

ON THE SHANNON ENTROPY OF THE NUMBER OF VERTICES WITH ZERO IN-DEGREE IN RANDOMLY ORIENTED HYPERGRAPHS

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Suppose that you have n colours and m mutually independent dice, each of which has r sides. Each die lands on any of its sides with equal probability. You may colour the sides of each die in any way you wish, but there is one restriction: you are not allowed to use the same colour more than once on the sides of a die. Any other colouring is allowed. Let X be the number of different colours that you see after rolling the dice. How should you colour the sides of the dice in order to maximize the Shannon entropy of X ? In this article we investigate this question. It is shown that the entropy of X is at most $\frac{1}{2} \log\left(\frac{n}{2} + \frac{1}{6}\right) + \frac{1}{2} \log(\pi e)$ and that the bound is tight, up to a constant additive factor, in the case of there being equally many coins and colours. Our proof employs the differential entropy bound on discrete entropy, along with a lower bound on the entropy of binomial random variables whose outcome is conditioned to be an even integer. We conjecture that the entropy is maximized when the colours are distributed over the sides of the dice as evenly as possible.

1. Prologue and main results

This work is motivated by the following entropy-maximization problem: Fix positive integers m, n, r such that $m \geq n > r \geq 2$. Suppose you are given n

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colours and m mutually independent dice, each of which has r sides. Each die lands on any of its sides with equal probability. You can colour the sides of the dice in any way you want, but there is only one restriction: you are not allowed to use the same colour more than once on the sides of a die. All other colourings are allowed. Let X be the number of different colours that you see after rolling the dice. In what way should you colour the sides of the dice in order to maximize the Shannon entropy of X ?

The *Shannon entropy* (or *entropy*, for short) of a random variable X that takes values on a finite set S is defined as

$$H(X) = - \sum_{s \in S} \mathbb{P}[X = s] \cdot \log \mathbb{P}[X = s],$$

with the convention $0 \log 0 = 0$. Throughout the text, $\log(\cdot)$ denotes logarithm with base 2. Shannon entropy may be thought of as the "amount of information", or the "amount of surprise", that is evidenced by a random variable and, in a certain sense, random variables with large entropy are less "predictable". Entropy enjoys several interesting properties which render itself as a useful tool for several problems in enumeration, statistics and theoretical computer science, among several others. We refer the reader to [2, 3] for excellent textbooks on the topic. A central theme that motivates the development of the theory of entropy concerns the so-called *maximum entropy principle*: within a given class of random variables, find one that has maximum entropy (see [2] for a whole chapter devoted to the topic). It is well-known that for any random variable, X , taking values in a finite set S , it holds

$$H(X) \leq \log(|S|).$$

This is a consequence of Jensen's inequality. Notice that the bound is attained by the random variable that takes each value from S with equal probability. The following result, due to James Massey, is referred to as the *the differential entropy bound on discrete entropy* and may be seen as a refinement of the aforementioned bound.

Theorem 1.1 (Massey). *Let X be a discrete random variable with finite variance, denoted $\text{Var}(X)$. Then*

$$H(X) \leq \frac{1}{2} \log(2\pi e(\text{Var}(X) + 1/12)).$$

In other words, upper bounds on the variance of discrete random variables imply upper bounds on their entropy. A proof of Theorem 1.1 can be found in [4] (see also [2, p. 258]) and a refinement can be found in [5].

This work introduces the problem of maximizing the entropy of the random variable that counts the number of different colours after a roll of m fair dice whose r sides have been coloured using n colours, subject to the condition that it is not allowed to use a colour more than once on the sides of a die; we refer to this condition as a *proper colouring*. The random variable that counts the number of different colours after a toss of m properly coloured coins (i.e., when $r = 2$) has been previously studied in [6, 7], in the setting of maximising its median. Similarly to the setting of the median, we conjecture that a proper colouring over the dice that maximizes Shannon entropy is such that the colours have been distributed as evenly as possible over the sides of the dice. In order to be more precise, we need some extra piece of notation which allows to express the problem under consideration in terms of hypergraphs. Recall that a hypergraph, \mathcal{H} , is a pair (V, \mathcal{E}) where V is a finite set, and referred to as the *vertex set*, and \mathcal{E} is a collection of (not necessarily distinct) of subsets of V , and referred to as the *edge set*.

