$\Gamma_{\text{cl}}(2^4 S_6)$	1, 4, 3	Example 6.3	
(4.8)	$\Gamma(2^{13} 3 M_{22})$	1, 20, 3	Example 3.2	
(4.9)	$\Gamma(2^{12} 3 M_{22})$	1, 20, 3	Example 3.2	quotient of (4.8)

- (5) [open case]  $\Gamma$  belongs to  $c.AG^*$  with orders  $1, s, q - 1$  where  $q > 2$ ,  $s = q^{d-1} + \dots + q^2 + q$  for  $d > 2$ , and the stabilizer in  $\text{Aut}(\Gamma)$  of a 0-element  $p$  induces a 1-dimensional group on  $\text{Res}_\Gamma(p)$ .

- (6) [open case]  $\Gamma$  belongs to one of the following diagrams:  $c.c^*$ ,  $AG.AG^*$ ,  $H.H^*$ ,  $W.W^*$ ,  $Af.W^*$ .
- (7) [open case]  $\Gamma$  is an unknown flag-transitive geometry for one of the following diagrams:  $U.U^*$ ,  $c.1D^*$ ,  $W.1D^*$ ,  $AG.1D^*$ ,  $1D.1D^*$ .

*Proof.* The proof is entirely contained in the literature. Firstly, in view of Theorem 1.2, we have 14 cases to consider, namely:  $PG.PG^*$ ,  $AG.PG^*$ ,  $c.U^*$ ,  $c.AG^*$ ,  $c.c^*$ ,  $AG.AG^*$ ,  $H.H^*$ ,  $W.W^*$ ,  $Af.W^*$ ,  $U.U^*$ ,  $c.1D^*$ ,  $W.1D^*$ ,  $AG.1D^*$ ,  $1D.1D^*$ . Cardinali and Pasini [27] have proved that in case  $PG.PG^*$  we have (1). By [62] (see also Huybrechts and Pasini [47]), (2) holds in case  $AG.PG^*$ . In the  $c.U^*$ -case Huybrechts and Pasini [46] have proved that  $\Gamma(J_2)$  is the unique possibility.

The special case  $c.Af^*$  of  $c.AG^*$  has been classified by Baumeister et al. [6]: the only possibilities are those mentioned in (4) (items (4.1), (4.2),..., (4.7)) and, besides them, the two biaffine geometries of rank 3 and order 2 and the flat quotient of the one of incident type. We have not mentioned the latter in (4) as they are  $c.c^*$ -geometries, hence included in (6). The  $c.AG^*$ -case with  $\{1, 2\}$ -residues of dimension  $d > 2$  is considered by Huybrechts and Pasini [48], where it is proved that either we have (4.8) or (4.9), or  $\Gamma$  is as in (5), or  $q = 2$  and  $\Gamma$  is a  $c.c^*$ -geometry. No classification is known for the remaining diagrams.  $\square$

**Remark 8.1.** The two gluings of  $AS(5, 2)$  with  $AS(3, 5)$  mentioned in Example 4.1 are the only two examples we know for case (5) of Theorem 8.1. We conjecture that no more example exists for that case.

**Remark 8.2.** Examples have been described in the previous sections for each of the diagrams mentioned in case (6) of Theorem 8.1.

In view of the variety of examples known for the  $c.c^*$ -case, I don't believe that a complete detailed classification will ever be reached for that case. However, something like a classification, group-theoretically tailored, has been obtained. We refer the reader to Baumeister [3], [4] and Baumeister and Buekenhout [5] for it.

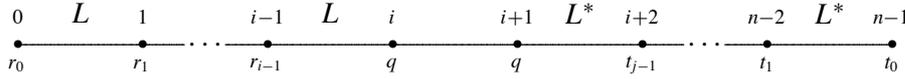
Not so much is known on flag-transitive  $AG.AG^*$ -geometries. Only those obtained as gluings are classified (Propositions 4.2 and 4.3). Partial results on flag-transitive  $Af.Af^*$ -geometries where IP fails to hold have been obtained by Del Fra and Pasini [36], but they are not sufficient for a classification. (We recall that, by Proposition 2.9,  $Af.Af^*$ -geometries with IP are biaffine).

No attempt at all has ever been done to classify flag-transitive geometries of type  $H.H^*$ ,  $W.W^*$  and  $Af.W^*$ . Even less is known on the diagrams mentioned in (7): we do not even know any example for any of them.

**9. Cases of rank  $n \geq 4$ .**

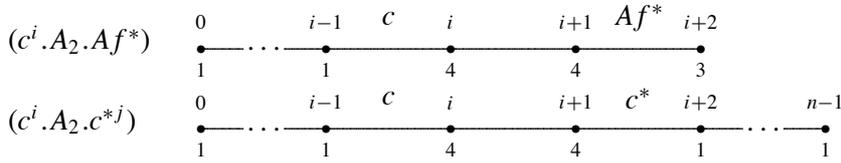
In this section we consider those families of  $L_h.L^*$ -diagrams for which something can be said without leaving too many gaps. The remaining cases will be considered in the next section. Henceforth,  $\Gamma$  is a flag-transitive locally finite  $L_h.L_k$ -geometry of rank  $n \geq 4$ .

**Theorem 9.1.** *Suppose that  $\Gamma$  has diagram as follows, where  $i + j + 2 = n$  and  $r_0, t_0 < s$ :*



Then  $\Gamma$  is either a (possibly non-proper) quotient of a biaffine geometry or an isomorphic copy of the exceptional geometry  $\Gamma(HS)$  of Example 6.8.

*Proof.* In view of Theorem 1.1, either  $\Gamma$  is an  $Af.A_{n-2}.Af^*$ -geometry or it belongs to one of the following diagrams:



with  $1 \leq i, j \leq 3$ . If  $\Gamma$  is an  $Af.A_{n-2}.Af^*$ -geometry then the conclusion follows from Proposition 2.9. Suppose  $\Gamma$  is not of type  $Af.A_{n-2}.Af^*$ . As proved by Del Fra, Pasini and Shpectorov [37],  $\Gamma$  cannot belong to  $c^i.A_2.Af^*$ . Hence  $\Gamma$  belongs to  $c^i.A_2.c^*j$ .

**Claim 1.** If  $\Gamma$  belongs to  $c.A_2.c^*$ , then  $\Gamma \cong \Gamma(HS)$ .

Indeed, if  $\Gamma$  belongs to  $c.A_2.c^*$ , then IP holds in it (Pasini and Yoshiara [64], Theorem 8.4). The conclusion follows from Proposition 2.9. The following, combined with Claim 1, is sufficient to finish the proof of the theorem.

**Claim 2.** No flag-transitive geometry exists for  $c^2.A_2.c$ .

Suppose the contrary: Let  $\Gamma$  be a flag-transitive geometry belonging to  $c^2.A_2.c^*$ . By Claim 1,  $\text{Res}_\Gamma(a) \cong \Gamma(HS)$  for every 0-element  $a$  of  $\Gamma$ . We firstly prove that  $\Gamma$  satisfies IP. Indeed, if not, by [56], Lemma 7.25, and the fact that the residues of the 0-elements of  $\Gamma$  are isomorphic to  $\Gamma(HS)$ , where IP holds,  $\Gamma$  admits pairs of distinct 1-elements incident with the same pair of 0-elements. Accordingly, the relation ‘being incident with the same pair of 0-elements’ is a non-trivial equivalence relation on the set of 1-elements of  $\Gamma$ .

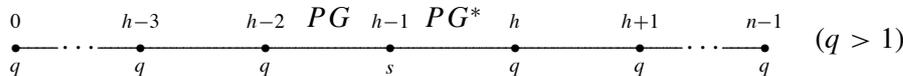
Hence the stabilizer  $G_a$  in  $\text{Aut}(\Gamma)$  of a 0-element  $a$  acts imprimitively on the set of 1-elements of  $\text{Res}_\Gamma(a)$ . However, this is a contradiction. Indeed  $G_a$  induces  $HS$  or  $\text{Aut}(HS)$  on  $\text{Res}_\Gamma(a) \cong \Gamma(HS)$ , and either of these groups acts primitively on the set of 0-elements of  $\Gamma(HS)$ . Therefore,  $\Gamma$  satisfies IP.

