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ON GRAPH ENERGY, MAXIMUM DEGREE AND VERTEX COVER NUMBER

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For a simple graph G with n vertices and m edges having adjacency eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, the energy $\mathscr{E}(G)$ of G is defined as $\mathscr{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. We obtain the upper bounds for $\mathscr{E}(G)$ in terms of the vertex covering number τ, the number of edges m, maximum vertex degree d_1 and second maximum vertex degree d_2 of the connected graph G. These upper bounds improve some recently known upper bounds for $\mathscr{E}(G)$. Further, these upper bounds for $\mathscr{E}(G)$ imply a natural extension to other energies like distance energy and Randić energy associated to a connected graph G.

1. Introduction

Let G be a finite simple graph with n vertices and m edges having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Throughout this paper by a graph G, we mean the graph $G = (V, E)$ having n vertices and m edges, unless otherwise stated. The adjacency matrix $A = (a_{ij})$ of G is a $(0,1)$-square matrix of order n and the spectrum of the adjacency matrix is the spectrum of the graph G.

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If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, the energy of G is defined as

$$\mathscr{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This concept introduced by Gutman [12] is intensively studied in Chemistry, since it can be used to approximate the total π-electron energy of a molecule, see [15] and the references therein. This spectrum-based graph invariant has been much studied in both chemical and mathematical literature. Among the pioneering results on graph energy are the lower and upper bounds for energy, see [1, 10, 14, 15, 19] and the references therein. For more information about the energy of a graph, we refer to the most recent papers [1, 14, 19].

A subset S of the vertex set $V(G)$ is said to be a *covering set* of G if every edge of G is incident to at least one vertex in S. A covering set with minimum cardinality among all covering sets is called the *minimum covering set* of G and its cardinality, denoted by $\tau = \tau(G)$, is called *vertex covering number* of the graph G. If H is a subgraph of the graph G, we denote the graph obtained by removing the edges in H from G by $G \setminus H$ (that is, only the edges of H are removed from G).

The neighbourhood of a vertex $v_i \in V(G)$, denoted by $N(v_i)$ is defined as the set of all the vertices of G which are adjacent to v_i, that is, $N(v_i) = \{v_j : v_i v_j \in E(G)\}$. A graph G of order n with m edges is said to be c-cyclic, where $c \geq 0$ is an integer, if $m = n + c - 1$. For the eigenvalue λ of G, $\lambda^{[k]}$ denotes its multiplicity.

Further, as usual P_n, C_n, K_n and $K_{s,t}$, respectively, denote the path on n vertices, the cycle on n vertices, the complete graph on n vertices and the complete bipartite graph on $s+t$ vertices. For other undefined notations and terminology from spectral graph theory, the readers are referred to [4, 18].

The rest of the paper is organized as follows. In Section 2, we obtain some upper bounds for $\mathscr{E}(G)$ in terms of the vertex covering number τ, the number of edges m, maximum vertex degree d_1 and second maximum vertex degree d_2 of the connected graph G. These upper bounds improve some recently known upper bounds for the energy $\mathscr{E}(G)$ of a connected graph. In Section 3, we extend the upper bound obtained for $\mathscr{E}(G)$ in Section 2, to other energies like distance energy and Randić energy associated to a connected graph G.

2. Bounds for the energy of a graph

Given a complex $m \times n$ matrix M, its energy, denoted by $\mathscr{E}(M)$, is the sum of its singular values (the positive square roots of the eigenvalues of the matrix

M^*M), see [16]. If M is a real symmetric matrix of order n and if $s_i(M), x_i(M)$ denote the singular values and the eigenvalues of M, respectively, then $s_i(M) = |x_i(M)|$, for all $i = 1, 2, \ldots, n$. In the light of this definition, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the graph G, then the energy $\mathscr{E}(G) = \mathscr{E}(A(G))$, [16], is

$$\mathscr{E}(G) = \sum_{i=1}^{n} |\lambda_i|, \tag{1}$$

where λ_i are the eigenvalues of the graph G.