Let the positive integers m, n, r be such that $m \geq n > r \geq 2$. Suppose that \mathcal{C} is a configuration consisting of m dice all of whose r sides have been properly coloured using n colours. Let $X_{\mathcal{C}}$ be the number of different colours after a roll of the dice. One may associate a hypergraph, $\mathcal{H}_{\mathcal{C}} = (V, \mathcal{E})$, to this configuration: for every colour put a vertex in V and for every (properly) coloured die put an edge in \mathcal{E} containing all vertices corresponding to the colours on the sides of the die. Note that $|V| = n$ and $|\mathcal{E}| = m$. Moreover, notice that the hypergraph $\mathcal{H}_{\mathcal{C}}$ may have isolated vertices and, since the same coloured dice may appear more than once in the configuration \mathcal{C} , it may have edges that appear more than one time in \mathcal{E} ; i.e., it is a *multi-hypergraph*. Notice also that every edge $E \in \mathcal{E}$ has cardinality r or, in other words, the hypergraph $\mathcal{H}_{\mathcal{C}}$ is *r-uniform*. A 2-uniform (multi)hypergraph is just a (multi)graph. Here and later, the class consisting of all r -uniform multi-hypergraphs on n vertices and m edges is denoted by $\mathcal{D}_{n,m,r}$. The class $\mathcal{D}_{n,n,r}$, i.e, the class consisting of all r -uniform hypergraphs having n vertices and n edges, will be of particular interest. When $r = 2$, we write \mathcal{G}_n instead of $\mathcal{D}_{n,n,2}$.

Now rolling the dice corresponds to choosing an element from each edge $\mathcal{H}_{\mathcal{C}}$ uniformly at random, where each choice is done independently of all previous choices; we refer to this sampling procedure by saying that each edge in $\mathcal{H}_{\mathcal{C}}$ is *randomly oriented towards one of its elements*. Here and later, the phrase *random orientation on the edges* of a hypergraph \mathcal{H} expresses the fact that each edge in \mathcal{H} is oriented towards one of its elements with probability $\frac{1}{r}$, where each edge is oriented independently of all other edges. For every $v \in V$, let $\deg_{\mathcal{H}}^-(v)$ denote the *in-degree* of v , i.e., the number of edges in \mathcal{E} that are oriented towards

$v \in V$ after a random orientation on the edges of \mathcal{H} . Notice that

$$X_{\mathcal{C}} = |\{v \in V : \deg_{\mathcal{H}}^-(v) > 0\}|.$$

In other words, $X_{\mathcal{C}}$ is the number of vertices with non-zero in-degree after a random orientation on the edges of $\mathcal{H}_{\mathcal{C}}$ and the above mentioned question on coloured dice can be equivalently expressed as follows.

Problem 1.2. Fix positive integers n, m, r such that $m \geq n > r \geq 2$. For every $\mathcal{H} \in \mathcal{D}_{n,m,r}$ let $X_{\mathcal{H}}$ be the random variable that counts the number of vertices with non-zero in-degree after a random orientation on the edges of \mathcal{H} . Find a hypergraph $\mathcal{H} \in \mathcal{D}_{n,m,r}$ such that

$$H(X_{\mathcal{H}}) \geq H(X_{\mathcal{F}}), \text{ for all } \mathcal{F} \in \mathcal{D}_{n,m,r}.$$

We conjecture that the hypergraph that maximizes entropy is such that the degrees of its vertices are as equal as possible. More precisely, we believe that the following holds true. Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a vertex $v \in V$, we denote by $\deg_{\mathcal{H}}(v)$ the number of edges in \mathcal{E} that contain v .