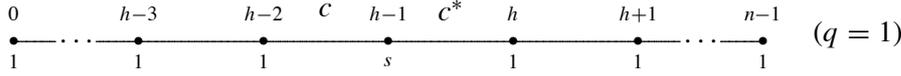
Let  $\mathcal{G}$  be the  $(0, 1)$ -space  $\mathcal{S}_{0,1}(\Gamma)$  of  $\Gamma$ , but regarded as a graph. The graph  $\mathcal{G}$  is regular of valency 100, as 100 is the number of 0-elements of  $\Gamma(HS)$ . We shall now prove that  $\mathcal{G}$  has diameter  $\delta \leq 2$ . Assume the contrary and let  $(a_1, b_1, b_2, a_2)$  be a path of  $\mathcal{G}$  with  $d(a_1, a_2) = 3$ . Let  $l$  be the 1-element incident to  $\{b_1, b_2\}$  and, for  $i = 1, 2$ , let  $l_i$  be the 1-element incident to  $\{a_i, b_i\}$ . Put  $\mathcal{G}_i := \mathcal{S}_{1,2}(\text{Res}_\Gamma(b_i))$ , which can also be regarded as a graph. It is clear from the description of  $\Gamma(HS) (\cong \text{Res}_\Gamma(b_i))$  that the graph  $\mathcal{G}_i$  is strongly regular and the vertices of  $\mathcal{G}_i$  at distance 2 from  $l_i$  are the 22 1-elements of  $\text{Res}_\Gamma(b_i, u_i)$ , for a 4-element  $u_i \in \text{Res}_\Gamma(b_i)$  uniquely determined by  $l_i$ . As  $d(a_i, b_j) = 2$  in  $\mathcal{G}$  for  $\{i, j\} = \{1, 2\}$ , we have  $d(l, l_i) = 2$  in  $\mathcal{G}_i$ . Hence  $l$  is one of the 22 1-elements of  $\text{Res}_\Gamma(b_i)$  incident to  $u_i$ . Pick a 4-element  $u$  on  $l$  different from either of  $u_1$  and  $u_2$ . At most 6 of the 22 1-elements incident to  $b_i$  and  $u$  are also incident to  $u_i$ . Therefore, at least 16 of those 1-elements, say  $l_{i,1}, l_{i,2}, \dots, l_{i,s_i}$  ( $s_i \geq 16$ ) are non-incident with  $u_i$ , whence adjacent to  $l_i$  in  $\mathcal{G}_i$ . However,  $u$  is incident with exactly 23 0-elements. Therefore,  $l_{1,j_1}$  and  $l_{2,j_2}$  are incident with a common 0-element  $c$ , for some choice of  $j_1 \leq s_1$  and  $j_2 \leq s_2$ . As  $l_{i,j_i}$  is adjacent to  $l_i$  in  $\mathcal{G}_i$ ,  $c$  and  $a_i$  are adjacent in  $\mathcal{G}$ . This forces  $d(a_1, a_2) \leq 2$ .

Next, let  $(a, b, c)$  be a path of  $\mathcal{G}$  and, denoted by  $l$  and  $m$  the 1-elements incident to  $\{a, b\}$  and, respectively,  $\{b, c\}$ , suppose that  $d(l, m) = 2$  in  $\mathcal{G}_b := \mathcal{S}_{1,2}(\text{Res}_\Gamma(b))$ . Let  $X$  be the group induced on  $\text{Res}_\Gamma(b)$  by the stabilizer of  $a$  and  $b$  in  $\text{Aut}(\Gamma)$  and  $u$  be the 4-element of  $\text{Res}_\Gamma(b)$  incident to the 22 1-elements of  $\text{Res}_\Gamma(b)$  at distance 2 from  $l$  in  $\mathcal{G}_b$ . Then  $X$  is isomorphic to  $M_{22}$  or  $\text{Aut}(M_{22})$ , it stabilizes  $l$  and  $u$  and permutes the 22 1-elements of  $\text{Res}_\Gamma(b, u)$  transitively. It follows from this remark that  $\mathcal{G}$  is either a complete graph with  $101 = 1 + 100$  vertices or a strongly regular graph with parameters  $k = 100, \lambda = 77$  and a divisor of 2200 as  $\mu$ .

For  $i = 0, 1, 2, 3, 4$ , let  $N_i$  be the number of  $i$ -elements of  $\Gamma$ . By elementary counting one can see that  $100N_0 = 23N_4$  and  $3850N_0 = 3N_2$ . Hence  $N_0$  is a multiple of  $69 = 3 \cdot 23$ . As 69 does not divide 101,  $\mathcal{G}$  is not a complete graph. Hence it is strongly regular, with  $N_0 = 101 + 2200/\mu$  vertices. However, no divisor  $\mu$  of 2200 exists such that  $101 + 2200/\mu$  is a multiple of 69. We have reached a final contradiction.  $\square$

**Theorem 9.2.** *Suppose that  $\Gamma$  belongs to diagram  $\text{Tr}_\uparrow(D_N^{h,k})_q$  of Subsection 2.1:*



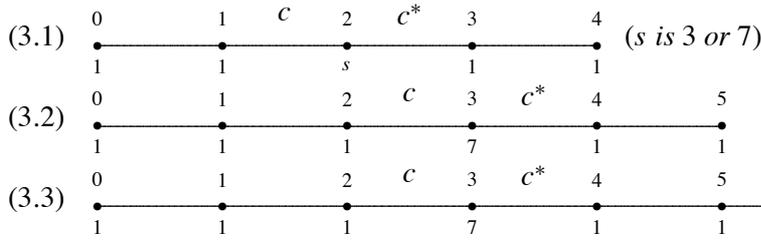


(We recall that  $h + k + 1 = n$  and  $h, k \geq 2$ .) Assume  $h \geq k$ , as we may. Then one of the following holds:

- (1)  $\Gamma$  is 2-covered by  $\text{Tr}_\uparrow(\Delta)$  for a  $D_N^{h,k}$ -building  $\Delta$  (and, if  $D_N^{h,k}$  is spherical and  $q > 1$ , then  $\Gamma = \text{Tr}_r \uparrow(\Delta)$ ).
- (2) We have  $q = 1$  and  $\Gamma$  is one of the following exceptional  $c^i.c^*$ -geometries:

	name	$n$	$s$	reference	notes
(2.1)	$\Gamma_{12}(M_{11})$	4	3	Example 6.4	
(2.2)	$\Gamma_{\text{Gol}}(2M_{11})$	4	7	Example 6.5	
(2.3)	$\Gamma_{\text{Gol}}(2M_{12})$	5	7	Example 6.5	
(2.4)	$\Gamma_{11}(M_{11})$	4	7	Example 6.5	quotient of (2.2)
(2.5)	$\Gamma_{12}(M_{12})$	5	7	Example 6.5	quotient of (2.3)

- (3) [open case]  $\Gamma$  belongs to one of the following diagrams:

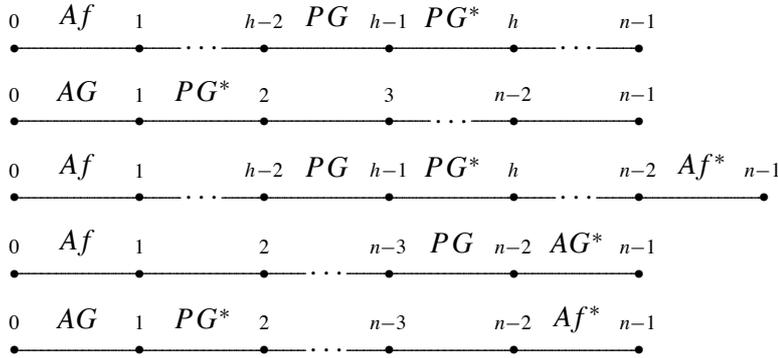


Residues of type  $c^2.c^*$ ,  $c.c^{*2}$ ,  $c^3$ ,  $c^*$  (in (3.2) and (3, 3)) and  $c.c^{*3}$  (in (3.3)) are as in (2).

