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EXTENSIONS OF RINGS OVER 2-PRIMAL RINGS

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For a set of endomorphisms $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and derivations $\Delta := \{\delta_1, \ldots, \delta_n\}$, we first introduce Σ-compatible ideals which are a generalization of Σ-rigid ideals and study the connections of the prime radical and the upper nil radical of R with the prime radical and the upper nil radical of the skew PBW extension. Let $A = R\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle$ be a bijective skew PBW extension of an (Σ, Δ)-compatible ring R. (i) It is shown that if R is a (semi)prime ring, then A is a (semi)prime ring. (ii) If R is a completely (semi)prime ring, then A is a completely (semi)prime ring. (iii) If R is a strongly (semi)prime ring, then A is a strongly (semi)prime ring. Also, we prove that R is 2-primal if and only if the skew PBW extension A is 2-primal if and only if $nil(R) = nil_*(R; \Sigma \cup \Delta)$ if and only if $nil(R)\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle = nil_*(A)$ if and only if every minimal (Σ, Δ)-prime ideal of R is completely prime.

1. Introduction

Let R denote an associative ring with identity. We use $nil_*(R)$, $nil^*(R)$ and $nil(R)$, to denote the lower nil radical (i.e., the intersection of all prime ideals), the upper nil radical (i.e. the sum of all nil ideals) and the set of all nilpotent elements of R, respectively.

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A ring R is called 2-*primal* if $nil_*(R) = nil(R)$ (see [4]). Every reduced ring is obviously 2-primal. Moreover, 2-primal rings have been extended to the class of rings which satisfy $nil^*(R) = nil(R)$, but the converse does not hold [5, Example 3.3]. Observe that R is a 2-primal ring if and only if $nil_*(R) = nil^*(R) = nil(R)$, if and only if $nil_*(R)$ is a completely semiprime ideal (i.e., $a^2 \in nil_*(R)$ implies that $a \in nil_*(R)$ for $a \in R$) of R. We refer to [4, 5, 11, 12, 14, 30, 31] for more details on 2-primal rings. Recall that a ring R is called *strongly prime* if R is prime with no nonzero nil ideals. An ideal P of R is *strongly prime* if R/P is a strongly prime ring. All (strongly) prime ideals are taken to be proper. We say an ideal P of a ring R is *minimal (strongly) prime* if P is minimal among (strongly) prime ideals of R. Note that (see [29])

$$nil^*(R) = \cap\{P \mid P \text{ is a minimal strongly prime ideal of } R\}.$$

Recall that an ideal P of R is *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. Every completely prime ideal of R is strongly prime and every strongly prime ideal is prime.

Let σ be an endomorphism of R and δ a σ-derivation of R (so δ is an additive map satisfying $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$). The general (left) Ore extension $R[x;\sigma,\delta]$ is the ring of polynomials over R in the variable x, with coefficients written on the left of x and with termwise addition, subject to the skew-multiplication rule $xr = \sigma(r)x + \delta(r)$ for $r \in R$. If σ is an injective endomorphism of R, then we say $R[x;\sigma,\delta]$ is an Ore extension of injective type. If σ is an identity map on R or $\delta = 0$, then we denote $R[x;\sigma,\delta]$ by $R[x;\delta]$ and $R[x;\sigma]$, respectively.

An endomorphism σ of R is called a *rigid endomorphism* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be σ-*rigid* if there exists a rigid endomorphism σ of R (for more details see [15]). According to Hong et al. [13], an σ-ideal I is called a σ-*rigid ideal* if $a\sigma(a) \in I$ implies $a \in I$ for $a \in R$. Hong et al. in [13] studied some connections between the σ-rigid ideals of R and the related ideals of Ore extensions. They also studied the relationship of $nil_*(R)$ (resp., $nil^*(R)$) and $nil_*(R[x;\sigma,\delta])$ (resp., $nil^*(R[x;\sigma,\delta])$), where $nil_*(R)$ (resp., $nil^*(R)$) is a σ-rigid ideal of R. They proved that if $nil_*(R)$ (resp., $nil^*(R)$) is a σ-rigid δ-ideal of R, then $nil_*(R[x;\sigma,\delta]) \subseteq nil_*(R)[x;\sigma,\delta]$ (resp., $nil^*(R[x;\sigma,\delta]) \subseteq nil^*(R)[x;\sigma,\delta]$).

Following [10], we say that R is σ-*compatible* if for each $a, b \in R$, $ab = 0$ if and only if $a\sigma(b) = 0$. Note that if R is σ-compatible, then σ is injective. Moreover, R is said to be δ-*compatible* if for each $a, b \in R$, $ab = 0$ implies that $a\delta(b) = 0$. If R is both σ-compatible and δ-compatible, we say that R is (σ,δ)-*compatible*. Note that (σ,δ)-compatible rings are a generalization of σ-rigid ring to the more general case where R is not assumed to be reduced. According to [8], an ideal I of R is called a σ-*compatible ideal* of R if for each $a, b \in R$, $ab \in$

I if and only if $a\sigma(b) \in I$. Moreover, I is called a *δ-compatible ideal* if for each $a, b \in R$, $ab \in I$ implies $a\delta(b) \in I$. If I is both σ-compatible and δ-compatible, we say that I is an *(σ, δ)-compatible ideal*. In [8], the connections between $nil_*(R)$ (resp., $nil^*(R)$) and that of $nil_*(R[x;\sigma,\delta])$ (resp., $nil^*(R[x;\sigma,\delta])$) are studied, where $nil_*(R)$ (resp., $nil^*(R)$) is an (σ, δ)-compatible ideal of R.

In [21], the author continued the study of the radicals of Ore extensions in case R is (σ, δ)-compatible. He proved that if R is (σ, δ)-compatible, then $R[x;\sigma,\delta]$ is 2-primal if and only if R is 2-primal if and only if $nil_*(R, \sigma, \delta) = nil(R)$ if and only if $nil(R)[x;\sigma,\delta] = nil_*(R[x;\sigma,\delta])$.

Other ring-theoretic extensions of a ring R are the Poincaré-Birkhoff-Witt (PBW for short) which were defined by Bell and Goodearl [3]. The skew Poincaré-Birkhoff-Witt (skew PBW for short) extensions introduced by Gallego and Lezema [6] are a generalization of PBW extensions, which are more general than Ore extensions of injective type. These extensions include several algebras which can not be expressed as Ore extensions (universal enveloping algebras of finite Lie algebras, diffusion algebras, etc.). More exactly, it has been shown that skew PBW extensions contain various well-known groups of algebras such as some types of Auslander-Gorenstein rings, some skew Calabi-Yau algebras, quantum polynomials, some quantum universal enveloping algebras, etc. (see [6, 22]).

In this paper, for a set of endomorphisms $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and derivations $\Delta := \{\delta_1, \ldots, \delta_n\}$, we first introduce Σ-compatible ideals which are a generalization of Σ-rigid ideals and study the connections of the prime radical and the upper nil radical of R with the prime radical and the upper nil radical of the skew PBW extension. Let $A = R\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle$ be a bijective skew PBW extension of an (Σ, Δ)-compatible ring R. (i) It is shown that if R is a (semi)prime ring, then A is a (semi)prime ring. (ii) If R is a completely (semi)prime ring, then A is a completely (semi)prime ring. (iii) If R is a strongly (semi)prime ring, then A is a strongly (semi)prime ring. Also, we prove that R is 2-primal if and only if the skew PBW extension A is 2-primal if and only if $nil(R) = nil_*(R; \Sigma \cup \Delta)$ if and only if $nil(R)\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle = nil_*(A)$ if and only if every minimal (Σ, Δ)-prime ideal of R is completely prime.

2. Definitions and basic properties of skew PBW extensions

We start by recalling the definition of (skew) *PBW* extensions and present some essential properties of these rings.

