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MAXIMAL SETS OF FACTORS

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In this paper, a survey is presented of results concerning maximal sets of factors in graphs. These factors at times satisfy additional structural constraints, such as being connected, or such as requiring each component of each factor to be isomorphic. Each set of factors occurs in some natural family of graphs, including complete graphs and complete multipartite graphs.

1. Introduction.

In this survey we consider some of the progress made in studying maximal sets of edge-disjoint factors in graphs of various types.

A set $S(G)$ of objects satisfying some property P defined on a graph G is said to be *maximal* if there exists no set $T(G)$ defined on G satisfying P such that $S(G)$ is a strict subset of $T(G)$. That is not to say $S(G)$ is necessarily the largest set of objects satisfying P in G; indeed if $|S(G)| \geq |S'(G)|$ for all sets of objects $S'(G)$ satisfying P then $S(G)$ is said to be a *maximum* set (with respect to P).

For example, if P is the property that $S(G)$ is a maximum matching M in G, then it is well known that this implies that G contains no M-augmenting path (i.e. a path in G beginning and ending with edges not in M in which every second edge is in M). Conversely, if M' is a maximal matching that is not maximum then G does contain an M'-augmenting path. Furthermore such a path can be used to form a matching in G that is bigger than M' (but of course does not contain M', since M' is maximal).

From an algorithmic point of view, maximal sets naturally arise as sets satisfying P that can be found using a greedy algorithm: form S by recursively looking for another object in G that can be added to S so that the resulting set also satisfies property P. Continuing our matching example, we can simply form S by recursively adding to S another edge in G that is not adjacent to any edge currently in S.

If one is really looking for a *maximum* set, then decisions made in greedily forming a maximal set may easily lead one astray! So a natural question arises: " What are the sizes of the maximal sets in G that satisfy property P? " Another way of thinking of this question is one more restricted view: "If we adopt a greedy approach, just how far away from a maximum set might be our result?" In the matching example, by using the observations about M-augmenting paths, it is easy to show that if M' is a maximal matching and M is a maximum matching, both in G, then $|M| \leq 2|M'|$. Furthermore, examples exist where equality holds, such as where each component of G is a path of length 3.

In this survey, we focus on this problem in the special case where property P requires at least that S be a set of edge-disjoint i-factors. Sometimes each i-factor will be required to be connected, so this takes us into the realm of hamilton cycles. The actual graphs G of interest will also vary. Most commonly G is considered to be either complete, or a complete multipartite graph. (We use $K_p(n)$ to denote the graph on np vertices, partitioned into p sets of size n each, where two vertices are joined if and only if they are in different parts of the partition. So $K_p(1) = K_p$.)

The methods used to study this problem are also of interest. There are two quite distinct phases leading to a complete solution. One is to construct maximal sets for various values, and this has been done using direct constructions, embeddings and amalgamations. The second phase is to show that no other sizes of maximal sets are possible. Sometimes this requires ad hoc arguments – otherwise known as " fiddling around with graphs "! Other times high powered graph theory must be involved, such as Tutte's f-factor theorem. This diversity of approaches makes the problem especially interesting, and hopefully makes this worth reading!

Let $K(s_1, \ldots, s_p)$ denote the complete p-partite graph with partition $\{V_1, \ldots, V_p\}$ of the vertex set in which $|V_i| = s_i$ for $1 \leq i \leq p$. As noted above, if all parts have the same size n, then we denote this by $K_p(n)$. If S is a set of edge-disjoint factors in G, then define the *deficiency* of S in G to be the graph formed from G by removing all edges of factors in S. If G is regular then the deficiency is regular of some degree, which we denote by d.

2. 1-Factors.

In this section we take a brief look at the possible sizes of S in the case where S is a set of maximal sets of edge-disjoint 1-factors in K_{2m}.

Of course, the *classic* 1-factorization of K_n, namely $\{\{\infty, i\}, \{i-1, i+1\}, \{i-2, i+2\}, \ldots, \{i-(m-1), i+(m-1)\} | i \in \mathbb{Z}_{2m-1}\}$ (reducing all calculations modulo $2m-1$) on the vertex set $\mathbb{Z}_{2m-1} \cup \{\infty\}$ produces the largest possible size of S, namely $|S| = 2m-1$.