*Proof.* If the  $\{h - 2, h - 1, h\}$ -residues of  $\Gamma$  are 2-covered by a truncated  $D_m^{2;2}$ -building ( $m = N - h - k + 4$ ) then we have (1) by Proposition 2.3. By Theorem 8.1(1) this is always the case when  $q > 1$ . Suppose  $q = 1$ . If  $k = 2$  then, by Meixner [54] and Ceccherini and Pasini [28], if (1) does not hold then (2) holds. Accordingly, if  $k > 2$  and (1) does not hold, then we have (3).  $\square$

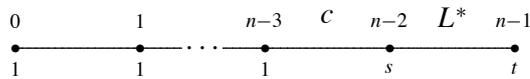
**Remark 9.1.** The four  $c^2.c^{*2}$ -geometries mentioned in Example 6.6 are the only examples we know for (3) of Theorem 9.2.

**Remark 9.2.** Diagrams  $\text{Tr}_\uparrow(Af.D_{N-1}^{h-1,k})$  and  $\text{Tr}_\uparrow(Af.D_{N-2}^{h-1,k-1}.Af^*)$  of subsections 2.2 and 2.3 would naturally come after  $\text{Tr}_\uparrow(D_N^{h,k})_q$ . We recall them here:



(Note that the last diagram is the dual of the fourth one.) By exploiting Theorem 8.1(1),(2) and imitating the proof of Proposition 2.3, one can prove that, if  $\Gamma$  belongs to  $\text{Tr}_\uparrow(Af.D_{N-1}^{h-1,k})$  or  $\text{Tr}_\uparrow(Af.D_{N-2}^{h-1,k-1}.Af^*)$ , then its chamber system is the  $\{n, n + 1, \dots, N - 1\}$ -truncation of a chamber system  $\mathcal{C}$  belonging to  $Af.D_{N-1}^{h-1,k}$  or  $Af.D_{N-2}^{h-1,k-1}.Af^*$ . However, the world of  $Af.D_{N-1}^{h-1,k}$ - and  $Af.D_{N-2}^{h-1,k-1}.Af^*$ -geometries is apparently too wild for we can obtain sharper conclusions in general. (Compare subsections 2.2, 2.3 and 3.2).

**Theorem 9.3.** *Suppose that  $\Gamma$  has diagram and orders as follows*



(We assume  $1 < t$  so that to keep the cases covered by the above diagram distinct from the special case  $c^{n-1}.c^*$  of  $\text{Tr}_\uparrow(D_N^{h,k})$ , considered in Theorem 9.2). Then one of the following holds:

- (1)  $n \leq 5$ ,  $(s, t) = (4, 3)$  and  $\Gamma$  is one of the exceptional  $c^i.Af^*$ -geometries  $\Gamma_{\text{far}}(M_{22})$ ,  $\Gamma_{\text{far}}(M_{23})$ ,  $\Gamma_{\text{cl}}(2^4A_7)$ ,  $\Gamma_{\text{cl}}(2^4A_8)$  described in examples 6.2 and 6.3.
- (2) [open case]  $n = 4$  and the  $\{3, 2\}$ -residues of  $\Gamma$  are 1-dimensional linear spaces or affine spaces of dimension  $d > 2$ . In the latter case, the stabilizer in  $\text{Aut}(\Gamma)$  of a  $\{0, 1\}$ -flag  $F$  induces a 1-dimensional group of  $\text{Res}_\Gamma(F)$  (as in (5) of Theorem 8.1).

*Proof.* In view of Theorem 1.2 and the assumption  $t > 1$ , the label  $L^*$  on rightmost stroke of the diagram can only stand for  $PG^*$  (in which case  $t = 2$ ),  $AG^*$  (in particular,  $Af^*$ ),  $U^*$  or  $1D^*$ . If  $L^* = Af^*$ , then we have (1) by [58]. Suppose  $L^* \neq Af^*$ . The following are sufficient to finish the proof of the theorem:

**Claim 1.** If  $L^* = AG^*$  with  $s > t + 1$ , then  $\Gamma$  is as in (2).

**Claim 2.** No flag-transitive  $c^3.L^*$ -geometry exists where the residues of the 0-elements are as in (2).

**Claim 3.**  $L^* \neq PG^*$ .

**Claim 4.**  $L^* \neq U^*$ .

We shall prove the following property of flag-transitive  $c^{n-2}.L^*$ -geometries before to turn to the proof of Claims 1, 2, 3 and 4.

**Claim 5.** The stabilizer  $G_p$  in  $\text{Aut}(\Gamma)$  of a 0-element  $p$  acts faithfully on  $\text{Res}_\Gamma(p)$ .

If  $n = 3$  then the above statement is Lemma 2.8 of Huybrechts and Pasini [46]. When  $n > 3$  (as we have assumed since the beginning of this subsection) we can prove Claim 5 by induction: Given a 0-element  $a$ , let  $l$  be a 1-element incident to  $a$ , and  $b$  be the 0-element of  $l$  different from  $a$ . For  $x = a, b, l$ , let  $K_x$  be the kernel of the action of  $G_x$  on  $\text{Res}_\Gamma(x)$ . Clearly,  $K_a$  is a normal subgroup of  $K_l$  and  $K_a K_b / K_a \leq K_l / K_a$ . However,  $K_l / K_a = 1$  by the inductive hypothesis on  $\text{Res}_\Gamma(a)$ . Therefore  $K_b \leq K_a$ . By symmetry,  $K_a = K_b$ . By the connectedness of  $\mathcal{S}_{0,1}(\Gamma)$ ,  $K_a = K_x$  for every 0-element  $x$ . Hence  $K_a = 1$ . Claim 5 is proved.

We are ready to prove Claim 1. Let  $\Gamma$  be as in the hypotheses of Claim 1 but assume that, for a  $\{0, 1\}$ -flag  $F = \{a, l\}$  of  $\Gamma$ , the group induced on  $\text{Res}_\Gamma(F)$  by the stabilizer  $G_F$  of  $F$  in  $G = \text{Aut}(\Gamma)$  is not 1-dimensional. Then, by Theorem 8.1,  $\text{Res}_\Gamma(a)$  is isomorphic to either  $\Gamma(2^{12}3M_{22})$  or  $\Gamma(2^{13}3M_{22})$  (cases (4.8) and (4.9) of Theorem 8.1). For a 3-element  $u \in \text{Res}_\Gamma(a)$ ,  $G_{u,a}/K_a$  is a non-split extension  $Z_3X$  with  $X$  isomorphic to either  $M_{22}$  or  $\text{Aut}(M_{22})$  and  $Z_3 = K_{a,u}/K_a$ , kernel of the action of  $G_{a,u}/K_a$  on  $\text{Res}_\Gamma(a, u)$  (Huybrechts and Pasini [48]). However,  $K_a = 1$  by Claim 5. Hence  $G_u = Z_3X$  with  $Z_3 = K_u$ . Therefore, as  $\text{Res}_\Gamma(u)$  contains exactly 23 elements of type 0,  $G_u$  can only be a non-split extension of  $M_{23}$  by  $Z_3$ . This is a contradiction, as  $M_{23}$  has trivial Schur multiplier [30]. So, Claim 1 is proved. We now turn to Claim 2. Suppose  $\Gamma$  is a flag-transitive  $c^3.L^*$ -geometry where the residues of the 0-elements are

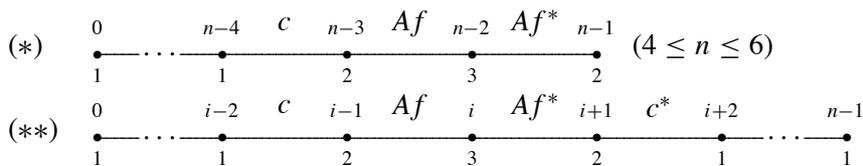
as in (3). Then, given a 4-element  $u$ ,  $G_u/K_u$  acts 4-transitively on the  $s + 4$  elements of  $\text{Res}_\Gamma(u)$  of type 0, with two-points stabilizer involved in  $Z_{q^d-1}Z_f$ , where  $f = dr$ ,  $q = p^r$ ,  $p$  prime. No 4-transitive permutation group exists with these properties. Claim 2 is proved.

