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BLOCKING SETS AND COLOURINGS IN STEINER SYSTEMS S(2, 4, v)

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A Steiner system S(2, 4, v) is a v-element set V together with a collection $\mathcal{B}$ of 4-subsets of V called *blocks* such that every 2-subset of V is contained in exactly one block. (Other names: Steiner 2-designs with $k = 4$; block designs with block size 4 and $\lambda = 1$; linear spaces with all lines of size 4). Hanani [7] was the first to show that a Steiner system S(2, 4, v) exists if and only if $v \equiv 1$ or 4 (mod 12); these values of v are *admissible*.

Although Steiner systems S(2, 4, v) are not as well studied as Steiner triple systems, there exists extensive literature devoted to S(2, 4, v)s as well as a host of interesting open questions, including many that apparently remain unexplored at all. We concentrate here on several types of subsets in Steiner systems S(2, 4, v) with specified properties, especially those related to colourings.

A set $S \subset V$ is *independent* if it contains no block. Let $\alpha(S)$ be the *independence number* of a Steiner system S(2, 4, v) $S = (V, \mathcal{B})$, i.e. the maximum cardinality of an independent set in S. A maximum size independent set in an S(2, 4, v) may contain as many as $\frac{2v+1}{3}$ elements. Indeed, if $v \equiv$ 4, 13 (mod 36), there exist Steiner systems S(2, 4, v) with $\alpha = \frac{2v+1}{3}$. This follows easily from applying the so-called $v \to 3v + 1$ rule for Steiner systems S(2, 4, v).

The v → 3 v + 1 rule. Let $(V, \mathcal{B})$ be an S(2, 4, v), $V = \{a_1, a_2, \dots, a_v\}$, and let $(X, \mathcal{C}, \mathcal{R})$ be a Kirkman triple system KTS($2v + 1$) (see [2]); $X \cap V =$

$\emptyset$; $\mathcal{R} = \{R_1, R_2, \ldots, R_v\}$. Put $\mathcal{D}_i = \{\{a_i, x, y, z\} : \{x, y, z\} \in R_i\}$, $\mathcal{D} = \bigcup_{i=1}^{v} D_i$. Then $(V \cup X, \mathcal{B} \cup \mathcal{D})$ is an S$(2, 4, 3v + 1)$.

One sees instantly that in *any* Steiner system S$(2, 4, 3v + 1)$ with a subsystem S$(2, 4, v)$, the complement of the subsystem (the set X in the above construction) is an independent set.

On the other hand, for sufficiently large orders v, a maximum size independent set may contain as few as $cv^{\frac{2}{3}}(log\ v)^{\frac{1}{3}}$ elements; this was shown in [5], [16]. More precisely, it was shown by Rödl and Šiňajová [16] that for sufficiently large v, there is a constant c such that for every S$(2, 4, v)$,

$$\alpha \geq cv^{\frac{2}{3}}(\log v)^{\frac{1}{3}},$$

and there is a constant c' such that there exist infinitely many S$(2, 4, v)$ with

$$\alpha \leq c'v^{\frac{2}{3}}(\log v)^{\frac{1}{3}}.$$

A *blocking set* in a Steiner system S$(2, 4, v)$ $(V, \mathcal{B})$ is a subset $X \subset V$ such that for any block $B \in \mathcal{B}$, we have $X \cap B \neq \emptyset$ but $X \not\supseteq B$. In other words, a blocking set is an independent set which intersects each block; equivalently, it is an independent set whose complement (in V) is also an independent set.

Not every Steiner system S$(2, 4, v)$ has a blocking set. In fact, it follows from [15] that for all admissible $v \geq 25$ there exists an S$(2, 4, v)$ without a blocking set. On the other hand, unlike for Steiner triple systems, a Steiner system S$(2, 4, v)$ with a blocking set exists for all admissible orders $v \equiv 1, 4 \pmod{12}$. This was shown in [8] for all admissible orders v except $v = 37, 40$, and 73, and in [3] for those three orders.

For a blocking set S, the *discrepancy* δ is the difference between the cardinalities of S and its complement $V \setminus S$ (which is also a blocking set):

$$\delta = \| S| - |V \setminus S\|$$

The blocking sets constructed in [8] and [3] all have discrepancy 0 or 1, according to whether $v \equiv 4 \pmod{12}$ or $v \equiv 1 \pmod{12}$. In 1990, Lo Faro [12] has shown that if $v \equiv 1 \pmod{12}$, the discrepancy of a blocking set *must* equal 1.

In the same paper [12] it was shown that if $v \equiv 4 \pmod{12}$, the discrepancy δ is either 0, or else $\delta \equiv 2 \pmod 4$. That is, in this case the cardinality of the blocking set is $\frac{v}{2}$ or else it is odd.

