For a simple graph $G$ with $n$ vertices and $m$ edges having adjacency eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, the energy $\mathcal{E}(G)$ of $G$ is defined as $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. We obtain the upper bounds for $\mathcal{E}(G)$ in terms of the vertex covering number $\tau$, the number of edges $m$, maximum vertex degree $d_1$ and second maximum vertex degree $d_2$ of the connected graph $G$. These upper bounds improve some recently known upper bounds for $\mathcal{E}(G)$. Further, these upper bounds for $\mathcal{E}(G)$ imply a natural extension to other energies like distance energy and Randić energy associated to a connected graph $G$.

1. Introduction

Let $G$ be a finite simple graph with $n$ vertices and $m$ edges having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Throughout this paper by a graph $G$, we mean the graph $G = (V, E)$ having $n$ vertices and $m$ edges, unless otherwise stated. The adjacency matrix $A = (a_{ij})$ of $G$ is a $(0, 1)$-square matrix of order $n$ and the spectrum of the adjacency matrix is the spectrum of the graph $G$. 

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Corresponding author: S. Pirzada
If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the adjacency eigenvalues of \( G \), the energy of \( G \) is defined as

\[
E(G) = \sum_{i=1}^{n} |\lambda_i|.
\]

This concept introduced by Gutman [12] is intensively studied in Chemistry, since it can be used to approximate the total \( \pi \)-electron energy of a molecule, see [15] and the references therein. This spectrum-based graph invariant has been much studied in both chemical and mathematical literature. Among the pioneering results on graph energy are the lower and upper bounds for energy, see [1, 10, 14, 15, 19] and the references therein. For more information about the energy of a graph, we refer to the most recent papers [1, 14, 19].

A subset \( S \) of the vertex set \( V(G) \) is said to be a covering set of \( G \) if every edge of \( G \) is incident to at least one vertex in \( S \). A covering set with minimum cardinality among all covering sets is called the minimum covering set of \( G \) and its cardinality, denoted by \( \tau = \tau(G) \), is called vertex covering number of the graph \( G \). If \( H \) is a subgraph of the graph \( G \), we denote the graph obtained by removing the edges in \( H \) from \( G \) by \( G \setminus H \) (that is, only the edges of \( H \) are removed from \( G \)).

The neighbourhood of a vertex \( v_i \in V(G) \), denoted by \( N(v_i) \) is defined as the set of all the vertices of \( G \) which are adjacent to \( v_i \), that is, \( N(v_i) = \{v_j : v_iv_j \in E(G)\} \). A graph \( G \) of order \( n \) with \( m \) edges is said to be \( c \)-cyclic, where \( c \geq 0 \) is an integer, if \( m = n + c - 1 \). For the eigenvalue \( \lambda \) of \( G \), \( \lambda^{[k]} \) denotes its multiplicity.

Further, as usual \( P_n, C_n, K_n \) and \( K_{s,t} \), respectively, denote the path on \( n \) vertices, the cycle on \( n \) vertices, the complete graph on \( n \) vertices and the complete bipartite graph on \( s + t \) vertices. For other undefined notations and terminology from spectral graph theory, the readers are referred to [4, 18].

The rest of the paper is organized as follows. In Section 2, we obtain some upper bounds for \( E(G) \) in terms of the vertex covering number \( \tau \), the number of edges \( m \), maximum vertex degree \( d_1 \) and second maximum vertex degree \( d_2 \) of the connected graph \( G \). These upper bounds improve some recently known upper bounds for the energy \( E(G) \) of a connected graph. In Section 3, we extend the upper bound obtained for \( E(G) \) in Section 2, to other energies like distance energy and Randić energy associated to a connected graph \( G \).

2. Bounds for the energy of a graph

Given a complex \( m \times n \) matrix \( M \), its energy, denoted by \( E(M) \), is the sum of its singular values (the positive square roots of the eigenvalues of the matrix
If $M$ is a real symmetric matrix of order $n$ and if $s_i(M), x_i(M)$ denote the singular values and the eigenvalues of $M$, respectively, then $s_i(M) = |x_i(M)|$, for all $i = 1, 2, \ldots, n$. In the light of this definition, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the graph $G$, then the energy $\mathcal{E}(G) = \mathcal{E}(A(G))$, [16], is

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_i$ are the eigenvalues of the graph $G$.