At the other extreme, one can use the theorems of Dirac, and then Tutte to show the values of $|S|$ which are too small for S to be maximal [7], [24].

Theorem 2.1. ([7]) *If G has n vertices and minimum degree at least $n/2$, then G is hamiltonian.*

Of course, if G has an even number of vertices and is hamiltonian, then taking every second edge of a hamilton cycle forms a 1-factor. So any set of at most $m-1$ 1-factors in K_{2m} cannot be maximal, since removing these 1-factors leaves an m-regular graph which is necessarily hamiltonian. It turns out that one can also show that the complement of $2r$ edge-disjoint 1-factors in K_{4r} must contain a 1-factor. So

$$|S| \geq 2\lfloor m/2 \rfloor + 1. \tag{1}$$

Theorem 2.2. ([24]) *H contains a 1-factor if and only if there is no subset of vertices W in H whose removal leaves more than $w = |W|$ components of odd size.*

We can obtain a stronger bound on $|S|$ in the case where the deficiency H of S on K_{2m} has odd degree, $d = 2m-1-|S|$. Since S is maximal, H has no 1-factor, so contains a set W of $w \geq 1$ vertices described in Theorem 2.2; of the odd components in $G - W$, say γ have size more than d, and β have size at most d. Since G has an even number of vertices, $\gamma + \beta \neq w+1$, so $G - W$ has $\gamma + \beta \geq w+2$ odd components. Being d-regular, each of the β small components must be joined to W by at least d edges (if such a component contains x vertices, with $1 \leq x \leq d-1$, then it is joined to W by at least $xd - x(x-1) = d + (x-1)(d-x) \geq d$ edges) and each of the γ big components is joined to W by an edge (assuming G is connected: we can consider each component of G in turn to apply this argument); so $\gamma + \beta d \leq wd$. Therefore $\beta \leq w$, with strict inequality if $\gamma > 0$. But we observed that $\gamma + \beta \geq w+2$, so in fact $\gamma \geq 2$, so we actually get $\beta \leq w-1$, and thus $\gamma \geq 3$! Finally, there are $2m \geq w + (d+2)\gamma + \beta$ vertices altogether, (since $d+1$ is even, each large component of G must actually have at least $d+2$

vertices) so $2m \geq 3d + 7$. So this leads us to the realization that S being maximal implies that if $d = 2m - 1 - |S|$ is odd then

$$|S| = 2m - 1 - d \geq 4(m + 1)/3. \tag{2}$$

Rees and Wallis [21] not only established these necessary conditions (1-2), but also showed that they are sufficient. The proof of the sufficiency follows both from some specific constructions, and from some recursive constructions when $|S|$ is large.

In particular, if $d = 2m - 1 - |S|$ is even then for all $|S| \geq 2\lfloor m/2 \rfloor + 1$ it is straightforward to construct a set S of edge-disjoint 1-factors for which the deficiency contains a component that is $K_{2m-|S|}$. This clearly contains no 1-factor since $2m - |S|$ is odd, so S is maximal. ($S = \{\{\{\infty_i, i + j\} | 1 \leq i \leq 2m - |S|\} \cup \{\{1 - i + j, 2m - |S| + i + j\} | 1 \leq i \leq |S| - m\} \mid 0 \leq j \leq |S| - 1\}$ will suffice).

On the other hand, if d is odd then S can be constructed in the smallest case, namely $2m = 3d + 7$ (by 2), such that the deficiency contains a cut-vertex whose removal leaves 3 odd components; so S is maximal. An embedding result then settles the problem whenever $2m \geq 6d + 14$ in the following way. Any 1-factorization $\{F_1', \ldots, F_{3d+6}'\}$ of K_{3d+7} can be embedded in a 1-factorization $\{F_1, \ldots, F_{2m-1}\}$ of K_{2m}; so $F_i' \subseteq F_i$ for $1 \leq i \leq 3d + 6$. This is equivalent to embedding a unipotent symmetric latin square of order $t = 3d + 7$ in a unipotent symmetric latin square of order $2m$, which is possible since $2m \geq 2t$ [23], [22]. Let $S' = \{f_1, \ldots, f_{2d+6}\}$ be a maximal set of 1-factors in K_{3d+7} (so the deficiency G' of S' has degree d). Then $S = \{F_1 \cup f_1 \setminus F_1', \ldots, F_{2d+6} \cup f_{2d+6} \setminus F_{2d+6}, F_{3d+7}, \ldots, F_{2m-1}\}$ is a maximal set (since one component in the deficiency of S is G') of 1-factors in K_{2m} with deficiency of S having degree d.