Let R and A be rings. According to Bell and Goodearl [3], we say that A is a Poincaré-Birkhoff-Witt extension (also called a PBW extension) of R, denoted by $A := R\langle x_1, \ldots, x_n\rangle$, if the following conditions hold:

1. $R \subseteq A$;
2. There exist elements $x_1, \ldots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $Mon(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n\}$;
3. $x_i r - r x_i \in R$ for each $r \in R$ and $1 \leq i \leq n$;
4. $x_i x_j - x_j x_i \in R + Rx_1 + \cdots + Rx_n$, for any $1 \leq i, j \leq n$.

Definition 2.1. [6, Definition 1] Let R and A be rings. We say that A is a *skew PBW extension of R* (also called a *σ-PBW extension*) if the following conditions hold:

1. $R \subseteq A$;
2. There exist elements $x_1, \ldots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $Mon(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n\}$;
3. For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$;
4. For each $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$.

Proposition 2.2. *[6, Proposition 3] Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension of R. For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$.*

Let A be a skew PBW extension of a ring R. By using Proposition 2.2, we denote A by $R\langle x_1, \ldots, x_n; \Sigma, \Delta \rangle$. We recall the following definition (cf. [6]).

Definition 2.3. Let $A = R\langle x_1, \ldots, x_n; \Sigma, \Delta \rangle$ be the skew PBW extension of a ring R.

1. A is called *quasi-commutative* if the conditions (3) and (4) in Definition 2.1 are replaced by ($3'$): for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$; ($4'$): for any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$
2. A is called *bijective* if σ_i is bijective for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Clearly any PBW extension is a skew PBW extension. Many important class of rings and algebras are skew PBW extensions, for example:

Example 2.4. 1. Any *skew polynomial ring* $R[x;\sigma,\delta]$, with σ injective, is a skew *PBW* extension; in this case, we have $R[x;\sigma,\delta] \cong \sigma(R)\langle x\rangle$. If additionally $\delta = 0$, then $R[x;\sigma]$ is quasi-commutative. Any iterated skew polynomial ring $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ is a skew-PBW extension if it satisfies the following conditions:

• for $1 \leq i \leq n$, σ_i is injective;

• for every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in R$;

• for $i < j$, $\sigma_j(x_i) = cx_i + d$, with $c,d \in R$ and c has a left inverse;

• for $i < j$, $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_n$.

then, $R[x_1;\sigma_1,\delta_1]\cdots R[x_n;\sigma_n,\delta_n]$ is a skew *PBW* extension. Under these assumptions, we have

$$R[x_1;\sigma_1,\delta_1]\cdots R[x_n;\sigma_n,\delta_n] \cong \sigma(R)\langle x_1,\dots,x_n\rangle .$$

In particular, any Ore extension $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ is a skew *PBW* extension, when for $1 \leq i \leq n$, σ_i is injective. Note that in Ore extensions for every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in R$, and for $i < j$, $\sigma_j(x_i) = x_i$ and $\sigma_j(x_i) = 0$. An important subclass of Ore extension are the *Ore algebras*, i.e., $R = k[t_1,\dots,t_m]$ and $m \geq 1$. Thus, we have

$$k[t_1,\dots,t_m][x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] \cong \sigma(k[t_1,\dots,t_m])\langle x_1,\dots,x_n\rangle .$$

2. Let k be a ring and $q \in k$ a central unit. Then the *quantum n-space* is the ring $\mathcal{O}_q(k^n)$, generated by k together with n additional elements $x_1,\dots,x_n$ which commute with all elements of k, and such that $x_ix_j = qx_jx_i$ for all $i < j$. Clearly $\mathcal{O}_q(k^2) = k[y][x;\sigma]$, where $k[y]$ is a polynomial ring and σ is the monomorphism of $k[y]$ such that $\sigma = 1$ on k and $\sigma(y) = qy$; we have also $\mathcal{O}_q(k^n) = k[x_1][x_2;\sigma_2][x_3;\sigma_3]\dots[x_n;\sigma_n]$, where $k[x_1]$ is an ordinary polynomial ring and σ_i is a monomorphism of $k[x_1][x_2;\sigma_2][x_3;\sigma_3]\dots[x_{i-1};\sigma_{i-1}]$, for $2 \leqslant i \leqslant n$.

3. Let k be a ring and $q \in k$ a central unit. Then the n^{th} *quantized Weyl algebra* over k is the ring $A_n(k,q)$ generated by k together with $2n$ additional elements $x_1,\dots,x_n,y_1,\dots,y_n$ which commute with all elements of k, and such that $x_iy_i - qy_ix_i = 1$ for $1 \leq i \leq n$, and $x_ix_j = x_jx_i,\ y_iy_j = y_jy_i,\ x_iy_j = y_jx_i$ for all $i,j,\ i \neq j$. Clearly $A_1(k,q) = k[y_1][x_1;\sigma,\delta]$, where $k[y_1]$ is a polynomial ring and σ is the monomorphism of $k[y_1]$ such that $\sigma = 1$ on k and $\sigma(y_1) = qy_1$, and δ is the q-difference operator [7, page 371]. The n^{th}

quantized Weyl algebra, $A_n(k,q)$ over k, can also be viewed as an iterated skew polynomial ring.

4. *Quantum plane* $\mathcal{O}_q(\mathbf{k}^2)$. Let $q \in \mathbf{k}^*$. The *quantized coordinate ring* of $\mathbf{k}^2$ is a $\mathbf{k}$-algebra, denoted by $\mathcal{O}_q(\mathbf{k}^2)$, presented by two generators x, y and the relation $xy = qyx$. We have $\mathcal{O}_q(\mathbf{k}^2) \cong \sigma(\mathbf{k})\langle x, y\rangle$.

5. *The algebra of q-differential operators* $D_{q,h}[x,y]$. Let $q, h \in \mathbf{k}, q \neq 0$; consider $\mathbf{k}[y][x;\sigma,\delta]$, where $\sigma(y) := qy$ and $\delta(y) := h$. By definition of skew polynomial ring we have $xy = \sigma(y)x + \delta(y) = qyx + h$, and hence $xy - qyx = h$. Therefore, $D_{q,h}[x,y] \cong \sigma(\mathbf{k})\langle x, y\rangle$.

6. *Algebra of linear partial differential operators*. The n-th Weyl algebra $A_n(\mathbf{k})$ over $\mathbf{k}$ coincides with the $\mathbf{k}$-algebra of linear partial differential operators with polynomial coefficients $\mathbf{k}[t_1, \ldots, t_n]$. As we have seen, the generators of $A_n(\mathbf{k})$ satisfy the following relations:

$$\begin{aligned} &t_i t_j = t_j t_i, \qquad \partial_i \partial_j = \partial_j \partial_i, \qquad && 1 \leq i < j \leq n,\\ &\partial_j t_i = t_i \partial_j + \delta_{ij}, && 1 \leq i, j \leq n, \end{aligned}$$

where δ_{ij} is the Kronecker symbol. Therefore $\sigma(\mathbf{k})\langle t_1, \ldots, t_n; \partial_1, \ldots, \partial_n\rangle$.

A detailed list of examples of skew PBW extensions is presented in [17, 18, 22, 23, 25, 26].

Now, we give some examples of skew PBW extensions which can not be expressed as Ore extensions (a more complete list can be found in [17, 22]).

Example 2.5.

1. Let k be a commutative ring and $\mathfrak{g}$ a finite dimensional Lie algebra over k with basis $\{x_1, \ldots, x_n\}$; the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $\mathcal{U}(\mathfrak{g})$, is a PBW extension of k (see [17]). In this case, $x_i r - r x_i = 0$ and $x_i x_j - x_j x_i = [x_i, x_j] \in \mathfrak{g} = k + kx_1 + \cdots + kx_n$, for any $r \in k$ and $1 \leq i, j \leq n$.

2. Let $k, \mathfrak{g}, \{x_1, \ldots, x_n\}$ and $\mathcal{U}(\mathfrak{g})$ be as in the previous example; let R be a k-algebra containing k. The *tensor product* $A := R \otimes_k \mathcal{U}(\mathfrak{g})$ is a PBW extension of R, and it is a particular case of a more general construction, the *crossed product* $R * \mathcal{U}(\mathfrak{g})$ of R by $\mathcal{U}(\mathfrak{g})$, that is also a PBW extension of R (see [20]).