Conjecture 1.3. Let the positive integers m, n, r be such that $m \geq n > r \geq 2$. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ from $\mathcal{D}_{n,m,r}$ for which it holds

$$H(X_{\mathcal{H}}) \geq H(X_{\mathcal{G}}), \text{ for all } \mathcal{G} \in \mathcal{D}_{n,m,r}$$

is such that

$$|\deg_{\mathcal{H}}(v_1) - \deg_{\mathcal{H}}(v_2)| \leq 1, \text{ for all } v_1, v_2 \in V.$$

In other words, the colours should be distributed over the dice as evenly as possible. In this note we obtain the following upper bound on the entropy of $X_{\mathcal{H}}$, for $\mathcal{H} \in \mathcal{D}_{n,m,r}$.

Theorem 1.4. Fix $\mathcal{H} \in \mathcal{D}_{n,m,r}$ and let $X_{\mathcal{H}}$ be the number of vertices with non-zero in-degree after a random orientation on the edges of \mathcal{H} . Then

$$H(X_{\mathcal{H}}) \leq \frac{1}{2} \log \left(\frac{n}{2} + \frac{1}{6} \right) + \frac{1}{2} \log(\pi e).$$

We prove Theorem 1.4 in Section 2. Moreover, we show that the bound is tight, up to an additive constant factor, when $m = n$ and $r = 2$. More precisely, we have the following.

Theorem 1.5. *Let C_n denote a cycle on $n \geq 3$ vertices. Let X_{C_n} be the number of vertices with non-zero in-degree after a random orientation on the edges of C_n . Then*

$$H(X_{C_n}) \geq \frac{1}{2} \log(n) + \frac{1}{2} \log(\pi e) - \frac{3}{2} - \frac{1}{2 \ln(2)(n-1)}.$$

Straightforward calculations show that the difference between the bounds provided in Theorem 1.4 and Theorem 1.5 is less than $0.73 + \frac{1}{2 \ln(2)(n-1)}$. We conjecture that $H(X_{C_n}) \geq H(X_G)$ for all $G \in \mathcal{G}_n$. We prove Theorem 1.5 in Section 3. The proof employs a lower bound on the entropy of a binomial random variable whose outcome is conditioned to be an even positive integer. Our article ends with Section 4 in which we state a conjecture.

2. Proof of Theorem 1.4

We show that $\text{Var}(X_{\mathcal{H}}) \leq n/4$. The result then follows from Theorem 1.1. For every vertex $v \in \mathcal{H}$, let \mathbf{I}_v be the indicator of the event $\{\text{deg}_{\mathcal{H}}^-(v) > 0\}$ and notice that $X_{\mathcal{H}} = \sum_{v \in V} \mathbf{I}_v$. We may therefore write

$$\begin{aligned} \text{Var}(X_{\mathcal{H}}) &= \sum_{v_1, v_2 \in V} (\mathbb{E}[\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2}] - \mathbb{E}[\mathbf{I}_{v_1}] \cdot \mathbb{E}[\mathbf{I}_{v_2}]) \\ &= \sum_{v \in V} (\mathbb{E}[\mathbf{I}_v] - \mathbb{E}[\mathbf{I}_v]^2) + \sum_{v_1 \neq v_2} (\mathbb{E}[\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2}] - \mathbb{E}[\mathbf{I}_{v_1}] \cdot \mathbb{E}[\mathbf{I}_{v_2}]). \end{aligned}$$