Turning to Claim 3, assume that  $L^* = PG^*$ . We may assume  $n = 4$ . By Theorem 1.2,  $t = 2$  and  $s = 2^d - 2$  for some  $d > 2$ . Therefore, for a 3-element  $u$  of  $\Gamma$ ,  $G_u/K_u$  acts a 3-transitive permutation group of degree  $2^d + 1$  on the set of 0-elements incident to  $u$ . However, if  $a$  is one of those 0-elements,  $G_{a,u}/K_u$  is isomorphic to  $AG(n, 2)$  (Huybrechts and Pasini [47]). No 3-transitive group exists with that degree and point-stabilizer like that. Claim 3 is proved. Finally, if  $\Gamma$  is a  $c^2.U^*$ -geometry then, given a  $\{0, 3\}$ -flag  $F = \{a, u\}$ , we have  $\text{Res}_\Gamma(a) \cong \Gamma(J_2)$  and  $G_{a,u} \cong Z_3X$ , where  $Z_3 = K_u$  and  $X$  is isomorphic to either  $PGL_2(9)$  or  $P\Gamma L_2(9)$ , acting on the 10 elements of  $\text{Res}_\Gamma(F)$  of type 1 as on the 10 points of  $PG(1, 9)$  (Huybrechts and Pasini [46]). Consequently,  $G_u/K_u$  is a 3-transitive group of degree 11 with point-stabilizer  $X$  as above. The group that gets closest to those requirements is  $M_{11}$  on 11 points, but the point-stabilizer of such a group is  $M_{10}$ , which is different from either of  $PGL_2(9)$  and  $P\Gamma L_2(9)$  (although it is a subgroup of  $P\Gamma L_2(9)$ ). Thus, Claim 4 is proved, too.  $\square$

**Remark 9.3.** Suppose that, as conjectured in Remark 8.1, the two gluings of  $AS(5, 2)$  with  $AS(3, 5)$  are the only possibilities for case (5) of Theorem 8.1. If so, it is not difficult to prove that  $L^* \neq AG^*$  in case (2) of Theorem 9.3.

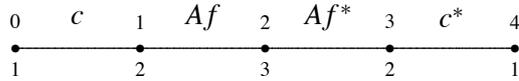
**Theorem 9.4.** *Suppose that at least one of the residues of  $\Gamma$  of rank 2 is isomorphic to  $AG(2, 3)$  or its dual and that  $\Gamma$  does not belong to any of the diagrams  $Af.A_{n-2}.Af^*$ ,  $\text{Tr}_\uparrow(Af.D_{N-1}^{h-1,k})$  or  $\text{Tr}_\uparrow(Af.D_{N-2}^{h-1,k-1}.Af^*)$ , already considered in Theorem 9.1 and Remark 9.2. Then  $n \leq 6$  and  $\Gamma$  is isomorphic to  $\Gamma(3^4\Gamma L_2(9))$ ,  $\Gamma(3^5M_{11})$  or  $\Gamma(3^62M_{12})$  (see Subsection 3.7), or to the dual of one of these geometries.*

*Proof.* By combining Theorem 1.1 with theorems 1.2, 8.1 and 9.3, we can see that the diagram of  $\Gamma$  or its dual can only be as in one of the following two pictures:



with  $5 \leq n \leq 9$  in (\*\*). It is proved in [63] that, in case (\*),  $\Gamma$  or its dual are isomorphic to  $\Gamma(3^4\Gamma L_2(9))$ ,  $\Gamma(3^5M_{11})$  or  $\Gamma(3^62M_{12})$ . It remains to prove that

case (\*\*) is empty. By way of contradiction, let  $\Gamma$  be a flag-transitive geometry with diagram and orders as follows:



We state the following conventions: For a type  $i$  and an element  $x$  of type  $t(x) \neq i$ ,  $\sigma_i(x)$  is the set of  $i$ -elements incident with  $x$ , and we put  $\sigma_i(x, y) := \sigma_i(x) \cap \sigma_i(y)$ . We denote by  $\mathcal{G}(\Gamma)$  the graph with the 0-elements of  $\Gamma$  as vertices, two 0-elements  $x, y$  being adjacent in  $\mathcal{G}(\Gamma)$  precisely when  $\sigma_1(x, y) \neq \emptyset$ . The adjacency relation of  $\mathcal{G}(\Gamma)$  will be denoted by  $\sim$ . The neighbourhood in  $\mathcal{G}(\Gamma)$  of a 0-element  $x$  will be denoted by  $x^\perp$ . Given two 0-elements  $x, y$ , we put  $\{x, y\}^\perp := x^\perp \cap y^\perp$ .

In view of what we know on case (\*), the residue  $\text{Res}(x)$  of a 0-element  $x$  is a subgeometry of an affine geometry  $\mathcal{A}(x) \cong AG(4, 3)$ . We denote by  $\mathcal{A}^\infty(x)$  the geometry at infinity of  $\mathcal{A}(x)$  and by  $\mathcal{A}^{\infty*}(x)$  the dual of  $\mathcal{A}^\infty(x)$ . The set of lines of  $\mathcal{A}(x)$  that (do not) correspond to elements of  $\sigma_2(x)$  will be denoted by  $\mathcal{L}^+(x)$  (respectively,  $\mathcal{L}^-(x)$ ). For every line  $L$  of  $\mathcal{A}(x)$ ,  $L^\infty$  stands for the point at infinity of  $L$ . Similarly, if  $X$  is an affine subspace of  $\mathcal{A}(x)$  of dimension  $> 1$ , we denote by  $X^\infty$  its space at infinity. Thus, the set  $O(x) := \{X^\infty\}_{X \in \sigma_4(x)}$  is an elliptic quadric of  $\mathcal{A}^{\infty*}(x)$  and  $O^*(x) := \{L^\infty\}_{L \in \mathcal{L}^-(x)}$  is the set of tangent planes of  $O(x)$ .

**Claim 1.** The graph  $\mathcal{G}(\Gamma)$  has diameter  $d \leq 2$ .

Suppose to the contrary that  $d > 2$  and let  $x, y$  be 0-elements at distance 3 in  $\mathcal{G}(\Gamma)$ . Let  $x = x_0 \sim x_1 \sim x_2 \sim x_3 = y$  be a 3-path of  $\mathcal{G}(\Gamma)$  from  $x$  to  $y$ . For  $i = 1, 2, 3$ , choose  $v_i \in \sigma_1(x_{i-1}, x_i)$  and, for  $i = 1, 2$ , let  $L_i$  be the line of  $\mathcal{A}(x_i)$  through  $v_i$  and  $v_{i+1}$ . As  $d(x_{i-1}, x_{i+1}) = 2$ ,  $L_i \in \mathcal{L}^-(x_i)$ . Let  $X_i$  be the unique 4-element of  $\sigma_4(v_i, v_{i+1})$ . This element is uniquely determined by the following condition:  $X_i$  is a 3-space of  $\mathcal{A}(x_i)$  and regarded  $L_i^\infty$  and  $X_i^\infty$  as a plane and a point of  $\mathcal{A}^{\infty*}(x_i)$ , we have  $X_i^\infty = L_i^\infty \cap O(x_i)$ . (Recall that  $L_i^\infty$  is a tangent plane of  $O(x_i)$ .) The 3-space  $X_i$  contains  $3^3(3 + 1)$  lines of  $\mathcal{L}^+(x_i)$ . For  $j \in \{i, i + 1\}$ , let  $\mathcal{L}^+(X_i, v_j)$  be the set of lines of  $\mathcal{L}^+(x_i)$  contained in  $X_i$  and passing through  $v_j$ . Then  $|\mathcal{L}^+(X_i, v_j)| = 3^2 + 3$  and every line of  $\mathcal{L}^+(X_i, v_{i+1})$  is concurrent with exactly 2 lines of  $\mathcal{L}^+(X_i, v_i)$ . More explicitly, for  $j \in \{i, i + 1\}$  we can partition  $\mathcal{L}^+(X_i, v_j)$  in 4 mutually disjoint subsets  $\mathcal{L}_{0,j}, \mathcal{L}_{1,j}, \mathcal{L}_{2,j}, \mathcal{L}_{3,j}$ , each of size 3, in such a way that, for  $k = 0, 1, 2, 3$ ,  $\mathcal{L}_{k,i}$  and  $\mathcal{L}_{k,i+1}$  are two bundles of lines (with centers  $v_i$  and  $v_{i+1}$ ) contained in the same plane  $\pi_k$  of  $\mathcal{A}(x_i)$ . The line at infinity  $\pi_k^\infty$  of  $\pi_k$ , regarded as a line of  $\mathcal{A}^{\infty*}(x_i)$ , is tangent to  $O(x_i)$  in  $X_i^\infty$  and the  $q$  planes of  $\mathcal{A}^{\infty*}(x_i)$  on  $\pi_k^\infty$