The remaining cases to consider, namely when $3d + 9 \leq 2m \leq 6d + 12$, can be solved using a mixture of edge-coloring techniques that are applied to specific graph decompositions. The interested reader is directed to [21] for details. Together, these observations produce the following result.

Theorem 2.3. ([21]) *There exists a maximal set of k 1-factors in K_{2m} if and only if*

(1) $2\lfloor m/2 \rfloor + 1 \leq k \leq 2m - 1$ *if* $2m - k$ *is odd, and*
(2) $4(m + 1)/3 \leq k \leq 2m - 4$ *if* $2m - k$ *is even.*

3. 2-Factors with no restrictions.

Next we turn to considering the possible sizes of S in the case where S is a maximal set of 2-factors in K_n. The following result of Petersen immediately handles half the cases.

Theorem 3.1. ([14]) *If G is regular of even degree then G has a 2-factorization.*

If n is odd, then the deficiency of any set S of edge-disjoint 2-factors is regular of even degree d, so by Theorem 3.1 contains a 2-factor unless $d = 0$. So if n is odd, $|S| = (n-1)/2$ and S is a 2-factorization of K_n.

So now suppose that $n = 2m$. If S is a maximal set of 2-factors in K_{2m}, then the deficiency G is regular of odd degree, d, containing no 2-factor. But it therefore follows that G cannot contain *any* factor: if G contained an i-factor, then it also contains a $(d-i)$-factor, and since one of i and $d-i$ must be even, G would also contain a 2-factor by Petersen's Theorem. So the first thing to do to settle this problem is to find the orders of d-regular graphs that contain no 2-factors. As in the last section, we again turn to Tutte, this time to his f-factor. Following a similar but more complicated approach to that used in Section 2, one can obtain the following result. The graph $G(d)$ is the unique simple graph on $d+2$ vertices in which one vertex has degree $d-1$ and the rest have degree d (so d is odd).

Theorem 3.2. ([11]) *Let G be a simple regular graph of degree d that has no d'-factor, where $1 \leq d' < d$. Then d is odd and G has at least $(d+1)^2$ vertices, with equality if and only if G is the graph $H(d)$ in Figure 1.*

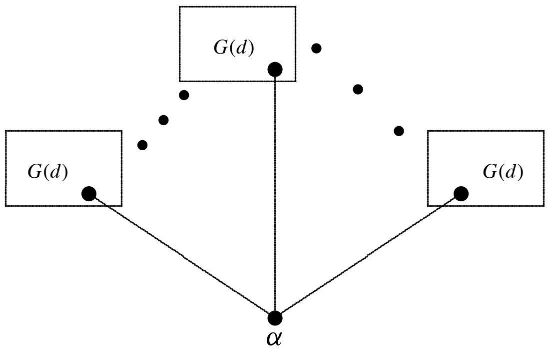

Figure 1: The unique smallest d-regular graph $H(d)$ that has no proper factors

Clearly each component $C_1, \ldots, C_d$ in $H(d) - \alpha$ has an odd number of vertices. This makes it easy to see why $H(d)$ has no proper factor. For if such a factor F existed, two of $C_1, \ldots, C_d$ would exist, say C_1 and C_2, such that

F contains the edge joining C_1 to α but does not contain the edge joining C_2 to α (because $d' < d$). But then in $F - \alpha$, the number of vertices of odd degree in the components induced by $V(C_1)$ and $V(C_2)$ would differ by 1, a contradiction. This same argument applies to the family of graphs formed by replacing one copy of $G(d)$ in $H(d)$ by any graph on $d + 2x$ vertices ($x \geq 1$) in which one vertex has degree $d - 1$ and the rest have degree d. Therefore we have the following result (since d is odd, there are no d-regular graphs on an odd number of vertices).

Theorem 3.3. ([11]) *There exists a d-regular simple graph on n vertices that contains no proper factor if and only if $n \geq (d+1)^2$, n is even and d is odd.*

So now, since by Petersen's Theorem the complement of such a graph has a 2-factorization, we have the following complete answer to our problem.