We now show that, whenever $v_1 \neq v_2$, we have

$$\mathbb{E}[\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2}] - \mathbb{E}[\mathbf{I}_{v_1}] \cdot \mathbb{E}[\mathbf{I}_{v_2}] \leq 0, \tag{1}$$

or, in other words, the indicators \mathbf{I}_{v_1} and \mathbf{I}_{v_2} are negatively correlated. This is clearly true when there exists no edge $E \in \mathcal{H}$ such that $\{v_1, v_2\} \subset E$, and we may therefore assume that v_1 and v_2 are both elements of some edge in \mathcal{H} . Let \mathcal{E}_{v_1} be the class consisting of those edges in \mathcal{H} that contain v_1 and do not contain v_2 and, similarly, let \mathcal{E}_{v_2} be the class consisting of those edges in \mathcal{H} that contain v_2 and do not contain v_1 . Finally, let \mathcal{E}_{v_1, v_2} be the subset of the edges in \mathcal{H} that contain both v_1 and v_2 . Notice that

$$\mathbb{E}[\mathbf{I}_{v_1}] \cdot \mathbb{E}[\mathbf{I}_{v_2}] = \left(1 - \left(\frac{r-1}{r}\right)^{\text{deg}_{\mathcal{H}}(v_1)}\right) \cdot \left(1 - \left(\frac{r-1}{r}\right)^{\text{deg}_{\mathcal{H}}(v_2)}\right)$$

and we proceed by working out the term $\mathbb{E}[\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2}]$.

Let A_1 be the event "there is no edge in \mathcal{E}_{v_1, v_2} which is oriented towards either v_1 or v_2 ", let A_2 be the event "no edge from \mathcal{E}_{v_1, v_2} is oriented towards v_2 ,

but some edge from \mathcal{E}_{v_1, v_2} is oriented towards v_1 ", let A_3 be the event "no edge from \mathcal{E}_{v_1, v_2} is oriented towards v_1 , but some edge from \mathcal{E}_{v_1, v_2} is oriented towards v_2 ", and, finally, let A_4 be the event "some edge from \mathcal{E}_{v_1, v_2} is oriented towards v_1 and some other edge from \mathcal{E}_{v_1, v_2} is oriented towards v_2 ". In particular, notice that the event A_4 has non-zero probability if and only if $|\mathcal{E}_{v_1, v_2}| \geq 2$. If $|\mathcal{E}_{v_1, v_2}| = 1$, then A_4 is empty and so $\mathbb{P}(A_4) = 0$. Moreover, when $r = 2$ the event A_1 is empty as well; thus $\mathbb{P}(A_1) = 0$.

Now we may write

$$\mathbb{E}[\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2}] = \sum_{i=1}^4 \mathbb{P}(A_i) \cdot \mathbb{P}(\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2} = 1 | A_i).$$

Let $d_1 = |\mathcal{E}_{v_1}|$, $d_2 = |\mathcal{E}_{v_2}|$ and $d_3 = |\mathcal{E}_{v_1, v_2}|$ and notice that $\deg_{\mathcal{H}}(v_1) = d_1 + d_3$ and $\deg_{\mathcal{H}}(v_2) = d_2 + d_3$. We compute

$$\mathbb{P}(A_1) \cdot \mathbb{P}(\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2} = 1 | A_1) = \left(\frac{r-2}{r}\right)^{d_3} \cdot \left(1 - \left(\frac{r-1}{r}\right)^{d_1}\right) \cdot \left(1 - \left(\frac{r-1}{r}\right)^{d_2}\right).$$

Moreover, since

$$\mathbb{P}(A_2) = \mathbb{P}(A_3) = \sum_{i=1}^{d_3} \binom{d_3}{i} \left(\frac{1}{r}\right)^i \left(\frac{r-2}{r}\right)^{d_3-i},$$

the binomial theorem yields

$$\mathbb{P}(A_2) \cdot \mathbb{P}(\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2} = 1 | A_2) = \frac{1}{r^{d_3}} \left((r-1)^{d_3} - (r-2)^{d_3} \right) \cdot \left(1 - \left(\frac{r-1}{r}\right)^{d_2}\right)$$

as well as

$$\mathbb{P}(A_3) \cdot \mathbb{P}(\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2} = 1 | A_3) = \frac{1}{r^{d_3}} \left((r-1)^{d_3} - (r-2)^{d_3} \right) \cdot \left(1 - \left(\frac{r-1}{r}\right)^{d_1}\right).$$