The following lemma can be seen in [6].

**Lemma 2.1.** Let $X, Y$ and $Z$ be square matrices of order $n$ such that $Z = X + Y$. Then

$$\sum_{i=1}^{n} s_i(Z) \leq \sum_{i=1}^{n} s_i(X) + \sum_{i=1}^{n} s_i(Y).$$

Moreover, equality holds if and only if there exists an orthogonal matrix $P$ such that $PX$ and $PY$ are both positive semidefinite matrices.

Wang and Ma [19] obtained the following upper bound for the energy $\mathcal{E}(G)$ in terms of the vertex covering number $\tau$ and the maximum vertex degree $\Delta$:

$$\mathcal{E}(G) \leq 2\tau \sqrt{\Delta},$$

with equality if and only if $G$ is the disjoint union of $\tau$ copies of $K_{1,\Delta}$ together with some isolated vertices.

Now, we obtain an upper bound for the energy $\mathcal{E}(G)$ in terms of the vertex covering number $\tau$ and the number of edges $m$ of the graph $G$.

**Theorem 2.2.** Let $G$ be a connected graph of order $n \geq 2$ and $m$ edges having vertex covering number $\tau$. Then

$$\mathcal{E}(G) \leq 2\sqrt{m\tau},$$

with equality if and only if $\tau = 1$ and $G \cong K_{1,n-1}$.

**Proof.** Let $G$ be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let $\tau$ be the vertex covering number and $C$ be the minimum vertex covering set of $G$. Without loss of generality, let $C = \{v_1, v_2, \ldots, v_\tau\}$.

Let $G_1, G_2, \ldots, G_\tau$ be the spanning subgraphs of $G$ corresponding to the
vertices \(v_1, v_2, \ldots, v_\tau\) of \(C\), having vertex set same as \(G\) and edge sets defined as follows.
\[
E(G_i) = \{v_iv_i : v_i \in N(v_i) \setminus \{v_1, v_2, \ldots, v_{i-1}\}\}, \quad i = 1, 2, \ldots, \tau.
\]

For \(i = 1, 2, \ldots, \tau\), let \(m_i = |E(G_i)|\). It is clear that \(E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_\tau)\) and \(G_i = \bar{K}_1, m_i \cup (n - m_i - 1)K_1\), for all \(i = 1, 2, \ldots, \tau\). Also, it is easy to see that the adjacency matrix \(A(G)\) now can be decomposed as
\[
A(G) = A(G_1) + A(G_2) + \cdots + A(G_\tau). \tag{4}
\]

The adjacency spectrum of \(G_i = \bar{K}_1, m_i \cup (n - m_i - 1)K_1\) is \(\{\pm \sqrt{m_i}, 0^{\lfloor n/2 \rfloor}\}\).

Therefore,
\[
\delta'(G_i) = \delta'(K_{1, m_i} \cup (n - m_i - 1)K_1) = 2\sqrt{m_i}, \quad \text{for all } i = 1, 2, \ldots, \tau. \tag{5}
\]

Now, applying Lemma 2.1 to equation (4) and using (1), (5) and Cauchy-Schwarz’s inequality, we have
\[
\delta'(G) \leq \delta'(G_1) + \delta'(G_2) + \cdots + \delta'(G_\tau)
\]
\[
= 2\sqrt{m_1} + 2\sqrt{m_2} + \cdots + 2\sqrt{m_\tau}
\]
\[
= 2 \sum_{i=1}^{\tau} \sqrt{m_i} \leq 2 \sqrt{\tau \sum_{i=1}^{\tau} m_i} = 2\sqrt{m_{\tau}},
\]
as \(E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_\tau)\) gives \(m = \sum_{i=1}^{\tau} m_i\). This proves the first part of the proof.

Assume that equality holds in (3), so all the inequalities above occur as equalities. Since \(G\) is connected, it is clear that equality occurs in (4) if and only if \(G \cong \bar{K}_{1, n-1}\). Also, since equality occurs in Cauchy-Schwarz’s inequality if and only if \(m_1 = m_2 = \cdots = m_\tau\), it follows that the equality occurs in (3) if and only if \(\tau = 1\) and \(G \cong \bar{K}_{1, n-1}\).