different from  $L_i^\infty$  are secant for  $O(x_i)$  (whence, regarded as points of  $\mathcal{A}^\infty(x_i)$ , they are points at infinity of lines of  $\mathcal{L}^+(x_i)$ ).

We now consider  $X_1$  and  $X_2$ . As  $v_2 \in \sigma_1(X_1, X_2)$ ,  $\sigma_3(X_1, X_2) \neq \emptyset$ . Indeed, an element  $Y \in \sigma_3(X_1, X_2)$  can be found in  $\text{Res}(x_2, v_2)$  ( $= \text{Res}(x_1, v_2)$ ); recall also that  $\text{Res}(x_2, v_2)$  is a dual inversive plane.) Choose a 2-element  $L \in \sigma_2(v_2, Y)$ . By the previous paragraph,  $L$  meets 2 lines of  $|\mathcal{L}^+(X_1, v_1)|$  in  $\mathcal{A}(x_1)$ . Similarly,  $L$  meets 2 lines of  $|\mathcal{L}^+(X_2, v_3)|$  in  $\mathcal{A}(x_2)$ . Accordingly, there exist 2 elements  $v_{1,1}, v_{2,1} \in \sigma_1(x_1, L)$  such that  $\sigma_2(v_{i,1}, v_1) \neq \emptyset$  for  $i = 1, 2$ , and 2 elements  $v_{1,2}, v_{2,2} \in \sigma_1(x_2, L)$  such that  $\sigma_2(v_{i,2}, v_3) \neq \emptyset$  for  $i = 1, 2$ . We now turn to  $\text{Res}(L)$ . The 0- and 1-elements incident with  $L$  form a circular space with 4 points. Clearly,  $\sigma_0(v_{i,1}, v_{j,2}) \neq \emptyset$  for at least one pair  $(i, j)$ . Assume that  $\sigma_0(v_{1,1}, v_{1,2}) \neq \emptyset$ , to fix ideas. Let  $z \in \sigma_0(v_{1,1}, v_{1,2})$ ,  $M_1 \in \sigma_2(v_{1,1}, v_1)$  and  $M_2 \in \sigma_2(v_{1,2}, v_3)$ . In  $\text{Res}(M_1)$  we see that  $z \sim x_0$  ( $= x$ ) and in  $\text{Res}(M_2)$  we see that  $z \sim x_3$  ( $= y$ ). Therefore,  $x$  and  $y$  have distance  $d(x, y) \leq 2$ . Claim 1 is proved.

**Claim 2.**  $|\sigma_1(x, y)| = 1$  for every edge  $\{x, y\}$  of  $\mathcal{G}(\Gamma)$ .

For an edge  $\{x, y\}$  of  $\mathcal{G}(\Gamma)$ , put  $\lambda := |\sigma_1(x, y)|$ . By the flag-transitivity of  $\Gamma$ , the number  $\lambda$  does not depend on the choice of the edge  $\{x, y\}$ . Suppose  $\lambda > 1$ . Then the  $\lambda$  elements of  $\sigma_1(x, y)$  are contained in a unique line of  $\mathcal{L}^-(x)$ . Let  $G_x$  be the stabilizer of  $x$  in  $\text{Aut}(\Gamma)$  and, given a 1-element  $v \in \sigma_1(x, y)$ , let  $G_{x,v}$  be the stabilizer of  $v$  in  $G_x$ . Then  $G_{x,v}$  stabilizes  $\sigma_1(x, y)$ , hence the group  $\overline{G}_{x,v}$  induced by  $G_{x,v}$  on  $\mathcal{A}(x)$  stabilizes  $L$ . However  $L^\infty$ , regarded as a plane of  $\mathcal{A}^{\infty*}(x)$ , is tangent to  $O(x)$ . Hence the group  $\overline{G}_{x,v}^\infty$  induced by  $\overline{G}_{x,v}$  on  $\mathcal{A}^{\infty*}$  stabilizes the point  $p^\infty := L^\infty \cap O(x)$ . On the other hand, the points of  $O(x)$ , regarded as planes of  $\mathcal{A}^\infty(x)$ , are the planes at infinity of the 3-spaces of  $\mathcal{A}(x)$  corresponding to the elements of  $\sigma_4(x)$ . Therefore  $G_{x,v}$  stabilizes the unique 4-element of  $\sigma_4(v)$  which, regarded as a 3-space of  $\mathcal{A}(x)$ , contains  $v$  and has  $p^\infty$  as the plane at infinity. This contradicts the flag-transitivity of  $\text{Aut}(\Gamma)$ . Claim 2 is proved.

**Claim 3.** The 4-truncation of  $\Gamma$ , obtained by removing all elements of  $\Gamma$  of type 4, satisfies the Intersection Property. In particular, for a 1-element  $u$  and a 2-element  $X$ , if  $\sigma_0(u) \subset \sigma_0(X)$  then  $u \in \sigma_1(X)$ .

This follows from Claim 2 and [56, Lemma 7.25].

**Claim 4.** The graph  $\mathcal{G}(\Gamma)$  is regular with valency 81.

This also follows from Claim 2.

**Claim 5.**  $|\{x, y\}^\perp| = 2(3^3 + 3) + \omega$  for every edge  $\{x, y\}$  of  $\Gamma$ , where  $\omega \in \{0, 1, 2\}$  does not depend on the particular choice of the edge  $\{x, y\}$  and

it is equal to the number of 0-elements  $z \in \{x, y\}^\perp$  such that  $z \notin \sigma_0(X)$  for any  $X \in \sigma_2(x, y)$ .

By Claim 3, for a 0-element  $z \in x^\perp$  we have  $\sigma_2(x, y, z) \neq \emptyset$  if and only if the 1-elements  $xy$  and  $xz$  incident with  $\{x, y\}$  and  $\{x, z\}$ , regarded as points of  $\mathcal{A}(x)$ , belong to a common line of  $\mathcal{L}^+(x)$ . Hence  $2(3^3 + 3)$  is the number of elements  $z \in \{x, y\}^\perp$  such that  $\sigma_0(x, y, z) \neq \emptyset$ . On the other hand, the stabilizer  $G_{x,y}$  of  $x$  and  $xy$  in  $\text{Aut}(\Gamma)$  induces a transitive action on  $O(x)$ . Accordingly, it acts transitively on the set of lines of  $\mathcal{L}^-(x)$  through  $xy$ . (Recall that the points at infinity of these lines, regarded as planes of  $\mathcal{A}^{\infty*}(x)$ , are the  $3^2 + 1$  tangent planes of  $O(x)$ .) Therefore, each of those lines contains the same number  $\omega < 2$  of points  $u = xz$  of  $\mathcal{A}(x)$  with  $z \in \{x, y\}^\perp$  (possibly,  $\omega = 0$ ). Claim 5 is proved, too.