Theorem 3.4. *There exists a maximal set of k 2-factors in K_n if and only if*

(a) $k = (n - 1/2)$ *when n is odd, and*
(b) $(n - \sqrt{n})/2 \leq k \leq (n-2)/2$ *when n is even.*

4. 2-Factors, each is connected: hamilton cycles.

In this section we focus on a restricted type of 2-factor, namely we require that S be a maximal set of *connected* 2-factors. So now each 2-factor in S must be a hamilton cycle, and the deficiency $G(S)$ of S in K_n must contain no hamilton cycles.

As in Section 2, we can again turn to Dirac to guide us on a lower bound on the size of S. By Theorem 2.1, if $G(S)$ has minimum degree at least $n/2$ then it must contain a hamilton cycle, and so S would not be maximal as defined in this section. So we immediately see that $|S| \geq (n-1)/4$. However, it turns out that if $n \equiv 1 \pmod 4$ then $G(S)$ still necessarily contains a hamilton cycle: Nash-Williams (for example, see Exercise 4.2.10 in [2]) showed that $G(S)$ contains so many edges that any cycle of length less than n can still be slightly altered to form a longer cycle in $G(S)$. So through the vagaries of modular arithmetic, removing this possibility forces $|S| \geq \lfloor (n+3)/4 \rfloor$.

This turns out to be the right lower bound, and in fact any integer greater than this up to $\lfloor (n-1)/2 \rfloor$ is the order of a maximal set of hamilton cycles in K_n.

Theorem 4.1. ([11]) *There exists a maximal set of k hamilton cycles in K_n if and only if $\lfloor (n+3)/4 \rfloor \leq k \leq \lfloor (n-1)/2 \rfloor$.*

The construction of k suitable hamilton cycles follows closely the embedding ideas of Hilton developed in [9]. Hilton used amalgamation techniques, described below, to obtain a beautiful result that gave necessary and sufficient conditions for an edge-colored complete graph K_s to be embedded in an edge-colored copy of K_t in such a way that each color class induces a hamilton cycle (so the colors on the edges in the K_s cannot be altered; all one is free to do is to color the edges in K_t that are not in the given K_s). This result was one of the true " breakthroughs " in graph decompositions, opening many interesting questions and providing a powerful new construction technique.

To get a feel for Hilton's result, the necessary conditions pave the way. Clearly t must be odd for a hamilton decomposition to exist. Also, each color class in K_s must consist of vertex disjoint paths, since all the embedding process can do is to join the paths together to create a hamilton cycle. But then each color class in K_s cannot contain too many components (including possible isolated vertices as a component), because the most efficient use of the added $t - s$ vertices can each only join 2 such components – take one end from each of two paths of some color in K_s, and join each of them to a " new " vertex in K_t with an edge of the same color. So each color class can have at most $t - s$ components if they are to be eventually all hooked up to form a single hamilton cycle.

So how are the hamilton cycles constructed, and what are amalgamations? One starts by adding a single vertex α, joining it with $t - s$ edges to each vertex in K_s, and adding $\binom{t-s}{2}$ loops to α. These edges are then colored so that each vertex in K_s is incident with exactly 2 edges of each color, then the loops are colored so that α is incident with exactly $2(t - s)$ " edge ends " (loops have 2 edge ends) of each color. (The necessary conditions guarantee that this manoeuvre is possible). The beauty of the technique is then revealed as, one by one, vertices are " peeled out " from α. This involves selecting exactly one edge incident with each vertex $v \neq \alpha$ that joins v to α, detaching it from α and then reattaching it to the new " peeled out " vertex instead. Notice that each vertex in K_s is originally joined to α with $t - s$ edges, so potentially $t - s$ vertices can be peeled out; just the right number to form a K_t containing the K_s! Also, if i vertices have been peeled out from α so far, then exactly $t - s - i$ of the loops on α are chosen to have one end detached from α and joined to the new vetex instead; so all vertices other than α are joined to α with exactly $t - s - i$ edges after step i. Notice that the number of loops on α to begin is $\binom{t-s}{2} = (t - s - 1) + (t - s - 2) + \ldots + 1$; just the right number! Finally, it must be noted that great care must be exercised in deciding which edges to detach in this process, because afterwards we need to check that

(a) the new vertex is incident with exactly two edges of each color, and
(b) each color class is still connected.