3. The *twisted or smash product differential operator ring* $k\#_\sigma\mathcal{U}(\mathfrak{g})$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra acting on k by derivations, and σ is Lie 2-cocycle with values in k.

Definition 2.6. [6, Definition 6] Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$ and σ_i-derivations δ_i as in Proposition 2.2.

1. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n, \sigma^\alpha := \sigma_1^{\alpha_1} \dots \sigma_n^{\alpha_n}, \delta^\alpha := \delta_1^{\alpha_1} \dots \delta_n^{\alpha_n}, |\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$; then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

2. For $X = x^\alpha \in \mathrm{Mon}(A), \exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ will denote a total order defined on $\mathrm{Mon}(A)$ (a total order on $\mathbb{N}_0^n$). For an element $x^\alpha \in \mathrm{Mon}(A), \exp(x^\alpha) := \alpha \in \mathbb{N}_0^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$.
   Every element $f \in A$ can be expressed uniquely as $f = a_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R \setminus \{0\}$, and $X_m \succ \cdots \succ X_1$. With this notation, we define $lm(f) := X_m$, the *leading monomial* of $f; lc(f) := a_m$, the *leading coefficient* of f; $lt(f) := a_m X_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $lm(0) := 0, lc(0) := 0, lt(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$.

Remark 2.7. [6, Remark 2]

1. Since that $Mon(A)$ is a R-basis for A, the elements $c_{i,r}$ and $c_{i,j}$ in the Definition 2.1 are unique.

2. If $r = 0$, then $c_{i,0} = 0$. Moreover, in Definition 2.1(4), $c_{i,i} = 1$.

3. Let $i < j$, there exist $c_{j,i}, c_{i,j} \in R$ such that $x_i x_j - c_{j,i} x_j x_i \in R + R x_1 + \cdots + R x_n$ and $x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n$, but since $Mon(A)$ is a R-basis then $1 = c_{j,i} c_{i,j}$, i.e., for every $1 \leq i < j \leq n$, $c_{i,j}$ has a left inverse and $c_{j,i}$ has a right inverse.

4. Each element $f \in A - \{0\}$ has a unique representation in the form $f = c_1 X_1 + \cdots + c_t X_t$, with $c_i \in R - \{0\}$ and $X_i \in Mon(A)$, $1 \leq i \leq t$.

Skew PBW extensions can be characterized in the following way.

Theorem 2.8. *[6, Theorem 7] Let A be a polynomial ring over R with respect to $\{x_1, \dots, x_n\}$. Then A is a skew PBW extension of R if and only if the following conditions are satisfied:*

1. *For each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}, p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.*

2. *For each* $x^{\alpha}, x^{\beta} \in \mathrm{Mon}(A)$ *there exist unique elements* $c_{\alpha,\beta} \in R$ *and* $p_{\alpha,\beta} \in A$ *such that* $x^{\alpha}x^{\beta} = c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}$, *where* $c_{\alpha,\beta}$ *is left invertible,* $p_{\alpha,\beta} = 0$ *or* $\deg(p_{\alpha,\beta}) < |\alpha+\beta|$ *if* $p_{\alpha,\beta} \neq 0$.

We remember also the following facts [6, Remark 8].

Remark 2.9.

1. A left inverse of $c_{\alpha,\beta}$ will be denoted by $c^{'}_{\alpha,\beta}$. We observe that if $\alpha = 0$ or $\beta = 0$, then $c_{\alpha,\beta} = 1$ and hence $c^{'}_{\alpha,\beta} = 1$.

2. We observe if A is a skew PBW extension quasi-commutative, then from Theorem 2.8, we conclude that $p_{\alpha,r} = 0$

   and $p_{\alpha,\beta} = 0$, for every $0 \neq r \in R$ and every $\alpha, \beta \in \mathbb{N}_0^n$.

3. From Theorem 2.8, we get also that if A is a bijective skew PBW extension, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}_0^n$.

In the next proposition, we will look more closely at the form of the polynomials $p_{\alpha,r}$ and $p_{\alpha,\beta}$ which appear in Theorem 2.8.

Remark 2.10. [24, Remark 2.10]

1. Let x_n be a variable and α_n an element of $\mathbb{N}_0$. Then we have

$$x_n^{\alpha_n} r = \sigma_n^{\alpha_n}(r)x_n^{\alpha_n} + \sum_{j=1}^{\alpha_n} x_n^{\alpha_{n-j}}\delta_n(\sigma_n^{j-1}(r))x_n^{j-1}, \quad \sigma_n^0 := id_R \tag{1}$$

and so

$$\begin{aligned} x_n^{\alpha_n} r = \sigma_n^{\alpha_n}(r)x_n^{\alpha_n} + x_n^{\alpha_n-1}\delta_n(r) + x_n^{\alpha_n-2}\delta_n(\sigma_n(r))x_n + x_n^{\alpha_n-3}\delta_n(\sigma_n^2(r))x_n^2 \\ + \cdots + x_n\delta_n(\sigma_n^{\alpha_n-2}(r))x_n^{\alpha_n-2} + \delta_n(\sigma_n^{\alpha_n-1}(r))x_n^{\alpha_n-1}, \quad \sigma_n^0 := id_R. \end{aligned}$$

Note that

$$\begin{aligned} p_{\alpha_n,r} =& x_n^{\alpha_n-1}\delta_n(r) + x_n^{\alpha_n-2}\delta_n(\sigma_n(r))x_n + x_n^{\alpha_n-3}\delta_n(\sigma_n^2(r))x_n^2 \\ &+ \cdots + x_n\delta_n(\sigma_n^{\alpha_n-2}(r))x_n^{\alpha_n-2} + \delta_n(\sigma_n^{\alpha_n-1}(r))x_n^{\alpha_n-1}, \end{aligned}$$

where $p_{\alpha_n,r} = 0$ or $\deg(p_{\alpha_n,r}) < |\alpha_n|$ if $p_{\alpha_n,r} \neq 0$. It is clear that $\exp(p_{\alpha_n,r}) \prec \alpha_n$. Again, using (2.1) in every term of the product $x_n^{\alpha_n} r$

above, we obtain

$$
\begin{aligned}
x_n^{\alpha_n} r = {} & \sigma_n^{\alpha_n}(r)x_n^{\alpha_n} + \sigma_n^{\alpha_n-1}(\delta_n(r))x_n^{\alpha_n-1} + \sum_{j=1}^{\alpha_n-1} x_n^{\alpha_n-1-j}\delta_n(\sigma_n^{j-1}(\delta_n(r)))x_n^{j-1} \\
& + \left[\sigma_n^{\alpha_n-2}(\delta_n(\sigma_n(r)))x_n^{\alpha_n-2} + \sum_{j=1}^{\alpha_n-2} x_n^{\alpha_n-2-j}\delta_n(\sigma_n^{j-1}(\delta_n(\sigma_n(r))))x_n^{j-1}\right] x_n \\
& + \left[\sigma_n^{\alpha_n-3}(\delta_n(\sigma_n^2(r)))x_n^{\alpha_n-3} + \sum_{j=1}^{\alpha_n-3} x_n^{\alpha_n-3-j}\delta_n(\sigma_n^{j-1}(\delta_n(\sigma_n^2(r))))x_n^{j-1}\right] x_n^2 \\
& + \cdots + \left[\sigma_n(\delta_n(\sigma_n^{\alpha_n-2}(r)))x_n + \delta_n(\delta_n(\sigma_n^{\alpha_n-2}(r)))\right] x_n^{\alpha_n-2} \\
& + \delta_n(\sigma_n^{\alpha_n-1}(r))x_n^{\alpha_n-1},
\end{aligned}
$$

which shows that

$$lc(p_{\alpha_n,r}) = \textstyle\sum_{p=1}^{\alpha_n} \sigma_n^{\alpha_n-p}(\delta_n(\sigma_n^{p-1}(r))).$$

In this way, we can see that $lc(p_{\alpha_n,r})$ involves elements obtained evaluating σ_n and δ_n in the element r of R.

2. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n, r \in R$ and $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Then

$$
\begin{aligned}
x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n} r = {} & \sigma_1^{\alpha_1}(\cdots(\sigma_n^{\alpha_n}(r)))x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \\
& + p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \\
& + x_1^{\alpha_1} p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}x_3^{\alpha_3}\cdots x_n^{\alpha_n} \\
& + x_1^{\alpha_1}x_2^{\alpha_2} p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))}x_4^{\alpha_4}\cdots x_n^{\alpha_n} \\
& + \cdots + x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-2}^{\alpha_{n-2}} p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)}x_n^{\alpha_n} \\
& + x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}} p_{\alpha_n,r}.
\end{aligned}
$$

Considering the leading coefficients of $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} r$ we can write this term as

$$
\begin{aligned}
&= \sigma_1^{\alpha_1}(\cdots(\sigma_n^{\alpha_n}(r)))x_1^{\alpha_1}\cdots x_n^{\alpha_n}\\
&+ \left[\sum_{p=1}^{\alpha_1}\sigma_1^{\alpha_1-p}(\delta_1(\sigma_1^{p-1}(\sigma_2^{\alpha_2}(\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))))))\right] x_1^{\deg(p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))})}\\
&\quad x_2^{\alpha_2}\cdots x_n^{\alpha_n}\\
&+ \left[\sum_{p=1}^{\alpha_2}\sigma_1^{\alpha_1}(\sigma_2^{\alpha_2-p}(\delta_2(\sigma_2^{p-1}(\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))))))\right] x_1^{\alpha_1}x_2^{\deg(p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))})}\\
&\quad x_3^{\alpha_3}\cdots x_n^{\alpha_n}\\
&+ \left[\sum_{p=1}^{\alpha_3}\sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\sigma_3^{\alpha_3-p}(\delta_3(\sigma_3^{p-1}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r))))))))\right] x_1^{\alpha_1}x_2^{\alpha_2}\\
&\quad x_3^{\deg(p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))})}x_4^{\alpha_4}\cdots x_n^{\alpha_n}+\cdots\\
&+ \left[\sum_{p=1}^{\alpha_{n-1}}\sigma_1^{\alpha_1}(\cdots(\sigma_{n-2}^{\alpha_{n-2}}(\sigma_{n-1}^{\alpha_{n-1}-p}(\delta_{n-1}(\sigma_{n-1}^{p-1}(\sigma_n^{\alpha_n}(r)))))))\right] x_1^{\alpha_1}\cdots x_{n-2}^{\alpha_{n-2}}\\
&\quad x_{n-1}^{\deg(p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)})}x_n^{\alpha_n}\\
&+ \left[\sum_{p=1}^{\alpha_n}\sigma_1^{\alpha_1}(\cdots(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n-p}(\delta_n(\sigma_n^{p-1}(r))))))\right] x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\deg(p_{\alpha_n,r})}\\
&+ \text{other terms of degree less than } \deg(p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))})\\
&+ \alpha_2+\cdots+\alpha_n\\
&+ \text{other terms of degree less than } \alpha_1+\deg(p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))})\\
&+ \alpha_3+\cdots+\alpha_n\\
&+ \text{other terms of degree less than } \alpha_1+\alpha_2+\deg(p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))})\\
&+ \alpha_4+\cdots+\alpha_n\\
&\ \vdots\\
&+ \text{other terms of degree less than } \alpha_1+\cdots+\alpha_{n-2}+\deg(p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)})\\
&+ \alpha_n\\
&+ \text{other terms of degree less than } \alpha_1+\cdots+\alpha_{n-1}+\deg(p_{\alpha_n,r}).
\end{aligned}
$$

Therefore we can see that the polynomials $p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))},\ldots,p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)}$, and $p_{\alpha_n,r}$ in the expression above for the term $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}r$, involve elements obtained evaluating σ's and δ's in the element r of R.

3. Let $X_i := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $Y_j := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$ and $a_i, b_j \in R$. Then

$$\begin{aligned} a_iX_ib_jY_j =& a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + a_ip_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j} \\ &+ a_ix_1^{\alpha_{i1}}p_{\alpha_{i2},\sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j} \\ &+ a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_4^{\alpha_{i4}}\cdots x_n^{\alpha_{in}}x^{\beta_j} \\ &+ \cdots + a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b_j)}x_n^{\alpha_{in}}x^{\beta_j} \\ &+ a_ix_1^{\alpha_{i1}}\cdots x_{i(n-1)}^{\alpha_{i(n-1)}}p_{\alpha_{in},b_j}x^{\beta_j} \end{aligned}$$

As we saw above, the polynomials $p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))}, \ldots, p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)}$, and $p_{\alpha_n,r}$, involve elements of R obtained evaluating σ_j and δ_j in the element r of R. So, when we compute every summand of $a_iX_ib_jY_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i.

3. Compatible ideals and radicals of skew PBW extensions

Let $A = R\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle$ be a bijective skew PBW extension of a ring R with a set of endomorphisms $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and derivations $\Delta := \{\delta_1, \ldots, \delta_n\}$.

According to Reyes [24], Σ is called a *rigid endomorphisms family* if $a\sigma^{\alpha}(a) = 0$ implies $a = 0$ for each $a \in R$ and $\alpha \in \mathbb{N}_0^n$, where σ^{α} is as mentioned in the Definition 2.6. A ring R is called Σ-*rigid* if there exists a rigid endomorphisms family Σ of R.

In [9] (and independently in [27]), the authors defined Σ-compatible rings, which are a generalization of Σ-rigid rings. A ring R is called Σ-*compatible* if for each $a, b \in R$ and $\alpha \in \mathbb{N}_0^n$, $ab = 0 \Leftrightarrow a\sigma^{\alpha}(b) = 0$, moreover, R is said to be Δ-*compatible* if for each $a, b \in R$ and $\alpha \in \mathbb{N}_0^n$, $ab = 0 \Rightarrow a\delta^{\alpha}(b) = 0$, where σ^{α} and δ^{α} are as mentioned in Definition 2.6. If R is both Σ-compatible and Δ-compatible, we say that R is (Σ, Δ)-*compatible*. In this case, clearly the endomorphism σ_i is injective for every $1 \le i \le n$. In [9, Lemma 3.3], the authors showed that R is Σ-rigid if and only if R is Σ-compatible and reduced. Thus Σ-compatible rings are a generalization of Σ-rigid rings to the more general case where R is not assumed to be reduced.

In this section, motivated by the above facts, for a set of endomorphisms $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ of a ring R, we first define Σ-compatible ideals in R which are a generalization of Σ-rigid ideals and investigate the relationship of $nil_*(R)$ and $nil^*(R)$ with the prime radical and the upper nil radical of the skew PBW extension of R, respectively. Then, we prove our main result by providing a necessary and sufficient condition that a skew PBW extension is (semi)prime, completely (semi)prime and strongly (semi)prime. For some other related results, we refer the reader to consult [1].

Definition 3.1. Let $A = R\langle x_1, \dots, x_n; \Sigma, \Delta\rangle$ be a skew PBW extension of a ring R, where $\Delta := \{\delta_1, \dots, \delta_n\}$ and $\Sigma := \{\sigma_1, \dots, \sigma_n\}$.

1. We say that an ideal I of R is Σ-*ideal* if $\sigma^{\alpha}(I) \subseteq I$; I is Δ-*ideal* if $\delta^{\alpha}(I) \subseteq I$; I is Σ-*invariant* if $\sigma^{-\alpha}(I) = I$ for each $\alpha \in \mathbb{N}_0^n$, where σ^{α} and δ^{α} is as mentioned in the Definition 2.6. If I is both Σ and Δ-ideal, we say that I is (Σ, Δ)-*ideal*.

2. For an Σ-ideal I, we say that I is Σ-*rigid* if $a\sigma^{\alpha}(a) \in I$ implies $a \in I$ for each $a \in R$ and $\alpha \in \mathbb{N}_0^n$.