Conversely, if \(\tau = 1\) and \(G \cong \bar{K}_{1, n-1}\), then it is easy to see that equality holds in (3).

\[\square\]

**Remark 2.3.** For a connected graph \(G\) of order \(n \geq 2\) having \(m\) edges, it can be seen that the upper bound (3) is better than the upper bound (2) for all graphs \(G\) with \(m \leq \tau \Delta\). In particular, if \(G \cong K_n\) or \(K_{r,s}, r \leq s\), then \(m = n(n-1)/2\), \(\tau = n - 1, \Delta = n - 1\) or \(m = rs, \tau = r, \Delta = s\). It can be seen that \(m \leq \tau \Delta\) always hold. If \(G\) is a \(c\)-cyclic graph, \(c \geq 0\), then \(m = n + c - 1\) and so \(m \leq \tau \Delta\) gives \(c \leq \tau \Delta - n + 1\). If \(G\) is a path \(P_n\), then \(c = 0, \tau = \lfloor n/2 \rfloor, \Delta = 2\) and so \(c \leq \tau \Delta - n + 1\) always hold. Similarly, it can be seen that for a cycle \(C_n\), we always have \(c \leq \tau \Delta - n + 1\). We
now give a construction of an infinite family of graphs for which the condition
\( m \leq \tau \Delta \) holds. Let \( S_\omega(a,a,\ldots,a,a) \), \( a \geq 1 \) be the family of connected
graphs of order \( n = \omega + a \omega \) with \( m \) edges having \( a \) pendent vertices attached to
each of the \( \omega \) vertices of the clique \( K_\omega \) (see Figure 1). For this graph it is clear
that \( \tau = \omega, \Delta = \omega - 1 + a \) and \( m = \frac{\omega(\omega - 1)}{2} + n - \omega \). We have \( \omega \geq 1 \), which
gives \( \omega(\omega - 1) \geq 0 \), further implies that \( 2\omega^2 - 4\omega + 2n \geq \omega^2 - 3\omega + 2n \), or
\( 2\omega^2 - 2(n - \omega) - 2\omega \geq \omega^2 - 3\omega + 2n \), or \( 2\omega(\omega - a + 1) \geq \omega^2 - 3\omega + 2n \), or
\( m \leq \tau \Delta \), as \( n = \omega = a \omega \).

We now obtain an upper bound for the energy \( \mathcal{E}(G) \) in terms of the vertex
covering number \( \tau \), the maximum vertex degree \( d_1 \) and the second maximum
vertex degree \( d_2 \) of the graph \( G \).

**Theorem 2.4.** Let \( G \) be a connected graph of order \( n \geq 2 \) and \( m \) edges hav-
ing vertex covering number \( \tau \). If \( d_1 \) and \( d_2 \) are the maximum and the second
maximum vertex degrees in \( G \), then

\[
\mathcal{E}(G) \leq 2\sqrt{d_1} + 2(\tau - 1)\sqrt{d_2},
\]

with equality if and only if \( \tau = 1 \) and \( G \cong K_{1,n-1} \).

**Proof.** Let \( G \) be a connected graph with vertex set \( V(G) = \{v_1,v_2,\ldots,v_n\} \) and
edge set \( E(G) = \{e_1,e_2,\ldots,e_m\} \). Let \( \tau \) be the vertex covering number and \( C \) be the minimum vertex covering set of \( G \). Without loss of generality let
\( C = \{v_1,v_2,\ldots,v_\tau\} \). Let \( G_1,G_2,\ldots,G_\tau \) be the spanning subgraphs of \( G \) cor-
responding to the vertices \( v_1,v_2,\ldots,v_\tau \) of \( C \) as defined in Theorem 2.2. Now,
proceeding similarly as in Theorem 2.2, we arrive at

\[
\mathcal{E}(G) \leq \mathcal{E}(G_1) + \mathcal{E}(G_2) + \cdots + \mathcal{E}(G_\tau)
= 2\sqrt{m_1} + 2\sqrt{m_2} + \cdots + 2\sqrt{m_\tau}.
\]

Let \( d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n \) be the degree sequence of the graph \( G \), where
d\( d_i = d(v_i) \), for all \( i \) is the degree of the vertex \( v_i \). Since \( C \) is a covering set with
minimum cardinality, we can pick the vertices in \( C \) as follows.