Suppose $(V, \mathcal{B})$ is an S$(2, 4, v)$, $v \equiv 4 \pmod{12}$, $v = 12t + 4$. Let $S \subset V$ be a blocking set with $|S| = 6t + 2 - s$, $|\bar{S}| = 6t + 2 + s$, and assume $s > 0$,

s odd; $s = \frac{\delta}{2}$ is the *half-discrepancy*. Let a, b, and c, respectively, be the number of blocks B in $\mathcal{B}$ such that $|B \cap S| = 3, 2$, and 1, respectively (and thus $|B \cap \bar{S}| = 1, 2$, and 3, respectively). Counting the number of pairs of elements which are both in S, both in $\bar{S}$, or one in S and one in $\bar{S}$, we get the following equalities:

$$a + b + c = \frac{1}{6}\binom{12t+4}{2} = 12t^2 + 7t + 1$$

$$3a + b = \frac{(6t+2-s)(6t+1-s)}{2}$$

$$b + 3c = \frac{(6t+2+s)(6t+1+s)}{2}$$

$$3a + 4b + 3c = (6t+2-s)(6t+2+s) = (6t+2)^2 - s^2$$

Solving for a, b, c, we obtain $a = 6t^2 - (2s-2)t + \binom{s}{2}$, $b = 3t + 1 - s^2$, $c = 6t^2 + (2s+2)t + \binom{s+1}{2}$. Furthermore, clearly either $b = 0$ or $b \geq 6$ which implies $t = \frac{s^2-1}{3}$ or $t \geq \frac{s^2+5}{3}$.

No nontrivial example with $b = 0$ is known. The smallest possibility occurs at $v = 100$ (with $t = 8, s = 5$); this would be the "century design" mentioned by M. J. de Resmini [14].

The smallest nontrivial case where a Steiner system S(2, 4, v) with a blocking set of half-discrepancy $s = 1$ can exist occurs when $t = 2$ and $v = 28$. Such a design does indeed exist; it was first constructed in [9].

In order to show how we can construct an S(2, 4, v) having a blocking set of discrepancy $\delta = 2$ for all $v \equiv 4 \pmod{24}$, $v \geq 100$, we need a definition.

Let S be a set with $t.n$ elements, let $\{S_1, \ldots, S_n\}$ be a partition of S where $|S_i| = t$. A *skew Room frame* of type t^n is a $t.n \times t.n$ array R indexed by S such that

(1) every cell of R is either empty or contains an unordered pair of elements of S;
(2) the subarrays $S_i \times S_i$ ("holes") are empty;
(3) each element of $S \setminus S_i$ occurs exactly once in row (column) s where $s \in S_i$;
(4) the pairs $\{s, t\}$ in R are precisely those where s, t are from different holes; and
(5) of any two cells (s, t), (t, s) where s, t are in different holes, exactly one is empty.

Fig.1 shows an example of a skew Room frame of type 4^4.

					12,13	11,15		7,16	6,14				5,10		8,9
				11,14			12,16	5,13	8,15			6,9		7,10	
				9,13			10,15			5,14	8,16		6,11		7,12
					10,14	9,16				7,15	6,13	5,12		8,11	
10,16			12,15					2,14		3,13		4,11	1,9		
	9,15	11,16							1,13		4,14	2,10	3,12		
	10,13	12,14						1,15		4,16				2,9	3,11
9,14			11,13						2,16		3,15			4,12	1,10
		6,15	8,14		3,16		4,13						2,7	1,5	
		7,13	5,16	4,15		3,14						1,8			2,6
8,13	6,16				2,15		1,14					3,7			4,5
5,15	7,14			1,16		2,13							4,8	3,6	
7,11		8,10				1,12	3,9	4,6			2,5				
	8,12		7,9			4,10	2,4		3,5	1,6					
6,12		5,9		3,10	1,11				4,7	2,8					
	5,11		6,10	2,12	4,9			3,8			1,7				

Figure 1: Skew Room frame of type 4^4

We can now describe a following

Construction. Let X be a set, $|X| = 4t$, let R be a skew Room frame of type 4^t based on X, with holes $H = \{h_1, h_2, \ldots, h_t\}$, $|h_i| = 4$. Let $S = \{a, b, c, d\} \cup X \times \{1, 2, 3, 4, 5, 6\}$ where $\{a, b, c, d\}$ is a block of an S(2, 4, 28). For $i = 1, 2, \ldots, t$, let $(\{a, b, c, d\} \cup \{h_i \times \{1, 2, 3, 4, 5, 6\}, \mathcal{B}_i)$ be an S(2, 4, 28) with blocking sets $\{a\} \cup \{h_i \times \{1, 2, 3\}\}$ and $\{b, c, d\} \cup \{h_i \times \{4, 5, 6\}\}$. Place the blocks of $\mathcal{B}_i$, $i = 1, \ldots, t$ in $\mathcal{B}$. If x and y belong to different holes of H, place the six blocks $\{(x, i), (y, i), (r, i+1), (c, i+4)\}$ in $\mathcal{B}$ where $i \in \{1, 2, 3, 4, 5, 6\}$ (second coordinates reduced mod 6) and $\{x, y\}$ is in the cell (r, c) of R.