3. For an ideal I, we say that I is Σ-*compatible* if for each $a, b \in R$ and $\alpha \in \mathbb{N}_0^n$, $ab \in I \Leftrightarrow a\sigma^{\alpha}(b) \in I$. Moreover, I is said to be Δ-*compatible ideal* if for each $a, b \in R$ and $\alpha \in \mathbb{N}_0^n$, $ab \in I \Rightarrow a\delta^{\alpha}(b) \in I$, where σ^{α} and δ^{α} are as mentioned in the Definition 2.6. If I is both Σ-*compatible* and Δ-*compatible*, then we say that I is (Σ, Δ)-*compatible*.

Clearly, R is a Σ-rigid ring if and only if $\{0\}$ is Σ-rigid ideal of R. Also, R is a Σ-compatible (resp., Δ-compatible) ring if and only if $\{0\}$ is Σ-compatible (resp., Δ-compatible) ideal of R.

Let $I \subseteq R$. We denote the set of all elements of A with coefficients in I by $I\langle x_1, \dots, x_n\rangle$. If A is a skew PBW extension of a ring R and I is an (Σ, Δ)-ideal of R, then by using Remark 2.10, one can show that $I\langle x_1, \dots, x_n\rangle$ is an ideal of A and we denote it by $I\langle x_1, \dots, x_n; \Sigma, \Delta\rangle$.

If I is an (Σ, Δ)-ideal of R, then for every $1 \leq i \leq n$, $\sigma_i : R/I \to R/I$ defined by $\overline{\sigma_i}(a + I) = \sigma_i(a) + I$ is an endomorphism and $\overline{\delta_i} : R/I \to R/I$ defined by $\overline{\delta_i}(a + I) = \delta_i(a) + I$ is an $\overline{\sigma_i}$-derivation.

Lemma 3.2. *Let I be an Σ-compatible ideal of a ring R. Then I is Σ-invariant.*

Proof. It is sufficient to prove that $\sigma_t^{-1}(I) = I$ for every $1 \leq t \leq n$. Let $a \in \sigma_t^{-1}(I)$. Then $\sigma_t(a) \in I$. Since I is Σ-compatible ideal, $1\sigma_t(a) \in I$ implies $a \in I$. Thus I is an Σ-invariant ideal of R. $\square$

Proposition 3.3. *Let I be an (Σ, Δ)-compatible ideal of R and $a, b \in R$. Then we have the following:*

1. *If $ab \in I$ then $a\sigma^{\alpha}(b) \in I$ and $\sigma^{\alpha}(a)b \in I$ for each $\alpha \in \mathbb{N}_0^n$.*

2. *If $ab \in I$ then $\sigma^{\alpha}(a)\delta^{\beta}(b), \delta^{\beta}(a)\sigma^{\alpha}(b) \in I$ for each $\alpha, \beta \in \mathbb{N}_0^n$.*

3. *If* $a\sigma^{\theta}(b) \in I$ *or* $\sigma^{\theta}(a)b \in I$ *for some* $\theta \in \mathbb{N}_0^n$*, then* $ab \in I$.

Proof. (1) It is sufficient to prove that if $ab \in I$ then $a\sigma_t(b) \in I$ and $\sigma_t(a)b \in I$ for every $1 \leq t \leq n$. If $ab \in I$, then $\sigma_t(a)\sigma_t(b) \in I$, since I is an Σ-ideal. Hence by Σ-compatibility of I, $\sigma_t(a)b \in I$ for every $1 \leq t \leq n$.

(2) It is sufficient to prove that $\delta_t(a)\sigma_t(b) \in I$ for every $1 \leq t \leq n$. If $ab \in I$, then by (1) and Δ-compatibility of I, $\sigma_t(a)\delta_t(b) \in I$ for every $1 \leq t \leq n$. Hence $\delta_t(a)b = \delta_t(ab) - \sigma_t(a)\delta_t(b) \in I$ for every $1 \leq t \leq n$. Thus $\delta_t(a)b \in I$ for every $1 \leq t \leq n$. Since I is an Σ-compatible ideal, $\delta_t(a)\sigma_t(b) \in I$ for every $1 \leq t \leq n$.

(3) It is enough to show that $a\sigma_t(b) \in I$ or $\sigma_t(a)b \in I$ for every $1 \leq t \leq n$, then $ab \in I$. Let $\sigma_t(a)b \in I$ for every $1 \leq t \leq n$. Then by Σ-compatibility of I, $\sigma_t(a)\sigma_t(b) \in I$ and so $\sigma_t(ab) \in I$ for every $1 \leq t \leq n$. Hence $ab \in I$, since I is Σ-invariant, by Lemma 3.2. $\square$

Theorem 3.4. *Let* $A = R\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle$ *be a bijective skew PBW extension of* R *and* I *an* (Σ, Δ)*-compatible semiprime ideal of* R*. Assume that* $f = a_0 + a_1X_1 + \cdots + a_mX_m,\ g = b_0 + b_1Y_1 + \cdots + b_tY_t \in A$*. Then the following are equivalent:*

1. $fAg \subseteq I\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle$*;*

2. $a_iRb_j \subseteq I$ *for each* i, j.

Proof. $(1) \Rightarrow (2)$. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m,\ g = b_0 + b_1Y_1 + \cdots + b_tY_t \in A$, where $a_i \in R$, $1 \leq i \leq m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $X_m \succ X_{m-1} \succ \cdots \succ X_1$, and $b_j \in R$, $1 \leq j \leq t$, $b_t \neq 0$, with $Y_j = x^{\alpha_j} = x_1^{\alpha_{j1}} \cdots x_n^{\alpha_{jn}}$, $Y_t \succ Y_{t-1} \succ \cdots \succ Y_1$. Assume that $fAg \subseteq I\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle$. Then

$$(a_0 + a_1X_1 + \cdots + a_mX_m)c(b_0 + b_1Y_1 + \cdots + b_tY_t) \in I\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle \qquad (2)$$

for each $c \in R$, and hence

$$\text{“other terms of order less than”} + a_mX_mcb_tY_t \in I\langle x_1, \ldots, x_n; \Sigma, \Delta\rangle.$$

By Theorem 2.8, we have

$$\begin{aligned} a_mX_mcb_tY_t &= a_m[\sigma^{\alpha_m}(cb_t)x^{\alpha_m} + p_{\alpha_m, cb_t}]x^{\beta_t} \\ &= a_m\sigma^{\alpha_m}(cb_t)x^{\alpha_m}x^{\beta_t} + a_mp_{\alpha_m, cb_t}x^{\beta_t} \\ &= a_m\sigma^{\alpha_m}(cb_t)[c_{\alpha_m, \beta_t}x^{\alpha_m+\beta_t} + p_{\alpha_m, \beta_t}] + a_mp_{\alpha_m, cb_t}x^{\beta_t} \\ &= a_m\sigma^{\alpha_m}(cb_t)c_{\alpha_m, \beta_t}x^{\alpha_m+\beta_t} + a_m\sigma^{\alpha_m}(cb_t)p_{\alpha_m, \beta_t} + a_mp_{\alpha_m, cb_t}x^{\beta_t}, \end{aligned}$$

where $p_{\alpha_m, cb_t} = 0$ or $\deg(p_{\alpha_m, cb_t}) < |\alpha_m|$ if $p_{\alpha_m, cb_t} \neq 0$ and $p_{\alpha_m, \beta_t} = 0$ or $\deg(p_{\alpha_m, \beta_t}) < |\alpha_m + \beta_t|$ if $p_{\alpha_m, \beta_t} \neq 0$. Since A is bijective so by using Remark 2.9, from the equality $lc(fAg) = a_m\sigma^{\alpha_m}(cb_t)c_{\alpha_m, \beta_t} \in I$ we obtain $a_m\sigma^{\alpha_m}(cb_t) \in I$

and hence $a_m cb_t \in I$, since I is an (Σ,Δ)-compatible ideal. Also by Remark 2.10, we can see that the polynomial p_{α_m, cb_t} involve elements obtained evaluating σ's and δ's in the element cb_t of R. Thus $a_m p_{\alpha_m,cb_t}$, $a_m\sigma^{\alpha_m}(cb_t)p_{\alpha_m,\beta_t} \in I$, by Proposition 3.3. If we replace c by $cb_{t-1}da_me$ in Eq. (2), where $c,d,e \in R$, then we get

$$(a_0 + a_1X_1 + \cdots + a_mX_m)cb_{t-1}da_me(b_0 + b_1Y_1 + \cdots + b_tY_t) \in I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle .$$