It turns out that the first property can be achieved easily by using edge-coloring of bipartite multigraphs that satisfy several notions of balance. More work is required to maintain color classes that are connected, but the vital observation Hilton made was to see that the method could guarantee that if the edge ends incident with α were paired up in *any* way, then it was possible to ensure that *at most one* edge end from each pair would be detached from α. Disconnecting a color class would require selecting both of the *only* 2 edges joining some component of some color class to detach from α, so pairing up such edges made sure that it is *never* the case that both are selected. This paper of Hilton's is a gem, and certainly worth the read!

So how does that idea fit in here?

Whenever $2s \geq t \geq 1$, it is not too hard to find a $(2s-t)$-regular subgraph G of K_{2s}, the edges of which can be partitioned into s sets $S_1, \ldots, S_s$, each of size $2s - t$ and each of which induces a subgraph consisting of vertex-disjoint paths. (Using the classic hamilton decomposition of K_{2s} is one approach that works well). Using the Hilton embedding approach, each such subgraph is then incorporated into a hamilton decomposition of $G \vee K_t^c$ as follows. (K_t^c is the complement of K_t, and $\vee$ is the join operation where each vertex in G is joined to each vertex in K_t^c).

Name the s subgraphs $G_1, \ldots, G_s$, and color the edges of G_i with color i for $1 \leq i \leq s$. Add one new vertex α, and for each vertex $v \in V(G)$ and for $1 \leq i \leq s$, join v to α with $2 - y_{i,v}$ edges colored i, where $y_{i,v}$ is the number of edges in G colored i incident with v.

Notice that since G_i is acyclic, it contains $2s - (2s - t) = t$ components, so α is incident with exactly $2t$ edges colored i. So as α is peeled out into t vertices, it is conceivable that each such vertex is incident with exactly 2 edges of each color – that is a good thing if we expect each color class to be a hamilton cycle!

Notice also that since the t vertices peeled out from α in our case are intended to induce a copy of K_t^c, we do not want any loops on α. This, as you might guess, makes the choice of edge ends to detach from α at each step easier then in Hilton's result. It also points out the flexibility in his technique in that starting with a complete graph is by no means necessary for the procedure to work.

Returning to our problem, one can see that the hamilton cycles that each color class form must together be a maximal set, because their complement in K_{2s+t} is the disjoint union of G^c and K_t, a disconnected graph (G^c is the complement of G in K_{2s}). So a proof of Theorem 4.1 is obtained.

5. Hamilton cycles in $K_p(n)$.

Here we consider the situation where again S is a set of hamilton cycles, but each is in the complete multipartite graph $K_p(n)$; so S is maximal in the sense that $K_p(n) - E(S)$ contains no hamilton cycles ($E(S)$ is the set of edges occurring in the hamilton cycles in S).

So what is the lower bound on $|S|$? If $p = 2$, so $K_p(n)$ is bipartite, we use the fact that every connected subgraph of $K_{n,n}$ with minimum degree at least $n/2$ is hamiltonian (see [4], for example). This means that if $E(S)$ induces a graph that is regular of degree at most $n/2$, then the complement in $K_{n,n}$ would be regular of degree at least $n/2$; all such graphs except one are clearly connected, so are hamiltonian implying that S is not maximal. The one exception is the disjoint union of two copies of $K_{n/2,n/2}$; but this cannot arise in our case since clearly its complement in $K_{n,n}$ is also disconnected, an impossibility since we know it has a hamilton decomposition.

If $p \geq 3$, then another beautiful theorem kicks in. Bill Jackson proved the following result, and it is exactly what is needed here.

Theorem 5.1. *If G is a 2-connected d-regular graph on v vertices with $d \geq v/3$ then G is hamiltonian.*

Notice that requiring G to be 2-connected is not surprising since the result looks for hamilton cycles; this, together with the regularity condition, permits the improvement of the lower bound on the minimum degree from $n/2$ in Theorem 2.1 to $n/3$ in this result.