Then $(S, \mathcal{B})$ is a Steiner system $S(2, 4, 24t+4)$ with blocking sets of sizes $12t + 1$ and $12t + 3$.

Chen and Zhu [1] have shown that a skew Room frame of type 4^t exists for all $t \geq 4$. This, together with our Construction above, yields the following theorem (cf. Theorem 3 of [9]).

Theorem. *There exists a Steiner system S*$(2, 4, v)$ *with blocking sets of sizes* $\frac{v}{2} - 1$ *and* $\frac{v}{2} + 1$ *(i.e. of discrepancy* $\delta = 2$*) for all* $v \equiv 4 \pmod{24}$*,* $v \geq 28$*, except possibly for* $v \in \{52, 76\}$.

This still leaves following open problems.

Problem 1. Do there exist Steiner systems S$(2, 4, v)$ with blocking sets of discrepancy $\delta = 2$ if $v \equiv 16 \pmod{24}$?

Problem 2. Do there exist Steiner systems S$(2, 4, v)$ with blocking sets of discrepancy $\delta \geq 6$?

The smallest order for which there may exist a Steiner system S$(2, 4, v)$ with a blocking set of discrepancy 6 is $v = 64$.

Maximum arcs in a Steiner system S$(2, 4, v)$ provide another example of sets with interesting properties. A set S in S$(2, 4, v)$ such that S intersects each block in 0 or 2 points is called a *maximum arc* or *hyperoval* (or a set of type $(0, 2)$, see [14]). For a maximum arc to exist, we must have $v \equiv 4 \pmod{12}$, and $|S| = \frac{v+2}{3}$. It was shown recently in [6] (and also independently in [10]) that for each $v \equiv 4 \pmod{12}$ there exists an S$(2, 4, v)$ with a maximum arc. For an application of maximum arcs to a special type of colourings (colourings of type AC), see below.

A (clasical, weak) *colouring* of a Steiner system S$(2, 4, v)$, $S = (V, \mathcal{B})$, is a mapping $f : V \to C$ such that $f^{-1}(c)$ is an independent set for each $c \in C$ (no block is monochromatic). The elements of C are *colours*, and for each $c \in C$, $f^{-1}(c)$ is a *colour class*. The *chromatic number* $\chi = \chi(V, \mathcal{B})$ is the smallest integer $m = |C|$ such that S admits a colouring with m colours [17]. An S$(2, 4, v)$ is *m-colourable* if it admits a colouring with m colours, and is *m-chromatic* if $\chi = m$.

An S$(2, 4, v)$ is 2-chromatic if and only if it admits a blocking set; the colour classes in any 2-colouring are blocking sets. It follows from a clasical result of Erdös and Hajnal, together with Ganter's embedding result for partial S$(2, 4, v)$s that there exist Steiner systems S$(2, 4, v)$ with an arbitrarily high chromatic number. In [15] it is shown that a 3-chromatic S$(2, 4, v)$ exists for all admissible $v \geq 25$, and in [11] it is shown that for all $m \geq 2$ there exists v_m such that for all $v \geq v_m$, $v \equiv 1, 4 \pmod{12}$, there exists an m-chromatic S$(2, 4, v)$. Still, many open problems remain.

Voloshin's mixed hypergraph colouring concept has motivated an examination of more specific type colourings for hypergraphs and designs in general, and for Steiner systems S(2, 4, v) in particular. A *block colour pattern* is a partition of the block size, in our case of the number 4. The five possible partitions of 4, and the corresponding block colour paterns, are $A = 4$, $B = 3 + 1$, $C = 2 + 2$, $D = 2 + 1 + 1$, $E = 1 + 1 + 1 + 1$. For S a nonempty subset of $\{A, B, C, D, E\}$, a colouring of type S colours the elements of S(2, 4, v) in such a way that each block is coloured according to a pattern from S. This may lead to a consideration of 31 different types of colourings; however, not all of these are very interesting, and some of these are easily dealt with.

Since in general the existence of a colouring of type S is no longer guaranteed, the main questions asked here are those about colourability, and then about the *spectrum* for colourings of type S, i.e. the set Ω_S (defined for individual systems, $\Omega_S(V, \mathcal{B})$, and also for admissible orders, $\Omega_S(v) = \cup\Omega_S(V, \mathcal{B})$ where the union is taken over all Steiner systems S(2, 4, v) of order v) of integers m such that there exists an m-colouring of type S; unlike for classical colourings, it is essential here that all colours must be used (cf. [13]).