Finally, notice that

$$\mathbb{P}(A_4) \cdot \mathbb{P}(\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2} = 1 | A_4) = \mathbb{P}(A_4) = 1 - \mathbb{P}(A_1) - \mathbb{P}(A_2) - \mathbb{P}(A_3).$$

Now straightforward calculations show that

$$\begin{aligned} \mathbb{E}[\mathbf{I}_{v_1} \cdot \mathbf{I}_{v_2}] - \mathbb{E}[\mathbf{I}_{v_1}] \cdot \mathbb{E}[\mathbf{I}_{v_2}] &= \left(\frac{r-2}{r}\right)^{d_3} \left(\frac{r-1}{r}\right)^{d_1+d_2} - \left(\frac{r-1}{r}\right)^{d_1+d_2+2d_3} \\ &\leq 0. \end{aligned}$$

Thus (1) holds true and we have shown

$$\text{Var}(X_{\mathcal{H}}) \leq \sum_{v \in V} (\mathbb{E}[\mathbf{I}_v] - \mathbb{E}[\mathbf{I}_v]^2).$$

Now write

$$\begin{aligned} \mathbb{E}[\mathbf{I}_v] - \mathbb{E}[\mathbf{I}_v]^2 &= \left(1 - \left(\frac{r-1}{r}\right)^{\deg_{\mathcal{H}}(v)}\right) - \left(1 - \left(\frac{r-1}{r}\right)^{\deg_{\mathcal{H}}(v)}\right)^2 \\ &= \left(\frac{r-1}{r}\right)^{\deg_{\mathcal{H}}(v)} \left(1 - \left(\frac{r-1}{r}\right)^{\deg_{\mathcal{H}}(v)}\right) \\ &\leq \frac{1}{4}, \end{aligned}$$

where the last estimate follows from the inequality $x(1-x) \leq 1/4$, for $x \in [0, 1]$. Hence $\text{Var}(X_{\mathcal{H}}) \leq n/4$ and Theorem 1.4 follows from Theorem 1.1.

3. Proof of Theorem 1.5

Throughout this section, we denote by $\text{Bin}(n, 1/2)$ the binomial distribution of parameters n and $1/2$ and we occasionally identify a random variable with its distribution. Given two random variables Z, W , the notation $Z \sim W$ indicates that they have the same distribution. Recall that we work within the class \mathcal{G}_n and that, given $G \in \mathcal{G}_n$, we denote by X_G the number of vertices with non-zero in-degree after a random orientation on the edges of G . A non-negative integer which is equal to zero *mod* 2 is referred to as an *even* integer. Finally, $\text{Bin}(n, e)$ denotes a $\text{Bin}(n, 1/2)$ random variable conditional on the event that it is *even*. In particular, notice that

$$\mathbb{P}[\text{Bin}(n, e) = k] = \binom{n}{k} \frac{1}{2^{n-1}}, \text{ for even } k,$$

a fact that is immediate upon observing that the probability that a $\text{Bin}(n, 1/2)$ random variable is even equals $1/2$ (see also [6, Lemma 1] for a more general result). We begin by showing that the entropy of X_{C_n} , where C_n denotes a cycle on n vertices, equals the entropy of a $\text{Bin}(n, e)$ random variable. The following result is a direct consequence of [7, Theorem 4], but we include here an independent proof (which borrows ideas from [7]) for the sake of completeness.