Applying Theorem 5.1 to our situation shows that if $|S|$ is small enough such that each vertex is joined to *at most half* the vertices in the *other* parts, and if the deficiency H of S in $K_p(n)$ is 2-connected, then H is hamiltonian and S is not maximal. Notice that when $p > 3$ (or $p = 3$), being joined to at most half the vertices in the other (two) parts is equivalent to saying the degree in the deficiency is at least $d > v/3$ (or $d = v/3$ respectively), so Theorem 5.1 is the perfect tool. Of course, there is the issue of making sure the deficiency is 2-connected before the lower bound is finalized. With one exception this can be shown using arguments showing that the degree of regularity is too high for a cut-vertex or two components to exist. (The one exception is when each vertex is joined to exactly half the vertices in the other parts and n is even. In this case a maximal set of size $|S| = n(p-1)/4$ always seems to exist). So we have the following start.

Lemma 5.1. ([6]) *Let $p \geq 3$. If $k \leq n(p-1)/4$, with n being even if equality holds, then there does not exist a maximal set of k hamilton cycles in $K_p(n)$.*

Proving the sufficiency of the condition in Lemma 5.2 is not quite done yet! A series of three papers [6], [8], [13] has whittled the myriad of cases left to consider down to just the single smallest possible value of $|S|$ for each odd value of p when $n = 3$.

The first paper [6] completely settled the case where the maximal set S of hamilton cycles used every edge joining a vertex among the " Top " $\lfloor n/2 \rfloor$ vertices in each part to a vertex among the " Bottom " $\lceil n/2 \rceil$ vertices in each other part, thinking of the vertices in each part depicted in vertical columns. More formally, for $1 \leq i \leq p$, V_i is partitioned into two sets $V_{i,T}$ and $V_{i,B}$ with $|V_{i,T}| = \lfloor n/2 \rfloor$ and $|V_{i,B}| = \lceil n/2 \rceil$. Then for $1 \leq i, j \leq p$ with $i \neq j$, each edge joining a vertex in $V_{i,T}$ to a vertex in $V_{j,B}$ occurs in a hamilton cycle in S. Therefore S is clearly maximal, since the complement is disconnected: it contains no path joining vertices in $V_{i,T}$ to vertices on $V_{i,B}$. This result turns out to prove that the condition described in Lemma 5.2 is also sufficient for S to exist, except possibly if $n = 2$, or if all three of the following are satisfied:

$$n \text{ is odd, } p \text{is odd, and } |S| \leq ((n+1)(p-1)-2)/4.$$

You'll notice that this last number is close to $n(p-1)/4$, the lower bound in Lemma 5.1. Considering these possible exceptional cases in the size of $|S|$ must involve a different approach. We now consider how almost all of these cases were handled.

Next to be settled was the case when $n = 2$. Notice that in this case $K_p(n)$ is just a complete graph with a 1-factor removed. It turns out that in a paper by Fu, Logan and Rodger [8], two very different proofs are presented that characterize when maximal sets of hamilton cycles exist in this graph. The first method is no surprise: amalgamations! Still, there is something new and interesting in its application here, because for the first time it was possible to amalgamate vertices from the *different* parts into one vertex, and still prove that the disentangling of the vertices could be accomplished. This advance made the initial amalgamated decomposition *much* easier to find. The second approach was more direct, using difference methods and doubling procedures in a clever way. Carefully selected and very carefully named hamilton decompositions of K_p were used to find maximal sets of hamilton decompositions of $G = K_{2p} - F$, where F is a 1-factor of G.

This case when $n \geq 5$ (see [13]) was dealt with in an interesting way because, for the first time, the edge-coloring was handled in a much better fashion. Excellent use was made of evenly-equitable edge-colorings; that is, edge-colorings with the properties that:

1. each vertex in each color-class has even degree, and
2. at each vertex, the number of edges of each color differs by at most two from the number of edges of each other color.

Hilton proved [10] that for every $m \geq 1$ and for every Eulerian graph G, there exists an evenly-equitable m-edge coloring of G. This approach made the construction of the amalgamated graph much easier than in any previous related paper. Even so, there were several cases of careful counting required to prove that the constructed graph indeed satisfied the conditions for being an amalgamated hamilton decomposition of $K_p(n)$.