Classical colourings in this setting become colourings of type $BCDE$ (no monochromatic blocks) while Voloshin-type colourings are those of type BCD (no monochromatic or polychromatic blocks). Several other types of colourings have been recently investigated: bicolourings (type BC, [4]), colourings of type B, AC etc. [13], with complete results available for some types, and only partial results for others.

Unlike in the classical case, it may happen that for a given colouring type S and a given system $(V, \mathcal{B})$, the spectrum $\Omega_S(V, \mathcal{B}) = \emptyset$, that is, $(V, \mathcal{B})$ is S-uncolourable. If $(V, \mathcal{B})$ is S-uncolourable then we must have $S \subseteq \{B, C, D\}$.

But do there indeed exist systems S(2, 4, v) which are BCD-uncolourable, i.e. have no Voloshin-type colouring? It is not hard to see that if the largest independent set in a Steiner system S(2, 4, v) has cardinality less than $\frac{v}{6}$ then it is BCD-uncolourable. The results of [5] and [16] mentioned earlier guarantee that infinitely many such systems S(2, 4, v) exist. In fact, there exists a constant v^* such that for all $v \geq v^*$, $v \equiv 1, 4 \pmod{12}$, there exists a BCD-uncolourable S(2, 4, v).

From among the 31 potential colouring types for Steiner systems S(2, 4, v), those that admit only a single block colour pattern may perhaps appear to be the most appealing. But one discovers instantly that colourings of type A or E are trivial and utterly uninteresting, and colouring of type C exists only for the trivial design with $v = 4$. This leaves types B and D which, on the other hand, are all but uninteresting.

If $B \in S$ then the $v \to 3v + 1$ rule given earlier shows that $m \in \Omega_S(v)$ implies $m + 1 \in \Omega_S(3v + 1)$. Starting with the trivial design with $v = 4$ which obviously admits a colouring of type B we obtain that for every order $v = \frac{3^m-1}{2}$

there exists a Steiner system S(2, 4, v) with an m-colouring of type B.

But do there exist S(2, 4, v)s of other orders v admitting colourings of type B? In an S(2, 4, v) with an m-colouring of type B, with colour classes X_i, $|X_i| = x_i, = 1, \dots, m$, we have:

(i) $x_i \equiv 1, 3 \pmod 6$, $i = 1, \dots, m$;

(ii) $\sum \binom{x_i}{2} = \sum x_i . x_j = \frac{1}{4} v(v-1)$;

(iii) $x_i \leq \frac{2v+1}{3}$.

Also, in an m-colouring of type B there is exactly one colour class with $x_i \equiv 1 \pmod 6$.

It turns out that for $v \leq 121$, we get only three additional solutions $(x_1, \dots, x_k)$ satisfying these necessary conditions:

(1) $v = 61, (x_1, x_2, x_3) = (3, 19, 39)$;
(2) $v = 100, (x_1, x_2) = (45, 55)$;
(3) $v = 109, (x_1, x_2, x_3) = (1, 45, 63)$.

No such S(2, 4, v) admitting a colouring of type B is known! Note that under (2) we again encountered the "century design" mentioned earlier.

Colourings of type D (each block is 3-coloured) are also quite interesting (cf. [13]). First of all, a 3-colouring of type D exists only for the trivial S(2, 4, v) with $v = 4$. No 4-colouring of type D exists for any S(2, 4, v) whatsoever, and a 5-colouring of type D of an S(2, 4, v) exists only if $v \in \{13, 16, 25\}$. If there is an m-colouring of type D of an S(2, 4, v) and $v > 25$ then $m \geq 6$. One has $\Omega_D(13) = \{5, 6\}$, $\Omega_D(16) = \{5, 6, 7\}$.

An example of an S(2, 4, 25) with a 5-colouring of type D is given by the following: $V = Z_5 \times Z_5$, $\mathcal{B} = \{\{00, 01, 10, 22\}, \{00, 02, 20, 44\}\} mod(5, 5)$, with colour classes $Z_5 \times \{i\}$, $i \in Z_5$. But, curiously, we also have the following stronger 'converse':

Let $m \geq 2$ be arbitrary, and assume there is an m-colouring of S(2, 4, v) of type D in which all colour classes have the same cardinality. Then $m = 5$ and $v = 25$.

Finally, let us conclude with a result which was obtained as a consequence of the result on the existence of maximum arcs in Steiner systems S2, 4, v) mentioned earlier. This concerns colourings of type AC. Since an S(2, 4, v) with a maximum arc admits a 2-colouring of type AC, one obtains a complete characterization of the spectrum $\Omega_{AC}(v)$: it equals $\{1\}$ for $v \equiv 1 \pmod{12}$, and it equals $\{1, 2\}$ for $v \equiv 4 \pmod{12}$.

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