Lemma 3.1. *Let C_n be a cycle on $n \geq 3$ vertices. Then $H(X_{C_n}) = H(\text{Bin}(n, e))$.*

Proof. Let Z be the number of vertices with zero in-degree after the random orientation on the edges of C_n , and let E be the number of vertices with even in-degree. Notice that $X_{C_n} = n - Z$ and thus

$$H(X_{C_n}) = H(Z).$$

For $i \in \{1, 2\}$, let $V_i := \{v \in C_n : \deg_{C_n}^-(v) = i\}$. The in-degree sum formula yields

$$n = \sum_{v \in C_n} \deg_{C_n}^-(v) = |V_1| + 2|V_2|.$$

Since C_n has n vertices, we also have

$$n = Z + |V_1| + |V_2|.$$

If we subtract the last two equations we get $Z = |V_2|$ and thus, since $E = Z + |V_2|$, we conclude

$$E = 2 \cdot Z.$$

In particular, this implies that E is even and

$$H(Z) = H(E).$$

Hence it is enough to determine the entropy of E . We claim that

$$\mathbb{P}(E = 2k) = \binom{n}{2k} \frac{1}{2^{n-1}}, \text{ for } k \leq n/2.$$

This is clearly true for $k = 0$, so assume that $k > 0$ is such that $k \leq n/2$. Now notice that if we fix an orientation of the edges of C_n for which $E = 2k$ then between any two vertices with zero in-degree there exists a vertex whose in-degree equals two and, conversely, between any two vertices with in-degree equal to two there exists a vertex whose in-degree equals zero. This implies that if we fix $2k$ vertices $\{v_1, \dots, v_{2k}\}$, whose in-degree is even, and we fix the in-degree of vertex v_1 , say $\deg_G^-(v_1) = 0$, then the in-degrees of all other vertices are determined (that is, $v_2 \in V_2, v_3 \in V_0, \dots, v_{2k} \in V_2$ and all other vertices have in-degree one). Since there are two ways to choose the in-degree of v_1 , the claim follows. Summarising, we have shown that E has the same distribution as a $\text{Bin}(n, e)$ random variable and so their entropies are equal. \square

Using Lemma 3.1, we obtain a lower bound on the entropy of X_{C_n} which is expressed in terms of the entropy of a binomial random variable.

Lemma 3.2. *Let C_n be a cycle of $n \geq 3$ vertices. Then*

$$H(X_{C_n}) \geq H(\text{Bin}(n-1, 1/2)) - 1.$$

Proof. Lemma 3.1 implies that it is enough to show that

$$H(\text{Bin}(n, e)) \geq H(\text{Bin}(n-1, 1/2)) - 1.$$

We begin by showing that an outcome of $\text{Bin}(n, e)$ can be obtained as follows: First draw from a $\text{Bin}(n-1, 1/2)$ random variable. If the outcome is even, then add 0 to the outcome. If the outcome is odd, then add 1. Formally, let $X \sim \text{Bin}(n-1, 1/2)$ and define the random variable δ_X by

$$\delta_X = \begin{cases} 0, & \text{if } X \text{ is even} \\ 1, & \text{if } X \text{ is odd.} \end{cases}$$

We claim that

$$\text{Bin}(n, e) \sim X + \delta_X \tag{2}$$

To prove (2), notice that $X + \delta_X$ is always even and fix an even integer k from $\{0, 1, \dots, n\}$. Then, using the convention $\binom{a}{b} = 0$, whenever $b < 0$ or $a < b$, and the relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we may write

$$\begin{aligned} \mathbb{P}(X + \delta_X = k) &= \mathbb{P}(\text{Bin}(n-1, 1/2) = k) + \mathbb{P}(\text{Bin}(n-1, 1/2) = k-1) \\ &= \binom{n}{k} \frac{1}{2^{n-1}} \\ &= \mathbb{P}(\text{Bin}(n, e) = k), \end{aligned}$$

and (2) follows. Next, we claim that

$$H(X|X + \delta_X) \leq 1 \tag{3}$$

To prove (3), write

$$H(X|X + \delta_X) = \sum_{k \text{ even}} H(X|X + \delta_X = k) \cdot \mathbb{P}(X + \delta_X = k).$$

and notice that conditional on the event $\{X + \delta_X = k\}$ it follows that $X \in \{k-1, k\}$. This implies that

$$H(X|X + \delta_X = k) \leq \log 2 = 1$$

and (3) follows upon observing that

$$H(X|X + \delta_X) \leq \sum_{k \text{ even}} \mathbb{P}(X + \delta_X = k) = 1.$$