The following lemma can be seen in [6].

Lemma 2.1. *Let X,Y and Z be square matrices of order n such that $Z = X + Y$. Then*

$$\sum_{i=1}^{n} s_i(Z) \leq \sum_{i=1}^{n} s_i(X) + \sum_{i=1}^{n} s_i(Y).$$

Moreover, equality holds if and only if there exists an orthogonal matrix P such that PX and PY are both positive semidefinite matrices.

Wang and Ma [19] obtained the following upper bound for the energy $\mathscr{E}(G)$ in terms of the vertex covering number τ and the maximum vertex degree Δ:

$$\mathscr{E}(G) \leq 2\tau\sqrt{\Delta}, \tag{2}$$

with equality if and only if G is the disjoint union of τ copies of $K_{1,\Delta}$ together with some isolated vertices.

Now, we obtain an upper bound for the energy $\mathscr{E}(G)$ in terms of the vertex covering number τ and the number of edges m of the graph G.

Theorem 2.2. *ALet G be a connected graph of order $n \geq 2$ and m edges having vertex covering number τ. Then*

$$\mathscr{E}(G) \leq 2\sqrt{m\tau}, \tag{3}$$

with equality if and only if $\tau = 1$ and $G \cong K_{1,n-1}$.

Proof. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let τ be the vertex covering number and C be the minimum vertex covering set of G. Without loss of generality, let $C = \{v_1, v_2, \ldots, v_\tau\}$.

Let $G_1, G_2, \ldots, G_\tau$ be the spanning subgraphs of G corresponding to the vertices $v_1, v_2, \dots, v_\tau$ of C, having vertex set same as G and edge sets defined as follows.

$$E(G_i) = \{v_i v_t : v_t \in N(v_i) \setminus \{v_1, v_2, \dots, v_{i-1}\}\}, \ \ i = 1, 2, \dots, \tau.$$

For $i = 1, 2, \dots, \tau$, let $m_i = |E(G_i)|$. It is clear that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_\tau)$ and $G_i = K_{1,m_i} \cup (n - m_i - 1)K_1$, for all $i = 1, 2, \dots, \tau$. Also, it is easy to see that the adjacency matrix $A(G)$ now can be decomposed as

$$A(G) = A(G_1) + A(G_2) + \cdots + A(G_\tau). \tag{4}$$

The adjacency spectrum of $G_i = K_{1,m_i} \cup (n - m_i - 1)K_1$ is $\{\pm\sqrt{m_i}, 0^{[n-2]}\}$.

Therefore,

$$\mathscr{E}(G_i) = \mathscr{E}(K_{1,m_i} \cup (n - m_i - 1)K_1) = 2\sqrt{m_i}, \qquad \text{for all } i = 1, 2, \dots, \tau. \tag{5}$$

Now, applying Lemma 2.1 to equation (4) and using (1), (5) and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\mathscr{E}(G) &\leq \mathscr{E}(G_1) + \mathscr{E}(G_2) + \cdots + \mathscr{E}(G_\tau)\\ &= 2\sqrt{m_1} + 2\sqrt{m_2} + \cdots + 2\sqrt{m_\tau}\\ &= 2\sum_{i=1}^{\tau}\sqrt{m_i} \leq 2\sqrt{\tau\sum_{i=1}^{\tau} m_i} = 2\sqrt{m\tau},\end{aligned}$$

as $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_\tau)$ gives $m = \sum_{i=1}^{\tau} m_i$. This proves the first part of the proof.

Assume that equality holds in (3), so all the inequalities above occur as equalities. Since G is connected, it is clear that equality occurs in (4) if and only if $G \cong K_{1,n-1}$. Also, since equality occurs in Cauchy-Schwarz's inequality if and only if $m_1 = m_2 = \cdots = m_\tau$, it follows that the equality occurs in (3) if and only if $\tau = 1$ and $G \cong K_{1,n-1}$.