We can now finish the proof of the theorem. Given a 0-element  $x$ , let  $G_x$  be its stabilizer in  $\text{Aut}(\Gamma)$  and  $\overline{G}_x$  be the group induced by  $G_x$  in  $\mathcal{A}(x)$ . Then  $\overline{G}_x$  contains the full translation group  $T$  of  $\mathcal{A}(x)$ . Given a line  $L \in \mathcal{L}^-(x)$ , the stabilizer  $T_L$  of  $L$  in  $T$  acts transitively on the triple of unordered pairs of points of  $L$ . It follows that  $G_x$  is transitive on the set of unordered pairs  $\{u, v\}$  of 1-elements of  $\sigma_1(x)$  such that  $\sigma_2(u) \cap \sigma_2(v) = \emptyset$ . Therefore, with  $\omega$  as in Claim 5, one of the following holds:

- (A)  $\omega = 2$  and  $\mathcal{G}(\Gamma)$  is a complete graph;
- (B)  $\omega = 0$ ,  $\mathcal{G}(\Gamma)$  has diameter  $d = 2$  and  $\text{Aut}(\Gamma)$  acts transitively on the set of pairs  $(y, \{x, z\})$  where  $\{x, y\}$  and  $\{y, z\}$  are edges of  $\mathcal{G}(\Gamma)$  but  $z \notin x^\perp$ .

We shall prove that either of the above leads to a contradiction.

Case A. In this case  $\Gamma$  has  $3^4 + 1 = 82$  elements of type 0 and the group  $G := \text{Aut}(\Gamma)$  acts faithfully on them as a 2-transitive group. The stabilizer  $G_x$  of a 0-element  $x$  is a subgroup of the stabilizer of  $O(x)$  in  $\text{Aut}(\mathcal{A}(x))$  and contains a split extension  $T : L$ , where  $T$  is the translation group of  $\mathcal{A}(x)$  and  $L \cong SO^-(4, 3)$ . So,  $3^4 : SO^-(4, 3) \leq G_x \leq 3^4 : \Gamma O^-(4, 3)$ . However, there is no 2-transitive group of degree 82 with point-stabilizer as above (compare Cameron [25]). So, Case A is impossible.

Case B. Assume that case B holds. Then  $\mathcal{G}(\Gamma)$  is strongly regular with valency  $k = 3^4$  and  $\lambda = (3^3 + 3)2 = 60$ , where  $\lambda := |\{x, y\}^\perp|$  for an edge  $\{x, y\}$  of  $\mathcal{G}(\Gamma)$ . Put  $\mu := |\{x, z\}^\perp|$ , for two 0-elements  $x, z$  at distance 2 in  $\mathcal{G}(\Gamma)$ . Then

$$(1) \quad \mu \text{ divides } k(k - \lambda - 1) = 3^4(3^4 - 61) = 3^4 \cdot 20.$$

Clearly,

$$(2) \quad \mu \leq 3^4 = k.$$

Given two 0-elements  $x, z$  at distance 2 and  $y \in \{x, z\}^\perp$ , let  $yx$  and  $yz$  be the 1-elements incident with  $\{y, x\}$  and  $\{y, z\}$  respectively. Regarded  $yx$  and  $yz$  as points of  $\mathcal{A}(y)$ , by the same argument used in the first part of the proof of Claim 1 we see that exactly  $(3 + 1)3(3 - 1) = 24$  points of  $\mathcal{A}(y)$  are joined with either of  $yx$  and  $yz$  by lines of  $\mathcal{L}^+(y)$ . Therefore,

(3)  $25 \leq \mu$ .

By comparing (3) with (1) we see that

(4) 3 divides  $\mu$ .

By a well known condition on strongly regular graphs, the equation

$$(\circ) \quad t^2 + (\mu - 60)t + (\mu - 81) = 0$$

has integral solutions. (Recall that  $k = 81$  and  $\lambda = 60$ .) Let  $t$  be a solution of equation  $(\circ)$ . By (4) and the fact that  $t$  is integral, 3 divides  $t$ . Hence  $3^2$  divides  $\mu - 81$ . Therefore:

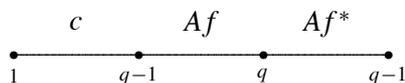
(5)  $3^2$  divides  $\mu$ .

By (1), (2), (3) and (5) we obtain the following possibilities for  $\mu$ :

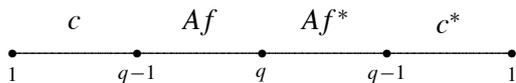
$$3^2 \cdot 2 = 18, \quad 3^2 \cdot 4 = 36, \quad 3^2 \cdot 5 = 45, \quad 3^3 \cdot 2 = 54, \quad 3^4 = 81.$$

It is easily seen that  $(\circ)$  admits integral solutions only if  $\mu = 3^4$  (and 0,  $-21$  are its solutions in that case). So,  $\mu = 3^4 = k$ . Hence  $\mathcal{G}(\Gamma)$  is a complete  $N$ -partite graph with  $1 + 81 + 20 = 102$  vertices and classes of size  $1 + 3^4 \cdot 20/\mu = 21$ . Needless to say,  $N = 102/21$ . However, 21 does not divide 102. We have reached a final contradiction.  $\square$

**Remark 9.4.** It is likely that all flag-transitive geometries belonging to the following diagram



are dually isomorphic to affine expansions of classical inversive planes. Some partial results pointing at that conjecture can be found in [63]. It is also likely that no flag-transitive geometry exists with diagram as follows where  $q > 2$ :



**10. Remaining open cases.**

**Proposition 10.1.** *Suppose that neither  $\Gamma$  nor its dual belong to any of the diagrams considered in theorems 9.1, 9.2, 9.3, 9.4 or Remark 9.2. Then  $\Gamma$  belongs to one of the following diagrams or its dual, where the symbol  $N$  is a specialization of  $1D$  and denotes the class of Netto triple systems.*

$$(1) \quad \begin{array}{ccccccc} 0 & c & 1 & AG & 2 & X^* & 3 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & & q-1 & & s & & t \end{array}$$

where  $X^*$  stands for  $AG^*$ ,  $W^*$ ,  $c^*$  or  $1D^*$  and  $s = q$  when  $X^* = W^*$ . Moreover, if  $X^* = c^*$  then the duals of the  $\{1, 2, 3\}$ -residues are as in (4.1), (4.2), ..., (4.7) or (5) of Theorem 8.1. In any case, the residues of type  $\{0, 1, 2\}$  are as in Theorem 1.1(6).

$$(2) \quad \begin{array}{ccccccc} 0 & c & 1 & N & 2 & X^* & 3 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & & 2 & & s & & t \end{array}$$

where  $X^*$  stands for  $AG^*$ ,  $W^*$ ,  $c^*$  or  $1D^*$ . Residues of type  $\{0, 1, 2\}$  are as in Theorem 1.1(7).

$$(3) \quad \begin{array}{cccccccc} 0 & c & 1 & X & 2 & Y^* & 3 & c^* & 4 \\ \bullet & \bullet \\ 1 & & r & & s & & t & & 1 \end{array}$$

where  $X$  stands for  $AG$  or  $N$  and  $Y^*$  stands for  $AG^*$  or the dual  $N^*$  of  $N$ . Residues of type  $\{0, 1, 2\}$  and  $\{2, 3, 4\}$  are as in cases (6) or (7) of Theorem 1.1.

$$(4) \quad \begin{array}{cccccccc} 0 & & 1 & c & 2 & X^* & 3 & c^* & 4 \\ \bullet & \bullet \\ 1 & & 1 & & s & & t & & 1 \end{array}$$

where  $X^*$  stands for  $AG^*$  or  $N^*$ . Residues of type  $\{0, 1, 2\}$  are as in Theorem 9.3(3) and those of type  $\{2, 3, 4\}$  are as in cases (6) or (7) of Theorem 1.1.