Since X determines $X + \delta_X$, we have $H(X + \delta_X|X) = 0$ and therefore the chain rule for entropy (see [2, p. 17]) implies

$$\begin{aligned} H(X + \delta_X) &= H(X) + H(X + \delta_X|X) - H(X|X + \delta_X) \\ &= H(\text{Bin}(n-1, 1/2)) - H(X|X + \delta_X) \\ &\geq H(\text{Bin}(n-1, 1/2)) - 1, \end{aligned}$$

where the inequality follows from (3). Finally, using (2), we deduce

$$H(\text{Bin}(n, e)) \geq H(\text{Bin}(n-1, 1/2)) - 1,$$

as desired. □

The proof of Theorem 1.5 is almost complete.

Proof of Theorem 1.5. It is known (see [1, p. 4]) that

$$H(\text{Bin}(n-1, 1/2)) \geq \frac{1}{2} \log(n-1) + \frac{1}{2} \log(\pi e) - \frac{1}{2}.$$

Hence, using the inequality $\log(n-1) \geq \log(n) - \frac{1}{(n-1)\ln 2}$, we conclude

$$H(\text{Bin}(n-1, 1/2)) \geq \frac{1}{2} \log(n) + \frac{1}{2} \log(\pi e) - \frac{1}{2} - \frac{1}{2\ln(2)(n-1)}$$

and Lemma 3.2 yields

$$H(X_{C_n}) \geq \frac{1}{2} \log(n) + \frac{1}{2} \log(\pi e) - \frac{3}{2} - \frac{1}{2\ln(2)(n-1)}.$$

The result follows. □

4. A conjecture

In this section we define a hypergraph from the class $\mathcal{D}_{n,n,r}$ which, we believe, maximizes the entropy of the number of vertices of non-zero in-degree, after a random orientation on its edges.

Fix a positive integer $r \geq 3$. A circular r -uniform hypergraph on $n > r$ vertices is defined as follows: Begin with a cycle C_n on n vertices, identified with \mathbb{Z}_n . A proper subset $I \subset \mathbb{Z}_n$ is a *path* in C_n if it induces a connected sub-graph of the graph C_n . The *size* of a path is the number of vertices in the corresponding induced sub-graph. We call a hypergraph *circular* if (up to isomorphism) its set of vertices is equal to \mathbb{Z}_n and its edges are paths of C_n of size r . We denote by $\mathcal{C}_{n,r}$ the circular hypergraph on n vertices whose edge set consists of all paths of

size r . Circular hypergraphs have attracted some attention due to the fact that they share similar properties with certain classes of matrices (see [8, 9]). Recall that $X_{\mathcal{C}_{n,r}}$ denotes the number of vertices with non-zero in-degree, after a random orientation on the edges of $\mathcal{C}_{n,r}$.

We conjecture that circular hypergraphs are such that

$$H(X_{\mathcal{C}_{n,r}}) \geq H(X_{\mathcal{H}}), \text{ for all } \mathcal{H} \in \mathcal{D}_{n,n,r}.$$

Moreover, we believe that the following holds true.

Conjecture 4.1. Fix a positive integer $r \geq 3$. Then for all $n > r$, it holds

$$H(X_{\mathcal{C}_{n,r}}) \geq \frac{1}{2} \log(n/r) - O(1).$$

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