Conversely, if $\tau = 1$ and $G \cong K_{1,n-1}$, then it is easy to see that equality holds in (3). □

Remark 2.3. For a connected graph G of order $n \geq 2$ having m edges, it can be seen that the upper bound (3) is better than the upper bound (2) for all graphs G with $m \leq \tau\Delta$. In particular, if $G \cong K_n$ or $K_{r,s}$, $r \leq s$, then $m = \frac{n(n-1)}{2}$, $\tau = n - 1$, $\Delta = n - 1$ or $m = rs$, $\tau = r$, $\Delta = s$. It can be seen that $m \leq \tau\Delta$ always hold. If G is a c-cyclic graph, $c \geq 0$, then $m = n + c - 1$ and so $m \leq \tau\Delta$ gives $c \leq \tau\Delta - n + 1$. If G is a path P_n, then $c = 0$, $\tau = \lfloor \frac{n}{2} \rfloor$, $\Delta = 2$ and so $c \leq \tau\Delta - n + 1$ always hold. Similarly, it can be seen that for a cycle C_n, we always have $c \leq \tau\Delta - n + 1$. We now give a construction of an infinite family of graphs for which the condition $m \leq \tau\Delta$ holds. Let $S_\omega(a,a,\ldots,a,\ldots,a)$, $a \geq 1$ be the family of connected graphs of order $n = \omega + a\omega$ with m edges having a pendent vertices attached to each of the ω vertices of the clique K_ω (see Figure 1). For this graph it is clear that $\tau = \omega$, $\Delta = \omega - 1 + a$ and $m = \frac{\omega(\omega-1)}{2} + n - \omega$. We have $\omega \geq 1$, which gives $\omega(\omega-1) \geq 0$, further implies that $2\omega^2 - 4\omega + 2n \geq \omega^2 - 3\omega + 2n$, or $2\omega^2 - 2(n-\omega) - 2\omega \geq \omega^2 - 3\omega + 2n$, or $2\omega(\omega - a + 1) \geq \omega^2 - 3\omega + 2n$, or $m \leq \tau\Delta$, as $n - \omega = a\omega$.

We now obtain an upper bound for the energy $\mathscr{E}(G)$ in terms of the vertex covering number τ, the maximum vertex degree d_1 and the second maximum vertex degree d_2 of the graph G.

Theorem 2.4. *Let G be a connected graph of order $n \geq 2$ and m edges having vertex covering number τ. If d_1 and d_2 are the maximum and the second maximum vertex degrees in G, then*

$$\mathscr{E}(G) \leq 2\sqrt{d_1} + 2(\tau - 1)\sqrt{d_2}, \tag{6}$$

with equality if and only if $\tau = 1$ and $G \cong K_{1,n-1}$.

Proof. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let τ be the vertex covering number and C be the minimum vertex covering set of G. Without loss of generality let $C = \{v_1, v_2, \ldots, v_\tau\}$. Let $G_1, G_2, \ldots, G_\tau$ be the spanning subgraphs of G corresponding to the vertices $v_1, v_2, \ldots, v_\tau$ of C as defined in Theorem 2.2. Now, proceeding similarly as in Theorem 2.2, we arrive at

$$\begin{aligned}\mathscr{E}(G) &\leq \mathscr{E}(G_1) + \mathscr{E}(G_2) + \cdots + \mathscr{E}(G_\tau) \\ &= 2\sqrt{m_1} + 2\sqrt{m_2} + \cdots + 2\sqrt{m_\tau}.\end{aligned}$$

Let $d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n$ be the degree sequence of the graph G, where $d_i = d(v_i)$, for all i is the degree of the vertex v_i. Since C is a covering set with minimum cardinality, we can pick the vertices in C as follows.

If v_1 has maximum degree in the graph G, we pick v_1 as the first vertex in C. If all the edges of the graph G are incident to v_1, then $C = \{v_1\}$ is the minimum vertex covering set, otherwise, if v_2 has the maximum degree in the graph $G - \{v_1\}$, where $G - \{v_1\}$ is the graph obtained from G by removing the vertex v_1 and all the edges incident to v_1, we pick v_2 as the second vertex in C. If all the edges of the graph $G - \{v_1\}$ are incident to v_2, then $C = \{v_1, v_2\}$ is the minimum vertex covering set, otherwise, we proceed similarly, to obtain the other elements $v_3, v_4, \ldots, v_\tau$ of the minimum vertex covering set C.