Hence by a similar argument as above, we have

$$a_m\sigma^{\alpha_m}(cb_{t-1}da_meb_{t-1})c_{\alpha_m,\beta_{t-1}} \in I,$$

and $a_m\sigma^{\alpha_m}(cb_{t-1}da_meb_{t-1}) \in I$. Then $a_mcb_{t-1}da_meb_{t-1} \in I$, since I is Σ-compatible. Thus $(Ra_mRb_{t-1})^2 \subseteq I$. Hence $(Ra_mRb_{t-1}) \subseteq I$, since I is semiprime. Continuing in this way, we obtain $a_mRb_k \subseteq I$, for $k = 0,1,\ldots,t$. Hence by (Σ,Δ)-compatibility of I, we get $(a_0 + a_1X_1 + \cdots + a_mX_m)A(b_0 + b_1Y_1 + \cdots + b_tY_t) \subseteq I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$. Then by using induction on $|\alpha_m + \beta_t|$, we obtain $a_iRb_j \subseteq I$ for each i, j.

$(2) \Rightarrow (1)$. It follows from Proposition 3.3. □

Corollary 3.5. *Let $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a bijective skew PBW extension of R and I a (semi)prime (Σ,Δ)-compatible ideal of R. Then $I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ is a (semi)prime ideal of A.*

Proof. Suppose that I is a prime (Σ,Δ)-compatible ideal of R. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m$, $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in A$ such that $fAg \subseteq I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$, where $a_i \in R$, $1 \le i \le m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$, $X_m \succ X_{m-1} \succ \cdots \succ X_1$, and $b_j \in R$, $1 \le j \le t$, $b_t \neq 0$, with $Y_j = x^{\alpha_j} = x_1^{\alpha_{j1}}\cdots x_n^{\alpha_{jn}}$, $Y_t \succ Y_{t-1} \succ \cdots \succ Y_1$. Then by Theorem 3.4, we have $a_iRb_j \subseteq I$ for each i,j. Now let $g \notin I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ and hence $b_j \notin I$ for some j. Since I is prime we have $a_i \in I$ for each $i = 0,1,\ldots,m$. Thus $f \in I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$. Therefore $I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ is a prime ideal of A. □

Corollary 3.6. *Let $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a bijective skew PBW extension of an (Σ,Δ)-compatible ring R. If R is a (semi)prime ring, then A is a (semi)prime ring.*

Theorem 3.7. *If each minimal prime ideal of R is (Σ,Δ)-compatible, then $nil_*(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ is (Σ,Δ)-compatible ideal of R and $nil_*(A) \subseteq nil_*(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$*

Proof. The result follows from Corollary 3.5. □

Theorem 3.8. *Let $A = R\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ be a bijective skew PBW extension of R and P a completely (semi)prime (Σ,Δ)-compatible ideal of R. Then $P\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ is a completely (semi)prime ideal of A.*

Proof. Assume that P is a completely prime ideal of R. So R/P is domain and hence it is a reduced ring. Then R/P is a $(\overline{\Sigma},\overline{\Delta})$-compatible ring and so R/P is a $\overline{\Sigma}$-rigid, by [9, Lemma 3.5]. Let $\overline{f} = \overline{a_0}+\overline{a_1}X_1+\cdots+\overline{a_m}X_m$, $\overline{g} = \overline{b_0}+\overline{b_0}Y_1+\cdots+\overline{b_0}Y_t \in R/P\langle x_1,\dots,x_n;\overline{\Sigma},\overline{\Delta}\rangle$ such that $\overline{f}\overline{g}=0$. Then by [24, Proposition 3.6], $\overline{f}=0$ or $\overline{g}=0$, which implies that $R/P\langle x_1,\dots,x_n;\overline{\Sigma},\overline{\Delta}\rangle$ is domain. It is easy to see that the mapping $\psi : R\langle x_1,\dots,x_n;\Sigma,\Delta\rangle \to R/P\langle x_1,\dots,x_n;\overline{\Sigma},\overline{\Delta}\rangle$ defined by $\psi(f)=\overline{f}$ is a ring homomorphism. Thus

$$R\langle x_1,\dots,x_n;\Sigma,\Delta\rangle / P\langle x_1,\dots,x_n;\Sigma,\Delta\rangle \cong R/P\langle x_1,\dots,x_n;\overline{\Sigma},\overline{\Delta}\rangle.$$

Therefore $P\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ is a completely prime ideal of A. □

Corollary 3.9. *Let $A = R\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ be a bijective skew PBW extension of an (Σ,Δ)-compatible ring R. If R is a completely (semi)prime ring, then A is a completely (semi)prime ring.*

Proof. It follows from Theorem 3.8. □

Corollary 3.10. *Let $A = R\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ be a bijective skew PBW extension of R and $nil_*(R)$ an Σ-rigid Δ-ideal of R. Then $nil_*(A) \subseteq nil_*(R)\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$.*

Theorem 3.11. *Let $A = R\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ be a bijective skew PBW extension of R and P a strongly (semi)prime (Σ,Δ)-compatible ideal of R. Then $P\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ is a strongly (semi)prime ideal of A.*

Proof. Notice that by Corollary 3.5, $P\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ is a prime ideal of A and hence

$$R\langle x_1,\dots,x_n;\Sigma,\Delta\rangle / P\langle x_1,\dots,x_n;\Sigma,\Delta\rangle \cong R/P\langle x_1,\dots,x_n;\overline{\Sigma},\overline{\Delta}\rangle$$

is a prime ring. Now, we show that $\{0\}$ is the only nil ideal of $R/P\langle x_1,\dots,x_n;\overline{\Sigma},\overline{\Delta}\rangle$. Let I be a nil ideal of $R/P\langle x_1,\dots,x_n;\overline{\Sigma},\overline{\Delta}\rangle$ and I_0 be the set of all leading coefficients of elements of I. First we prove that I_0 is an ideal of R/P. Clearly I_0 is a left ideal of R/P. Let $\overline{a}\in I_0$ and $\overline{r}\in R/P$. Then there exists $\overline{f} = \overline{a_0}+\overline{a_1}X_1+\cdots+\overline{a_m}X_m \in I$ with $\overline{a}=\overline{a_m}$. Hence $(\overline{f}\overline{r})^m = (\overline{a_0}\overline{r}+\overline{a_1}X_1\overline{r}+\cdots+\overline{a_m}X_m\overline{r})^m = 0$

for some non-negative integer m. By Theorem 2.8 and Remark 2.9, we have $\overline{a}\;\overline{\sigma}^{\alpha_m}(\overline{ra})\overline{\sigma}^{2\alpha_m}(\overline{ra})\dots\overline{\sigma}^{(m-1)\alpha_m}(\overline{ra})\overline{\sigma}^{m\alpha_m}(\overline{r})=0$, since it is the leading coefficient of $(\overline{f}\overline{r})^m=0$. Since R/P is $\overline{\Sigma}$-compatible, $(\overline{ar})^m=0$ and hence I_0 is an ideal of R/P. Also clearly I_0 is a nil ideal of R/P. Hence $I_0=0$ and so $I=0$. Therefore $P\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ is a strongly (semi)prime (Σ,Δ)-compatible ideal of A. □

Corollary 3.12. *Let $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a bijective skew PBW extension of an (Σ,Δ)-compatible ring R. If R is a strongly (semi)prime ring, then A is a strongly (semi)prime ring.*

Corollary 3.13. *If k is a completely (semi)prime and (Σ,Δ)-compatible ring, then the quantum n-space $\mathcal{O}_q(k^n)$ is a completely (semi)prime ring.*

Corollary 3.14. *If R is a strongly (semi)prime and (Σ,Δ)-compatible ring, then n^{th} quantized Weyl algebra over R is a strongly (semi)prime ring.*

Theorem 3.15. *If each minimal strongly prime ideal of R is (Σ,Δ)-compatible and $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ is a bijective skew PBW extension of a ring R, then $nil^*(A) \subseteq nil^*(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$.*

Proof. The result follows from Theorem 3.11. □

Corollary 3.16. *Let $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a bijective skew PBW extension and $nil^*(R)$ a Σ-rigid Δ-ideal of R. Then $nil^*(A) \subseteq nil^*(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$.*

4. Skew PBW extensions of 2-primal rings

In the theory of rings, it is an important issue to investigate the coincidence of certain radicals on a given class of rings. There are some papers in the literature where the property of being 2-primal has been studied for skew PBW extensions. Also, different radicals of these extensions have been characterized (see, [19, 28]).