If \( v_1 \) has maximum degree in the graph \( G \), we pick \( v_1 \) as the first vertex
in \( C \). If all the edges of the graph \( G \) are incident to \( v_1 \), then \( C = \{v_1\} \) is the
minimum vertex covering set, otherwise, if \( v_2 \) has the maximum degree in
the graph \( G - \{v_1\} \), where \( G - \{v_1\} \) is the graph obtained from \( G \) by removing the
vertex \( v_1 \) and all the edges incident to \( v_1 \), we pick \( v_2 \) as the second vertex in \( C \).
If all the edges of the graph \( G - \{v_1\} \) are incident to \( v_2 \), then \( C = \{v_1,v_2\} \) is
the minimum vertex covering set, otherwise, we proceed similarly, to obtain the
other elements \( v_3,v_4,\ldots,v_\tau \) of the minimum vertex covering set \( C \).

Let \( C = \{v_1,v_2,\ldots,v_\tau\} \) be the minimum vertex covering set obtained in this
way. It is clear that the degree of \( v_1 \) in \( G_1 = K_{m_1,1} \cup (n-m_1-1)K_1 \) is \( d_1 \), giving \( m_1 = d_1 \). Also, the degree of \( v_2 \) in \( G_1 = K_{m_2,1} \cup (n-m_2-1)K_1 \) is either \( d_2 \) or \( d_2 - 1 \), depending on whether \( v_1 \) and \( v_2 \) are non-adjacent or adjacent in \( G \), giving \( m_2 \leq d_2 \). Similarly, it can be seen that \( m_i \leq d_2 \) for all \( i = 3, 4, \ldots, \tau \). Therefore, it follows that

\[
\mathcal{E}(G) \leq 2\sqrt{m_1} + 2\sqrt{m_2} + \cdots + 2\sqrt{m_\tau} = 2\sqrt{d_1} + 2(\tau - 1)\sqrt{d_2},
\]

proving the first part. Equality case can be discussed similar to that as seen in Theorem 2.2.

**Remark 2.5.** For a connected graph \( G \) of order \( n \geq 2 \) with \( m \) edges having maximum and second maximum vertex degrees \( d_1 \) and \( d_2 \) respectively, the upper bound (6) is always better than the upper bound (2). If \( \tau = 1 \), then \( G \cong K_{n-1,1} \) and so the equality occurs in both the upper bounds. For \( \tau \geq 2 \), we have \( 2\sqrt{d_1} + 2(\tau - 1)\sqrt{d_2} \leq 2\tau\sqrt{d_1} \), giving \( d_1 \geq d_2 \), which is always true.

In a graph \( G \), for the adjacent vertices \( v_1 \) and \( v_2 \) with degrees respectively \( d_1 \) and \( d_2 \), we have the following observation, the proof of which follows on the similar lines as the proof of Theorem 2.4.

**Theorem 2.6.** Let \( G \) be a connected graph of order \( n \geq 2 \) and \( m \) edges having vertex covering number \( \tau \). If \( v_1 \) and \( v_2 \) are adjacent vertices in \( G \) having degrees respectively \( d_1 \) and \( d_2 \), then

\[
\mathcal{E}(G) \leq 2\sqrt{d_1} + 2(\tau - 1)\sqrt{d_2 - 1}.
\]

Equality holds if and only if \( \tau = 1 \) and \( G \cong K_{1,n-1} \).

**Remark 2.7.** For a connected graph \( G \) of order \( n \geq 2 \) with \( m \) edges, let \( v_1 \) and \( v_2 \) be the vertices of \( G \) having maximum and second maximum vertex degrees \( d_1 \) and \( d_2 \), respectively. If \( v_1 \) and \( v_2 \) are adjacent, it can seen that the upper bound given by Theorem 2.6 is always better than the upper bound given by (2).
3. Bounds for distance and Randić energy

It is well known that by using different structural properties one can associate various matrices like Laplacian matrix, signless Laplacian matrix, distance matrix, normalized Laplacian matrix, Randić matrix etc, to a graph \( G \). It is natural to define an energy like quantity for these matrices. The energy of distance matrix called distance energy is defined in [13], the energy of Randić matrix called Randić energy is defined in [11], the energy of normalized Laplacian matrix called normalized Laplacian energy is defined in [3]. In [11] it is shown that for a connected graph \( G \) the Randić energy and normalized Laplacian energy are same. For more on energy-like definitions of a graph with respect to various types of matrices and related results, we refer to [5, 7–9, 17] and the references therein.