Let $C = \{v_1, v_2 \ldots, v_\tau\}$ be the minimum vertex covering set obtained in this way. It is clear that the degree of v_1 in $G_1 = K_{m_1,1} \cup (n-m_1-1)K_1$ is d_1, giving $m_1 = d_1$. Also, the degree of v_2 in $G_1 = K_{m_2,1} \cup (n-m_2-1)K_1$ is either d_2 or d_2-1, depending on whether v_1 and v_2 are non-adjacent or adjacent in G, giving $m_2 \leq d_2$. Similarly, it can be seen that $m_i \leq d_2$, for all $i = 3,4,\dots,\tau$. Therefore, it follows that

$$\mathscr{E}(G) \leq 2\sqrt{m_1} + 2\sqrt{m_2} + \cdots + 2\sqrt{m_\tau} = 2\sqrt{d_1} + 2(\tau-1)\sqrt{d_2},$$

proving the first part. Equality case can be discussed similar to that as seen in Theorem 2.2. □

Remark 2.5. For a connected graph G of order $n \geq 2$ with m edges having maximum and second maximum vertex degrees d_1 and d_2 respectively, the upper bound (6) is always better than the upper bound (2). If $\tau = 1$, then $G \cong K_{n-1,1}$ and so the equality occurs in both the upper bounds. For $\tau \geq 2$, we have $2\sqrt{d_1} + 2(\tau-1)\sqrt{d_2} \leq 2\tau\sqrt{d_1}$, giving $d_1 \geq d_2$, which is always true.

In a graph G, for the adjacent vertices v_1 and v_2 with degrees respectively d_1 and d_2, we have the following observation, the proof of which follows on the similar lines as the proof of Theorem 2.4.

Theorem 2.6. *Let G be a connected graph of order $n \geq 2$ and m edges having vertex covering number τ. If v_1 and v_2 are adjacent vertices in G having degrees respectively d_1 and d_2, then*

$$\mathscr{E}(G) \leq 2\sqrt{d_1} + 2(\tau-1)\sqrt{d_2-1}.$$

Equality holds if and only if $\tau = 1$ and $G \cong K_{1,n-1}$.

Remark 2.7. For a connected graph G of order $n \geq 2$ with m edges, let v_1 and v_2 be the vertices of G having maximum and second maximum vertex degrees d_1 and d_2, respectively. If v_1 and v_2 are adjacent, it can seen that the upper bound given by Theorem 2.6 is always better than the upper bound given by (2).

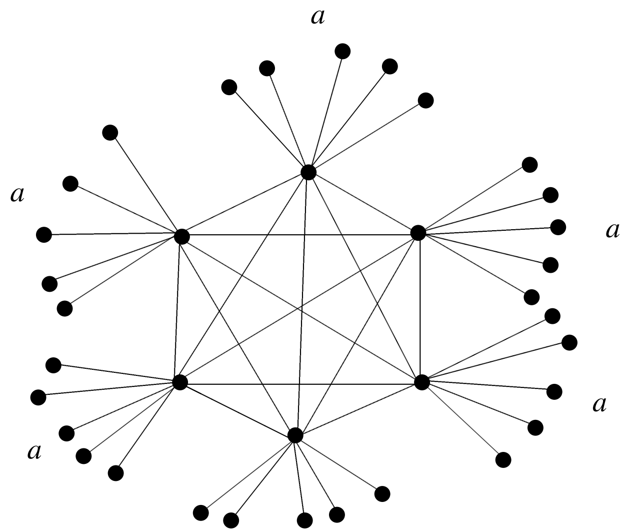

Figure 1: The graph $S_\omega(a,a,\ldots,a)$, $a \geq 1$, for $\omega = 6$.