*Proof.* We obtain a list of feasible diagrams by combining Theorem 1.1 with theorems 1.2 and 9.3. According to our hypotheses,  $\Gamma$  belongs neither to  $\text{Tr}_\uparrow(Af.D_{N-1}^{h-1,k})$  nor  $\text{Tr}_\uparrow(Af.D_{N-1}^{h-1,k-1}.Af^*)$ . Therefore, if the diagram of  $\Gamma$  contains a piece as follows

$$\dots \bullet \xrightarrow{i-1} \bullet \xrightarrow{AG} \bullet \xrightarrow{i} \bullet \xrightarrow{PG^*} \bullet \xrightarrow{i+1} \bullet \dots$$

(or dual of this), then that piece can only occur in the following context (or its dual):

$$\dots \bullet \xrightarrow{i-2} \bullet \xrightarrow{c} \bullet \xrightarrow{i-1} \bullet \xrightarrow{AG} \bullet \xrightarrow{i} \bullet \xrightarrow{PG^*} \bullet \xrightarrow{i+1} \bullet \dots$$

Accordingly, and in view of Theorem 1.1(6), if  $\mathcal{X}$  is an  $\{i - 1, i\}$ -residue of  $\Gamma$  and  $G_{i-1,i}$  is the group induced on it by its stabilizer in  $\text{Aut}(\Gamma)$ ,  $G_{i-1,i}$  is 1-dimensional. On the other hand, by Theorem 8.1(2), if  $q$  and  $s = q^{d-1} + \dots + q^2 + q$  are the order of  $\Gamma$  at  $i + 1$  and  $i$ , then  $G_{i-1,i}$  induces at least  $L_d(q)$  on the  $i$ -panels of  $\mathcal{X}$ . We have reached a contradiction. Therefore, the diagram of  $\Gamma$  does not contain any piece of the form  $AG.PG^*$ . By assumption, it neither contains a stroke corresponding to  $AG(2, 3)$  or its dual, as this situation is considered in Theorem 9.4. Accordingly the diagram of  $\Gamma$  can only be as said in the proposition.  $\square$

**Remark 10.1.** Examples are known for the diagrams gathered in picture (1) of Proposition 10.1, but only with  $X^* \neq 1D^*$  and  $s = q = 4$  when  $X^* = c^*$  (see Subsections 3.6 and 5.2). No example is known for any of the remaining cases of Proposition 10.1.

#### REFERENCES

- [1] B. Baumeister, *Fahnen transitiven Rang 3 Geometrien, die Lokal Vollständige Graphen Sind*, Diplomarbeit, FU Berlin, 1992.
- [2] B. Baumeister, *Two new sporadic semiplanes related to  $M_{22}$* , European J. Combin., 15 (1994), pp. 325–336.
- [3] N. Baumeister, *On flag-transitive  $c.c^*$ -geometries*, in Groups, Difference Sets and the Monster (K.T. Arasu et al. eds.), de Gruyter, Berlin (1996), pp. 3–21.
- [4] B. Baumeister, *On flag transitive  $c.c^*$ -geometries admitting a duality*, Bull. Soc. Math. Belgique-Simon Stevin, 6 (1999), pp. 443–454.
- [5] B. Baumeister - F. Buekenhout, *On the classification of flag-transitive  $c.c^*$ -geometries*, Geom. Dedicata, 75 (1999), pp. 1–18.
- [6] B. Baumeister - A. Del Fra - T. Meixner - A. Pasini, *Flag-transitive  $c.Af^*$ -geometries*, Beiträge zur Algebra und Geometrie, 37 (1996), pp. 231–258.
- [7] B. Baumeister - T. Meixner - A. Pasini,  *$GF(2)$ -expansions*, Geom. Dedicata, 67 (1997), pp. 163–180.
- [8] B. Baumeister - D. Pasechnik, *The universal covers of the sporadic semiplanes*, European J. Combin., 17 (1996), pp. 595–604.
- [9] B. Baumeister - D. Pasechnik, *The universal covers of certain semiplanes*, European J. Combin., 18 (1997), pp. 491–496.
- [10] B. Baumeister - A. Pasini, *On flat flag-transitive  $c.c^*$ -geometries*, J. Alg. Combin., 6 (1997), pp. 2–26.

- [11] B. Baumeister - A. Pasini, *On  $C_2.L$ -geometries*, *Geom. Dedicata*, 71 (1998), pp. 33–59.
- [12] B. Baumeister - G. Stroth, *The non-canonical gluings of two affine spaces*, in *Groups and Geometries* (L. Di Martino et al. eds.), Birkhäuser (1998), pp. 9–28.
- [13] R. J. Blok - A. E. Brouwer, *The geometry far from a residue*, in *Groups and Geometries* (L. Di Martino et al. eds.), Birkhäuser (1998), pp. 29–38.
- [14] A. E. Brouwer - A. M. Cohen, *Some remarks on Tits's geometries*, *Indag. Math.*, 45 (1983), pp. 393–402.
- [15] A. E. Brouwer - A. M. Cohen - A. Neumaier, *Distance-Regular Graphs*, Springer, Berlin, 1989.
- [16] F. Buekenhout, *Diagrams for geometries and groups*, *J. Combin. Th. Ser. A*, 27 (1979), pp. 121–151.
- [17] F. Buekenhout, *Geometries for the Mathieu group  $M_{12}$* , in *Combinatorial Theory* (D. Jungnickel and K. Vedder eds.), Springer Lecture Notes in Math. 969 (1982), pp. 74–85.
- [18] F. Buekenhout, *Diagram geometries for sporadic groups*, in *Finite Groups Coming of Age* (J. McKay ed.), *Contemp. Math.*, 45 American Math. Soc. (1985), pp. 1–32.
- [19] F. Buekenhout - A. Delandtsheer - J. Doyen, *Finite linear spaces with flag-transitive automorphism groups*, *J. Combin. Theory Ser. A*, 49 (1988), pp. 268–293.
- [20] F. Buekenhout - A. Delandtsheer - J. Doyen - P.B. Kleidman - M.W. Liebeck - J. Saxl, *Linear spaces with flag-transitive automorphism groups*, *Geom. Dedicata*, 36 (1990), pp. 89–94.
- [21] F. Buekenhout - C. Huybrechts - A. Pasini, *Parallelism in diagram geometry*, *Bull. Soc. Math. Belgique-Simon Stevin*, 1 (1994), pp. 355–397.
- [22] F. Buekenhout - C. Huybrechts, *A  $c.L^*$ -geometry for the sporadic group  $J_2$* , *Geom. Dedicata*, 73 (1998), pp. 225–235.
- [23] F. Buekenhout - A. Pasini, *Finite diagram geometries extending buildings*, Chap. 22 of *Handbook of Incidence Geometry* (F. Buekenhout ed.), North-Holland, Amsterdam (1995), pp. 1143–1254.
- [24] M. Buratti - A. Del Fra, *Semi-Boolean Steiner quadruple systems and dimensional dual hyperovals*, *Adv. Geometry*, special issue dedicated to A. Barlotti (2003), pp. S254–S270.
- [25] P.J. Cameron, *Finite permutation groups and finite simple groups*, *Bull. London Math. Soc.*, 13 (1981), pp. 1–22.
- [26] P.J. Cameron - G. Korchmaros, *One-factorizations of complete graphs with a doubly transitive automorphism group*, *Bull. London Math. Soc.*, 25 (1993), pp. 1–6.