The case when $n \geq 5$ is also interesting for another reason, namely that some assymetry had to be introduced into the partition of the vertices. So, the set of Top vertices $V_T = \cup_{i=1}^{p} V_{i,T}$ had to be partitioned into two sets: the " Left " set $V_{T,L} = \cup_{i=1}^{(p-1)/2} V_{i,T}$ and the " Right " set $V_{T,R} = \cup_{i=(p+1)/2}^{p} V_{i,T}$. Similarly the set of Bottom vertices is partitioned into the " Left " set $V_{B,L} = \cup_{i=1}^{(p-1)/2} V_{i,B}$ and the " Right " set $V_{B,R} = \cup_{i=(p+1)/2}^{p} V_{i,B}$. Then $V_{T,L}$ and $V_{B,R}$ each contained exactly $(n+1)/2$ vertices from each part, and $V_{T,R}$ and $V_{B,L}$ each contained exactly $(n-1)/2$ vertices from each part. This reduced the number of edges joining Top vertices to Bottom vertices to a low enough number that all could be included in the k hamilton cycles, even when k was at its lowest.

When $n = 3$, the edge-coloring approach used when $n = 5$ did not work well because it did not ensure that each color class was connected in the amalgamation. But there was an approach that did work well for the required edge-coloring to be obtained, namely by rotating the edge-cut in the deficiency by 90°. That is, the edges used in hamilton cycles included all the edges joining Left vertices to Right vertices. Of course this ensured that the deficiency was disconnected, thus ensuring that the set of hamilton cycles was maximal. Amalgamations, together with the " new " orientation of the cutset, allowed all cases to be completed except for the smallest possible value of k!

So it still remains to settle the case when $n = 3$, k is the smallest possible value allowed by Lemma 5.1, and p is odd. In this small tight case, we are finally forced to deal with both the asymmetry in the arrangement of the vertices, and finding some clever way to making sure that each color class is connected. Again, the asymetry manifests itself in the sizes of the Top and Bottom sets of vertices: the Left Top sets contain 2 vertices each, as do the Bottom Right sets; the Left Bottom and the Right Top sets each contain a single vertex (recall that we need only do this when $n = 3$). The relevant numbers suggest that this structural setup will allow the final case to be settled, but some details still remain to be established. (All other structural arrangements appear to require too many edges in the cutset to allow k to be as small as we require).

To summarize, all these results combine to produce the following theorem.

Theorem 5.2. ([4], [6], [11], [13]) *There exists a maximal set of k hamilton cycles in $K_p(n)$ (p parts of size n) if and only if*

1. $\lceil \frac{n(p-1)}{4} \rceil \leq k \leq \lfloor \frac{n(p-1)}{2} \rfloor$ *and*
2. $k > \frac{n(p-1)}{4}$ *if*
   (a) *n is odd and $p \equiv 1 \pmod 4$, or*
   (b) *$p = 2$, or*
   (c) *$n = 1$*

except possibly for the undecided case when $n = 3$, and k is the smallest value.

6. 2-factors, each component is a triangle.

A *triangle-factor* of a graph G is a 2-factor of G in which each component is a copy of K_3. In this section we consider the case where S is a maximal set of triangle-factors; so the deficiency of S contains no triangle-factors. We focus on the case where $G = K_n$. Clearly, if G is to have any triangle factors at all it must have a number of vertices that is divisible by 3. Two cases emerge in this study: $n = 6m$ and $n = 6m + 3$.

Amazingly enough, the result that provides the lower bound on $|S|$ was proved over 40 years ago by Corrádi and Hajnal [5].

Theorem 6.1. *Let G be a simple graph on $n = 3z$ vertices with minimum degree at least $2z$. Then G contains a triangle-factor.*

So the least one can expect $|S|$ to be is $n/6$ (for then the deficiency of S in G is regular of degree $3z-1-2(z/2) = 2z-1 < 2z$). At the other extreme, the case where $|S| = (n-1)/2$ (so n is necessarily odd; $n = 6m+3$) is equivalent to finding a Kirkman Triple System of order $6m+3$. This is a historic problem that is over 150 years old, and one that was eventually solved by Ray-Chaudhuri and Wilson [15]. The next largest value for $|S|$ is $|S| = (n-2)/2$ (so n is even), and this corresponds to a Nearly Kirkman Triple System on $6m$ vertices; this was solved in a series of papers [1], [3], [12], [20]. The remaining possible values of $|S|$ first came under consideration by Rees, Rosa and Wallis ten years ago [19], when they focussed on the values of $|S|$ in the range $\lceil n/6 \rceil < |S| \leq n/4$. So at that point, $|S| = \lceil n/6 \rceil$ and the large possible values of $|S|$ remained in doubt. Recently, a series of two papers by Rees [17], [18] has removed almost all that doubt! One paper looked at the cases when G has $6m$ vertices, and the other considered graphs with $6m + 3$ vertices. Put together, this work produces the current state of affairs described in the following.