The following theorem gives an upper for the distance energy \( D\mathcal{E}(G) \) in terms of the vertex covering number \( \tau \), the maximum degree \( d_1 \) and the second maximum degree \( d_2 \) of the graph \( G \).

**Theorem 3.1.** Let \( G \) be a connected graph of order \( n \geq 2 \) and \( m \) edges having vertex covering number \( \tau \). If \( d_1 \) and \( d_2 \) are the maximum and second maximum vertex degrees in \( G \), then

\[
D\mathcal{E}(G) \leq 2(d_1 - 1) + 2\sqrt{d_1^2 - d_1 + 1} + 2(\tau - 1)\left[d_2 - 1 + \sqrt{d_2^2 - d_2 + 1}\right],
\]

with equality if and only if \( \tau = 1 \) and \( G \cong K_{1,n-1} \).
Proof. Using the fact that the distance spectrum of $K_{1,n-1}$ is

$$\{n-2+\sqrt{n^2-3n+3}, n-2-\sqrt{n^2-3n+3}, -2^{[n-2]}\}$$

and proceeding similarly as in Theorem 2.4, the result follows.

The following upper bound for the distance energy $D\mathcal{E}(G)$ of a connected graph $G$ was obtained in [2]:

$$D\mathcal{E}(G) \leq \sqrt{2(n-1)\sum_{i<j} d_{ij}^2 + n|\text{det}(D(G))|\frac{2}{n}}, \quad (7)$$

where $d_{ij}$ is the distance between $i^{th}$ and $j^{th}$ vertex of $G$.

Remark 3.2. It is easy to see that for several families of connected graphs $G$ of order $n \geq 2$ having vertex covering number $\tau$, maximum degree $d_1$ and second maximum degree $d_2$, the upper bound given by Theorem 3.1 is better than the upper bound given by (7). If $G \cong K_{1,n-1}$, then $\tau = 1, d_1 = n-1, d_2 = 1$ and so the upper bound given by Theorem 3.1 is better than the upper bound given by (7). This is because equality occurs for $G \cong K_{1,n-1}$ in Theorem 3.1, while as the upper bound (7) is strict for $G \cong K_{1,n-1}$.

The following theorem gives an upper for the Randić energy $R\mathcal{E}(G)$, in terms of the vertex covering number $\tau$ of the graph $G$.

Theorem 3.3. Let $G$ be a connected graph of order $n \geq 2$ and $m$ edges having vertex covering number $\tau$. Then

$$R\mathcal{E}(G) \leq 2\tau,$$

with equality if and only if $\tau = 1$ and $G \cong K_{1,n-1}$.

Proof. Using the fact that the Randić spectrum of $K_{1,n-1}$ is

$$\{1,0^{[n-2]}, -1\}$$

and proceeding similarly as in Theorem 2.4, the result follows.

The following upper bound for the Randić energy $R\mathcal{E}(G)$ of a connected graph $G$ was obtained in [3]:

$$R\mathcal{E}(G) \leq \sqrt{\frac{15}{28}}(n+1). \quad (8)$$
Remark 3.4. It can be seen that for the connected graphs $G$ of order $n \geq 2$ having vertex covering number $\tau$, the upper bound given by Theorem 3.3 is better than the upper bound given by (8), for all $\tau \leq \sqrt{\frac{15}{28} \frac{n+1}{2}}$. In particular, if $G \cong K_{a,b}$, with $a \leq b$ and $a \leq \sqrt{\frac{15}{28} \frac{n+1}{2}}$, then the upper bound Theorem 3.3 is always better than the upper bound given by (8).

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HILAL A. GANIE

Department of Mathematics, University of Kashmir, Srinagar, Kashmir, India
e-mail: hilahmad1119kt@gmail.com

U. SAMEE

Department of Mathematics, Islamia College of Science and Commerce, Srinagar, Kashmir, India
e-mail: drumatulsamee@gmail.com

S. PIRZADA

Department of Mathematics, University of Kashmir, Srinagar, Kashmir, India
e-mail: pirzadasd@kashmiruniversity.ac.in