3. Bounds for distance and Randić energy

It is well known that by using different structural properties one can associate various matrices like Laplacian matrix, signless Laplacian matrix, distance matrix, normalized Laplacian matrix, Randić matrix etc, to a graph G. It is natural to define an energy like quantity for these matrices. The energy of distance matrix called distance energy is defined in [13], the energy of Randić matrix called Randić energy is defined in [11], the energy of normalized Laplacian matrix called normalized Laplacian energy is defined in [3]. In [11] it is shown that for a connected graph G the Randić energy and normalized Laplacian energy are same. For more on energy-like definitions of a graph with respect to various types of matrices and related results, we refer to [5, 7–9, 17] and the references therein.

The following theorem gives an upper for the distance energy $D\mathscr{E}(G)$ in terms of the vertex covering number τ, the maximum degree d_1 and the second maximum degree d_2 of the graph G.

Theorem 3.1. *Let G be a connected graph of order $n \geq 2$ and m edges having vertex covering number τ. If d_1 and d_2 are the maximum and second maximum vertex degrees in G, then*

$$D\mathscr{E}(G) \leq 2(d_1-1)+2\sqrt{d_1^2-d_1+1}+2(\tau-1)\left[d_2-1+\sqrt{d_2^2-d_2+1}\right],$$

with equality if and only if $\tau = 1$ and $G \cong K_{1,n-1}$.

Proof. Using the fact that the distance spectrum of $K_{1,n-1}$ is

$$\{n-2+\sqrt{n^2-3n+3}, n-2-\sqrt{n^2-3n+3},\ -2^{[n-2]}\}$$

and proceeding similarly as in Theorem 2.4, the result follows. □

The following upper bound for the distance energy $D\mathscr{E}(G)$ of a connected graph G was obtained in [2]:

$$D\mathscr{E}(G) \leq \sqrt{2(n-1)\sum_{i<j} d_{ij}^2 + n|det(D(G))|^{\frac{2}{n}}}, \tag{7}$$

where d_{ij} is the distance between i^{th} and j^{th} vertex of G.

Remark 3.2. It is easy to see that for several families of connected graphs G of order $n \geq 2$ having vertex covering number τ, maximum degree d_1 and second maximum degree d_2, the upper bound given by Theorem 3.1 is better than the upper bound given by (7). If $G \cong K_{1,n-1}$, then $\tau = 1$, $d_1 = n-1, d_2 = 1$ and so the upper bound given by Theorem 3.1 is better than the upper bound given by (7). This is because equality occurs for $G \cong K_{1,n-1}$ in Theorem 3.1, while as the upper bound (7) is strict for $G \cong K_{1,n-1}$.

The following theorem gives an upper for the Randić energy $R\mathscr{E}(G)$, in terms of the vertex covering number τ of the graph G.

Theorem 3.3. *Let G be a connected graph of order $n \geq 2$ and m edges having vertex covering number τ. Then*

$$R\mathscr{E}(G) \leq 2\tau,$$

with equality if and only if $\tau = 1$ and $G \cong K_{1,n-1}$.

Proof. Using the fact that the Randić spectrum of $K_{1,n-1}$ is

$$\{1, 0^{[n-2]}, -1\}$$

and proceeding similarly as in Theorem 2.4, the result follows. □

The following upper bound for the Randić energy $R\mathscr{E}(G)$ of a connected graph G was obtained in [3]:

$$R\mathscr{E}(G) \leq \sqrt{\frac{15}{28}}(n+1). \tag{8}$$

Remark 3.4. It can be seen that for the connected graphs G of order $n \geq 2$ having vertex covering number τ, the upper bound given by Theorem 3.3 is better than the upper bound given by (8), for all $\tau \leq \sqrt{\frac{15}{28}\frac{n+1}{2}}$. In particular, if $G \cong K_{a,b}$, with $a \leq b$ and $a \leq \sqrt{\frac{15}{28}\frac{n+1}{2}}$, then the upper bound Theorem 3.3 is always better than the upper bound given by (8).

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