Theorem 6.2. *Let* $n \equiv 0 \pmod 3$. *There exists a maximal set of* k *triangle-factors in* K_n *if and only if*

(a) $\lceil n/6 \rceil \leq k \leq \lfloor (n-1)/2 \rfloor$, *and*
(b) $(k, n) \notin \{(2, 9), (3, 9), (2, 12), (5, 12), (3, 15), (3, 18)\}$ *except possibly in the following unsettled cases:*
  (i) $k = \lfloor (n+3)/6 \rfloor$ *and* $n \equiv 0, 9$ *or* $12 \pmod{18}$,
  (ii) $k = 6 = \lfloor (n+3)/6 \rfloor$ *and* $n = 33$, *and*
  (iii) $n \in \{45, 57, 69, 81, 93, 237, 261, 309, 333, 381\}$.

Remark. For the 5 smallest values of n in (c), some values of k have been found for which there exists a set of k maximal triangle factors in K_n [18]:

The proofs of this result rely heavily on the existence of frames (a group divisible design whose blocks can be partitioned into *holey* parallel classes, each of which partitions all the points except for those in one of the groups). The interested reader is directed to the papers of Rees for more details.

But one particularly neat result that Rees uses to make great headway towards a solution is the following result [16]. Yet again revealing my liking of edge-colorings, I close this survey by presenting the proof below in that milieu. Let $G \otimes I_2$ denote the graph formed from 2 copies of G by joining x_i to y_j $(1 \leq i, j \leq 2)$ if and only if $\{x, y\} \in E(G)$ (denoting the two copies of $v \in V(G)$ by v_1 and v_2).

Theorem 6.3. *If there exists a triangle-factorization of* G *into* $2l$ *triangle factors then there exists a triangle-factorization of* $G \otimes I_2$.

Proof. Clearly it suffices to consider just $l = 1$. Let the triangle-factors of G be T_1 and T_2; let $T_i = \{t_{i,1}, \ldots, t_{i,n/3}\}$. The crucial observation in this proof is to show that the vertices of G can be partitioned into 3 sets V_1, V_2 and V_3, each of size $n/3$, such that each triangle in each triangle-factor has one vertex in each of these three sets.

To do so, we think of triangles in T_i as vertices in a bipartite graph. Let $T_i' = \{t_{i,1}', \ldots, t_{i,n/3}'\}$ for $1 \leq i \leq 2$. Form a bipartite graph B with bipartition T_1' and T_2' of the vertex set, and join t_{1,y_1}' to t_{2,y_2}' if and only if $t_{1,y_1} \cap t_{2,y_2} \neq \emptyset$.

Notice that each triangle in T_1 has at most one vertex in common with each triangle in T_2. Also, since T_1 and T_2 are each 2-factors of G for each vertex v in G there exist unique triangles $t_{1,z_1} \in T_1$ and $t_{2,z_2} \in T_2$ such that $t_{1,z_1} \cap t_{2.z_2} = \{v\}$. Therefore B is clearly 3-regular, so give it a proper 3-edge-coloring with colors 1, 2 and 3. Now place v in V_x if and only if the edge joining t'_{1,z_1} to t'_{2,z_2} is colored x in B.

So now, the 4 triangle factors in $G \otimes I_2$ are easily obtained as follows. For each triangle $\{a, b, c\} \subset T_1$ and each triangle $\{d, e, f\} \subset T_2$, named so that $\{a, d\} \subseteq V_1$, $\{b, e\} \subseteq V_2$, and $\{c, f\} \subseteq V_3$, let:

1. F_1 contain $\{a_1, b_1, c_1\}$ and F_2 contain $\{d_2, e_2, f_2\}$,
2. F_2 contain $\{a_1, b_2, c_2\}$ and F_2 contain $\{d_2, e_1, f_1\}$,
3. F_3 contain $\{a_2, b_1, c_2\}$ and F_2 contain $\{d_1, e_2, f_1\}$, and
4. F_4 contain $\{a_2, b_2, c_1\}$ and F_2 contain $\{d_1, e_1, f_2\}$.

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