- [27] I. Cardinali - A. Pasini, *A characterization of truncated  $D_n$ -buildings as flag-transitive  $PG.PG^*$ -geometries*, in *Finite Geometries* (J.W.P. Hirschfeld et al. eds.), Kluwer Academic Press, Dordrecht (2001), pp. 99–119.
- [28] G. Ceccherini - A. Pasini, *Extending locally truncated geometries*, *J. Combin. Th. Ser. A*, 94 (2001), pp. 289–338.
- [29] B.N. Cooperstein - J.A. Thas, *On generalized  $k$ -arcs in  $PG(2n, q)$* , *Annals of Combin.*, 5 (2001), pp. 141–152.
- [30] J.H. Conway - R.T. Curtis - S.P. Norton - R.A. Parker - R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.
- [31] A. Delandtsheer, *Finite (line,plane)-flag transitive linear spaces*, *Geom. Dedicata*, 41 (1992), pp. 145–153.
- [32] A. Delandtsheer, *Dimensional linear spaces whose automorphism group is (line-hyperplane)-flag-transitive*, *Designs, Codes and Cryptography*, 1 (1992), pp. 237–245.
- [33] A. Del Fra, *On two new classes of semiplanes*, *Discrete Maths.*, 174 (1997), pp. 107–116.
- [34] A. Del Fra, *On  $d$ -dimensional dual hyperovals*, *Geom. Dedicata*, 79 (2000), pp. 157–178.
- [35] A. Del Fra - D. Ghinelli, *Diameter bounds for locally partial geometries*, *European J. Combin.*, 12 (1991), pp. 293–307.
- [36] A. Del Fra - A. Pasini, *On  $Af.Af^*$ -geometries*, *J. Geometry*, 54 (1995), pp. 15–29.
- [37] A. Del Fra - A. Pasini - S. V. Shpectorov, *Geometries with bi-affine and bi-linear diagrams*, *European J. Combin.*, 16 (1995), pp. 439–459.
- [38] A. Del Fra - S. Yoshiara, *Dimensional dual hyperovals associated with Steiner systems*, *European J. Combin.*, 26 (2005), pp. 173–194.
- [39] G. Grams - T. Meixner, *Some results about flag-transitive diagram geometries using coset enumeration*, *Ars Combin.*, 33 (1994), pp. 129–146.
- [40] D. R. Hughes, *Biplanes and semi-biplanes*, in *Combinatorial Mathematics* (D. Holton and J. Seberry eds.), Springer Lecture Notes in Math., 686 (1978), pp. 55–58.
- [41] D.R. Hughes, *Semi-symmetric 3-designs*, in *Finite Geometries* (N.L. Johnson et al. eds.), Marcel Dekker, New York (1983), pp. 223–235.
- [42] C. Huybrechts, *Réduction de Géométries de Type  $L.L^*$* , Ph. D Thesis, Université Libre de Bruxelles, année académique, 1996.
- [43] C. Huybrechts,  *$L.L^*$ -geometries and  $D_n$ -buildings*, *Discrete Math.*, 197/198 (1999), pp. 431–451.
- [44] C. Huybrechts, *A unified construction for  $c^*.c.$  and  $L.L^*$ -geometries in projective spaces*, *J. Geometry*, 67 (2000) pp. 145–163.

- [45] C. Huybrechts, *Dimensional dual hyperovals in projective spaces and  $c.AG^*$ -geometries*, Discrete Maths., 255 (2002), pp. 193–223.
- [46] C. Huybrechts - A. Pasini, *A characterization of the Hall-Janko group  $J_2$  by a  $c.L^*$ -geometry*, in Groups and Geometries (L. Di Martino et al. eds.), Birkhäuser (1998), pp. 91–106.
- [47] C. Huybrechts - A. Pasini, *Flag-transitive extensions of dual projective spaces*, Bull. Soc. Math. Belgique-Simon Stevin, 5 (1998), pp. 341–352.
- [48] C. Huybrechts - A. Pasini, *Flag-transitive extensions of dual affine spaces*, Beiträge zur Algebra und Geometrie, 40 (1999), pp. 503–531.
- [49] A.A. Ivanov - C.E. Praeger, *On finite affine 2-transitive arcs*, European J. Combin., 14 (1993), pp. 421–444.
- [50] Z. Janko - T. Van Trung, *Two new semiplanes*, J. Combin. Th. Ser. A, 33 (1982), pp. 102–105.
- [51] D. Leemans, *On a rank five geometry of Meixner for the Mathieu group  $M_{12}$* , Geom. Dedicata, 85 (2001), pp. 273–281.
- [52] C. Lefèvre-Percsy - L. Van Nypelseer, *Finite rank 3 geometries with affine planes and dual affine point residues*, Discrete Math., 84 (1990), pp. 161–167.
- [53] A. Makowski - A. Shinzel, *Sur l' équation indéterminée de R. Goormagitch*, Mathesis, 68 (1959), pp. 128–142.
- [54] T. Meixner, *Two diagram geometries related to  $M_{11}$* , Atti Sem. Mat. Fis. Univ. Modena, 44 (1996), pp. 209–227.
- [55] A. Neumaier, *Rectagraphs, diagrams and Suzuki's sporadic simple groups*, Ann. Discrete Math., 15 (1982), pp. 305–318.
- [56] A. Pasini, *Diagram Geometries*, Oxford Univ. Press, 1994.
- [57] A. Pasini, *On the linearization of pure geometries in certain diagrams (cyclic diagrams and trees)*, Simon Stevin, 58 (1984), pp. 289–310.
- [58] A. Pasini, *Flag-transitive extensions of dual affine planes*, European J. Combin., 17 (1996), pp. 657–671.
- [59] A. Pasini, *Gluing two affine spaces*, Bull. Soc. Math. Belgique-Simon Stevin, 3 (1996), pp. 25–40.
- [60] A. Pasini, *Universal covers of geometries of far away type*, J. Alg. Combin., 18 (2003), pp. 211–243.
- [61] A. Pasini, *Extending locally truncated chamber systems by sheaves*, Adv. Geometry, special issue dedicated to A. Barlotti (2003), S75–S104.
- [62] A. Pasini, *A characterization of the geometry of large maximal cliques of the alternating forms graph*, to appear in Discrete Math.
- [63] A. Pasini, *A tower of geometries related to the ternary Golay codes*, Australasian J. Combin., 33 (2005), pp. 183–216.
- [64] A. Pasini - S. Yoshiara, *Flag-transitive Buekenhout geometries*, in Combinatorics '90 (A. Barlotti et al. eds.), North Holland, Amsterdam (1992), pp. 403–447.

- [65] A. Pasini - S. Yoshiara, *On a new family of flag-transitive semiplanes*, European J. Combin., 22 (2001), pp. 529–545.
- [66] A. Pasini - S. Yoshiara, *New distance regular graphs arising from dimensional dual hyperovals*, European. Combin., 22 (2001), pp. 547–560.
- [67] E.E. Shult, *Geometric hyperplanes of half-spin geometries arise from embeddings*, Bull. Soc. Math. Belgique-Simon Stevin, 1 (1994), pp. 439–454.
- [68] E.E. Shult, *On Veldkamp lines*, Bull. Soc. Math. Belgique-Simon Stevin, 4 (1997), pp. 299–316.
- [69] A. Sprague, *Extended dual affine planes*, Geom. Dedicata, 16 (1984), pp. 107–121.
- [70] A. Sprague, *A characterization of special Laguerre planes and extended dual affine planes*, Geom. Dedicata, 16 (1984), pp. 149–156.
- [71] H. Taniguchi,  *$d$ -Dimensional dual hyperovals in  $PG(d + n, 2)$  for  $d + 1 \leq n \leq 3d - 7$* , European J. Combin., 23 (2002), pp. 1079–1083.
- [72] J. Tits, *A local approach to buildings*, in The Geometric Vein (C. Davies et al. eds.), Springer, Berlin (1981), pp. 519–547.
- [73] L. Van Nypelseer, *Rank  $n$  geometries with affine hyperplanes and dual affine point residues*, European J. Combin., 12 (1991), pp. 561–566.
- [74] S. Yoshiara, *A family of  $d$ -dimensional dual hyperovals in  $PG(2d + 1, 2)$* , European J. Combin., 20 (1999), pp. 489–503.
- [75] S. Yoshiara, *Ambient spaces of dimensional dual arcs*, J. Alg. Combin., 19 (2004), pp. 5–23.

*Department of Mathematics,  
University of Siena,  
Via del Capitano 15, 53100 Siena (ITALY)  
e-mail: pasini@unisi.it*