Let $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a skew PBW extension of a ring R with a set of endomorphisms $\Sigma := \{\sigma_1,\ldots,\sigma_n\}$ and derivations $\Delta := \{\delta_1,\ldots,\delta_n\}$. In this section, we continue the study of the radicals of skew PBW extensions, in case R is (Σ,Δ)-compatible. Our main results in this section shows that the 2-primal condition on R is preserved by skew PBW extensions.

Proposition 4.1. *Let $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a skew PBW extension of an Σ-compatible ring R and $nil(R)$ a Δ-ideal of R. Then $nil(A) \subseteq nil(R)$ $\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$.*

Proof. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m$ be a nil element of A, where $a_i \in R$, $1 \le i \le m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$, $X_m \succ X_{m-1} \succ \cdots \succ X_1$. Then there exists $t \ge 0$ such that $f^t = 0$. Thus by Remark 2.9, we have $a_m\sigma^{\alpha_m}(a_m)\sigma^{2\alpha_m}(a_m)$ $\ldots\sigma^{(t-1)\alpha_m}(a_m) = 0$. Since R is Σ-compatible, $a_m^t = 0$ and hence $a_m \in nil(R)$. Now let us write $f = q + a_mx^{\alpha_m}$ with $q \in A$ and $\deg(q) < |\alpha_m|$. Then $0 = f^t = q^t + h$, for some $h \in A$. Note that, when we compute every summand of h we obtain products of the coefficient a_m in σ's and δ's depending of the coordinates of α_m. Since $a_m \in nil(R)$ and $nil(R)$ is (Σ,Δ)-ideal, $h \in nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$. Thus $q^t \in nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$, and so $a_{m-1}\sigma^{\alpha_{m-1}}(a_{m-1})\ldots\sigma^{(t-1)\alpha_{m-1}}(a_{m-1}) \in nil(R)$ and by using [9, Lemma 3.3], $a_{m-1} \in nil(R)$. By continuing this process, we get $a_i \in nil(R)$, for each $1 \le i \le m$, and the proof is complete. □

Corollary 4.2. *Keeping all of the notations from the Example 2.4, let R be an Σ-compatible ring and $nil(R)$ a Δ-ideal of R. Then $nil(R[x_1;\sigma_1,\delta_1]\ldots[x_n;\sigma_n,\delta_n]) \subseteq nil(R)[x_1;\sigma_1,\delta_1]\ldots[x_n;\sigma_n,\delta_n]$.*

Corollary 4.3. *[21, Proposition 2.2] Let R be an σ-compatible ring and $nil(R)$ a δ-ideal of R. Then $nil(R[x;\sigma,\delta]) \subseteq nil(R)[x;\sigma,\delta]$.*

Let R be a ring, $End(R;+)$ the ring of additive endomorphisms of R and Φ a subset of $End(R;+)$. Recall that an ideal I of R is Φ-ideal if $\varphi(I) \subseteq I$ for any $\varphi \in \Phi$. A Φ-ideal $P \neq R$ is a Φ-*prime* ideal if for any Φ-ideals I and J such that $IJ \subseteq P$, we have either $I \subseteq P$ or $J \subseteq P$. We shall use the notation $I \lhd_\Phi R$ (resp., $P \lhd'_\Phi R$) to express the fact that I is a Φ-ideal (resp., P is a Φ-prime ideal) of R. We write $\mathcal{P}_\Phi = Spec(R;\Phi)$ for the set of all Φ-prime ideals of R and $rad(R;\Phi) = \cap_{P\in\mathcal{P}_\Phi} P$ for the Φ-prime radical. By definition, R is Φ-prime (resp., Φ semiprime) if $\{0\}$ is Φ-prime (resp., if $rad(R;\Phi) = 0$)

A sequence $(a_0,a_1,\ldots,a_n,\ldots)$ of elements of R is called a Φ-m-sequence if for any $i \in \mathbb{N}$ there exist $\varphi_i,\varphi'_i \in \Phi$ and $r_i \in R$ such that $a_{i+1} = \varphi_i(a_i)r_i\varphi'_i(a_i)$. An element $a \in R$ is called strongly Φ-nilpotent if every Φ-m-sequence starting with a eventually vanishes. If $\Phi = id_R$ we recover the corresponding classical notions. In the following, we recall the definition of a lower nil radical by transfinite induction from (for more details see [16]).

- $L_0 = L_0(R;\Phi) = \{0\}$
- $L_1 = L_1(R;\Phi) = \sum_{I\in N_\Phi} I$ where $N_\Phi = \{I \lhd_\Phi R | I \text{ is nilpotent }\}$.

$(L_1 \lhd_\Phi R$ and any $\varphi \in \Phi$ induces an additive endomorphism of $R/L_1)$

- $L_\alpha = L_\alpha(R;\Phi) = \{r \in R | r + L_\beta(R;\Phi) \in L_1(R/L_\beta(R;\Phi);\Phi)\}$ if $\alpha = \beta+1$
- $L_\alpha = L_\alpha(R;\Phi) = \cup_{\beta<\alpha} L_\beta(R;\Phi)$ if α is a limit ordinal.

There exists an ordinal β such that $L_\beta(R;\Phi) = L_{\beta+1}(R;\Phi)$ and we put $L(R;\Phi) = L_\beta(R;\Phi)$.

Lemma 4.4. *[16, Proposition 1.11] Keeping the above notation, we have:*

$$L(R;\Phi) = rad(R;\Phi) = \{a \in R \,|\, a \text{ is strongly } \Phi\text{-nilpotent }\}.$$

Lemma 4.5. *Let $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a skew PBW extension of a ring R. Then $rad(R;\Sigma\cup\Delta)\langle x_1,\ldots,x_n\rangle \subseteq rad(A)$.*

Proof. We follow the ideas presented in [16, Lemma 5.1].

The inclusion can be proved by transfinite induction, using the description of $rad(R;\Sigma\cup\Delta)$ given before Lemma 4.4, as follows:

Let $f \in L_1 \langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$. Then we have $f \in I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ for some nilpotent (Σ,Δ)-ideal. So $I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle \subseteq rad(A)$, since $I\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ is itself a nilpotent ideal of $R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$. In particular $f \in rad(A)$ and $L_1\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle \subseteq rad(A)$. Now assume that $L_\alpha\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle \subseteq rad(A)$ for all $\alpha < \beta$.

Let $\beta = \alpha + 1$ for some α. Then we have the following chain of isomorphisms and inclusions

$$\begin{aligned}
&L_\beta\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle / L_\alpha\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle \cong L_\beta/L_\alpha\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle\\
&= L_1(R/L_\alpha)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle\\
&\subseteq rad\left(R/L_\alpha\langle x_1,\ldots,x_n;\Sigma\cup\Delta\rangle\right)\\
&\cong rad(R\langle x_1,\ldots,x_n;\Sigma\cup\Delta\rangle)/L_\alpha\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle.
\end{aligned}$$

Now, by induction hypothesis, it is not hard to see that $L_\beta\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle \subseteq rad(R\langle x_1,\ldots,x_n;\Sigma\cup\Delta\rangle)$.

If β is a limit ordinal, $L_\beta = \cup_{\alpha<\beta} L_\alpha$ and so, by induction hypothesis, we have $L_\beta\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle \subseteq rad(R\langle x_1,\ldots,x_n;\Sigma\cup\Delta\rangle)$. □

Proposition 4.6. *Let $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a skew PBW extension of an Σ-compatible ring R.*

1. *If $nil(R) = nil_*(R;\Sigma\cup\Delta)$, then A is 2-primal.*

2. *If $nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle = nil_*(A)$, then A is 2-primal.*

Proof. (1) Suppose that R is Σ-compatible and $nil(R) = nil_*(R;\Sigma\cup\Delta)$. Since $nil_*(R;\Sigma\cup\Delta)$ is (Σ,Δ)-ideal, so $nil(A) \subseteq nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle = nil_*(R;\Sigma\cup\Delta)\langle x_1,\ldots,x_n\rangle$, by Proposition 4.1. On the other hand, by Lemma 4.5, $nil_*(R;\Sigma\cup\Delta)\langle x_1,\ldots,x_n\rangle \subseteq nil_*(A)$ and then A is 2-primal.

(2) Suppose that $nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle = nil_*(A)$. Hence $nil(R)$ is an ideal of R. Let $a \in nil(R)$. Since A is skew PBW extension of R, we have $\sigma_i(a)x_i + \delta_i(a) = x_i a \in nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ for every $1 \le i \le n$, by Proposition 2.2. Hence $\delta_i(a) \in nil(R)$ for every $1 \le i \le n$, since $nil(R)$ is an Σ-ideal of R. This means that $\delta^\alpha(a) \in nil(R)$ and so $nil(R)$ is Δ-ideal. Therefore by Proposition 4.1, $nil(A) \subseteq nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle = nil_*(A)$, which implies that A is 2-primal. □

Theorem 4.7. *Let $A = R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a skew PBW extension of an (Σ,Δ)-compatible ring R. Then A is 2-primal if and only if $nil(R) = nil_*(R;\Sigma\cup\Delta)$ if and only if $nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle = nil_*(A)$.*

Proof. First we prove that A is 2-primal if and only if $nil(R) = nil_*(R;\Sigma\cup\Delta)$. If $nil(R) = nil_*(R;\Sigma\cup\Delta)$ then A is 2-primal, by Proposition 4.6. For the backward direction, assume that A is 2-primal. So by Lemma 4.5, $nil_*(R;\Sigma\cup\Delta)\langle x_1,\ldots,x_n\rangle\subseteq nil_*(A)=nil(A)$. Hence $nil_*(R;\Sigma\cup\Delta)\subseteq nil(R)$. Since A is 2-primal, R is 2-primal because each subring of a 2-primal ring is 2-primal, by [4]. Now, let $a\in nil(R)=nil_*(R)$. Thus a is strongly nilpotent and hence each m-sequence starting with a eventually vanishes. So each $\{\Sigma,\Delta\}$-m-sequence starting with a eventually vanishes, by [9, Lemma 3.3]. Thus a is strongly $\{\Sigma,\Delta\}$-nilpotent and $a\in nil_*(R;\Sigma\cup\Delta)$, by Lemma 4.4, as desired. Now, let $nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle=nil_*(A)$. Then by Proposition 4.6, A is 2-primal. Conversely, suppose that A is 2-primal. Hence $nil(R)=nil_*(R;\Sigma\cup\Delta)$ and so $nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle=nil_*(R;\Sigma\cup\Delta)\subseteq nil_*(A)$, by Lemma 4.5. On the other hand, by Proposition 4.1, $nil_*(A)=nil(A)\subseteq nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$, and the proof is complete. $\square$

Corollary 4.8. *Let $A=R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a skew PBW extension of an (Σ,Δ)-compatible ring R. Then A is 2-primal if and only if R is 2-primal and $nil(R)$ $\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle=nil(A)$.*

Proof. Let A be a 2-primal ring. Then R is 2-primal and $nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ $=nil(A)$, by Theorem 4.7. For the backward direction, by a similar argument as used in the proof of Theorem 4.7, we can see that $nil(R)\subseteq nil_*(R;\Sigma\cup\Delta)$. Thus by Lemma 4.5, $nil(A)=nil(R)\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle\subseteq nil_*(R;\Sigma\cup\Delta)\langle x_1,\ldots,x_n\rangle\subseteq nil_*(A)$, and the proof is complete. $\square$

By a similar proof that is employed in [21, Lemma 2.7], we can prove the following.

Lemma 4.9. *Assume that R is a reduced (Σ,Δ)-compatible ring. Let P be a minimal (Σ,Δ)-prime ideal of R. Then P is completely prime.*

Theorem 4.10. *Let $A=R\langle x_1,\ldots,x_n;\Sigma,\Delta\rangle$ be a skew PBW extension of an (Σ,Δ)-compatible ring R. Then A is 2-primal if and only if every minimal (Σ,Δ)-prime ideal of R is completely prime.*

Proof. Suppose that A is 2-primal. Then $\overline{R}=R/nil_*(R;\Sigma\cup\Delta)$ is reduced, by Theorem 4.7. Let P be a minimal (Σ,Δ)-prime ideal of R; then $\overline{P}$ is a minimal $(\overline{\Sigma},\overline{\Delta})$-prime ideal of $\overline{R}$. So by Lemma 4.9, P is completely prime, since $R/P\cong\overline{R}/\overline{P}$. Conversely, let every minimal (Σ,Δ)-prime ideal of R is completely prime. Assume that $\{P_i\}_{i\in I}$ be the family of all minimal (Σ,Δ)-prime ideals of R. So $nil_*(R;\Sigma\cup\Delta)=\bigcap_{i\in I}P_i$ and then $R/nil_*(R;\Sigma\cup\Delta)$ embeds in $\prod_{i\in I}R/P_i$. Thus $R/nil_*(R;\Sigma\cup\Delta)$ is reduced and hence $nil(R)\subseteq nil_*(R;\Sigma\cup\Delta)$. On the other hand, since $nil_*(R;\Sigma\cup\Delta)\langle x_1,\ldots,x_n\rangle\subseteq nil_*(A)\subseteq nil(A)$, then $nil_*(R;\Sigma\cup\Delta)\subseteq nil(R)$. Therefore $nil(R)=nil_*(R;\Sigma\cup\Delta)$ and the proof is complete by Theorem 4.7. $\square$

Theorem 4.11. *Let $A = R\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ be a skew PBW extension of an (Σ,Δ)-compatible ring R. Then A is 2-primal if and only if R is 2-primal.*

Proof. Suppose that R is 2-primal and let $a \in nil(R) = nil_*(R)$. So a is strongly nilpotent. Since R is (Σ,Δ)-compatible, a is strongly (Σ,Δ)-nilpotent. Then by Lemma 4.4, $a \in nil_*(R;\Sigma\cup\Delta)$ and so $nil(R) \subseteq nil_*(R;\Sigma\cup\Delta)$. Also clearly $nil_*(R;\Sigma\cup\Delta) \subseteq nil(R)$. Therefore, by using Theorem 4.7, the result follows. □

Corollary 4.12. *Let $A = R\langle x_1,\dots,x_n;\Sigma,\Delta\rangle$ be a skew PBW extension of an (Σ,Δ)-compatible ring R. Then the following are equivalent:*

1. *R is 2-primal.*
2. *A is 2-primal.*
3. *$nil(R) = nil_*(R;\Sigma\cup\Delta)$.*
4. *$nil(R)\langle x_1,\dots,x_n;\Sigma,\Delta\rangle = nil_*(A)$.*
5. *R is 2-primal and $nil(R)\langle x_1,\dots,x_n;\Sigma,\Delta\rangle = nil(A)$.*
6. *every minimal (Σ,Δ)-prime ideal of R is completely prime.*

Corollary 4.13. *If k is a 2-primal and (Σ,Δ)-compatible ring, then the quantum n-space $\mathcal{O}_q(k^n)$ is 2-primal.*

Corollary 4.14. *Keeping all of the notations from the Example 2.4, if R is a 2-primal and (Σ,Δ)-compatible ring, then the iterated skew polynomial ring $R[x_1;\sigma_1,\delta_1]\dots[x_n;\sigma_n,\delta_n]$ is 2-primal.*

Corollary 4.15. *If R is a 2-primal and (Σ,Δ)-compatible ring, then the n^{th} quantized Weyl algebra over R is 2-primal.*

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