LE MATEMATICHE
Vol. LIX (2004) – Fasc. I–II, pp. 393–402

COLORING MIXED HYPERGRAPHS: FROM COMBINATORICS TO PHILOSOPHY

VITALY VOLOSHIN

We survey recent results and open problems on the chromatic spectrum, planarity and colorability of mixed hypergraphs and their relations to such categories of philosophy as identity and difference.

1. Coloring Theory.

Most of classical coloring theory can be described using hypergraph coloring as introduced by P. Erdös and A. Hajnal in 1966 [4]. A *hypergraph* [2] $\mathcal{H} = (X, \mathcal{E})$ is a set of vertices X and a set of edges $\mathcal{E}$, where each edge is a subset of the vertex set. A *proper coloring* of a hypergraph is a coloring of the vertex set such that every edge has at least two vertices with distinct colors. This generalizes the condition for proper colorings of graphs.

The *chromatic number* of a hypergraph is the minimum number of colors used in a proper coloring. Historically, coloring theory was the theory on the minimum number of colors. The maximum number of colors becomes interesting and nontrivial when we introduce another type of constraint on colorings. Both these constraints are incorporated in the concept of mixed hypergraph.

More precisely, *mixed hypergraph* is a triple $\mathcal{H}$, where X is the *vertex set* and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$-*edges* and $\mathcal{D}$-*edges*,

---

Partially supported by Troy University Research Grants

respectively. A *proper k-coloring* of a mixed hypergraph is a mapping from the vertex set X into a set of k *colors* so that each $\mathcal{C}$-edge has two vertices with $\mathcal{C}$ommon color and each $\mathcal{D}$-edge has two vertices with $\mathcal{D}$istinct colors. A mixed hypergraph is *k-colorable* if it has a proper coloring with at most k colors. A *strict k-coloring* is a proper k-coloring using all k colors. The maximum number of colors in a strict coloring of $\mathcal{H}$ is the *upper chromatic number* $\bar{\chi}(\mathcal{H})$; the minimum number is the *lower chromatic number* $\chi(\mathcal{H})$.

We obtain classical hypergraph coloring in special case when $\mathcal{H} = (X, \emptyset, \mathcal{D})$, denoted by $\mathcal{H}_{\mathcal{D}}$ and called $\mathcal{D}$-hypergraph. In this way, the theory of $\mathcal{D}$-hypergraphs is a theory on the minimum number of colors. When $\mathcal{H} = (X, \mathcal{C}, \emptyset)$ we denote it $\mathcal{H}_{\mathcal{C}}$ and call $\mathcal{C}$-hypergraph. In contrast, the theory of $\mathcal{C}$-hypergraphs is a theory on the maximum number of colors. Thus, general mixed hypergraphs represent the structures where both the problems on the minimum and the maximum number of colors occur.

This talk surveys scientific results in the following subdirections which represent a part in general theory described in Fields Institute Monograph [18]. The monograph contains 12 chapters and 10 of them end with open problems. A number of problems since then has been solved, a range of new problems appeared. This direction is continuously extending. Many further important details (including current list of 133 publications) can be found at the Mixed Hypergraph Coloring Web Site, see http://math.net.md/voloshin.

2. Chromatic spectrum.

The set of values of k for which $\mathcal{H}$ has a strict k-coloring is the *feasible set*. For each k, let r_k be the number of partitions of the vertex set into k nonempty parts (color classes) such that the coloring constraint is satisfied on each $\mathcal{C}$- and each $\mathcal{D}$-edge. Such partitions are called *feasible partitions*. In fact, r_k equals the number of strict k-colorings if we don't count the permutations of colors as different colorings. The vector $R(\mathcal{H}) = (r_1, \dots, r_n)$ is the *chromatic spectrum* [21], [22], [27].

Given a colorable mixed hypergraph $\mathcal{H}$, it is natural to ask whether $\mathcal{H}$ has strict k-colorings for all k such that $\chi(\mathcal{H}) \leq k \leq \bar{\chi}(\mathcal{H})$. Open since the introduction of mixed hypergraphs, this question has been solved by Jiang, Mubayi, Tuza, Voloshin, West in [7], [8]. The answer is negative; there may indeed be gaps in the chromatic spectrum.

A mixed hypergraph has a *gap at k* if its feasible set contains elements larger and smaller than k but omits k. It was constructed for $2 \leq s \leq t-2$ a mixed hypergraph $\mathcal{H}_{s,t}$ with feasible set $\{s, t\}$. Furthermore, it was proved that

$\mathcal{H}_{s,t}$ has the fewest vertices among all s-colorable mixed hypergraphs that have a gap at $t-1$; this minimum number of vertices is $2t-s$.

The last raised the question of which sets of positive integers are feasible sets of mixed hypergraphs. It was proved that a finite set of positive integers is a feasible set if and only if it is an initial interval $\{1,\dots,t\}$ or does not contain the element 1.

Chromatic spectrum of some special families of mixed hypergraphs was investigated, in particular, it was shown that gaps can arise even when $\mathcal{C}=\mathcal{D}$ and all the edges have the same size; gaps cannot arise when each member of $\mathcal{C}$ and $\mathcal{D}$ is an interval in an underlying linear order on the vertices [7], or, more generally, is a subtree of a host tree [11]. Gaps also cannot arise if mixed hypergraph has a uniquely colorable separator and derived subhypergraphs do not have the gaps. The last implies that the so called pseudo-chordal mixed hypergraphs [26] have also gap-free chromatic spectra. Perhaps the most notable fact however, is that planar mixed hypergraphs may have gaps, see [27,] and block designs may also have gaps, see [5].

We will discuss further results in this direction, in particular, the results by D. Kral stating that for each vector of positive integers $(r_1, r_2, \dots, r_n)$, $r_1 \neq 0$, there exist a mixed hypergraph having this vector as a chromatic spectrum [9], and that mixed hypergraphs with vertex degree at most two have continuous chromatic spectrum [10].

3. Planarity.

Let $\mathcal{H}=(X,\mathcal{E})$ be a hypergraph. The *bipartite representation* of $\mathcal{H}$, denoted by $B(\mathcal{H})$, is a bipartite graph with vertex set $X\cup\mathcal{E}$. $x\in X$ is adjacent to $E\in\mathcal{E}$ (in $B(\mathcal{H})$) if and only if $x\in E$. The following definition is due to [20]: a hypergraph $\mathcal{H}$ is called *planar* iff $B(\mathcal{H})$ is a planar graph.

Thus planar graphs are the special case of planar hypergraphs in which all edges have size 2. As one may see, a planar hypergraph admits an embedding in the plane in such a way that each vertex corresponds to a point on the plane, and every edge corresponds to a closed region homeomorphic to a disk such that it contains the points corresponding to its vertices in the boundary and it does not contain the points corresponding to the other vertices. Furthermore two such regions intersect exactly in the points that correspond to the vertices in the intersection of the corresponding edges. In this way the connected regions of the plane which do not correspond to the edges form the faces of the embedding of the planar hypergraph.

Using properties of the bipartite representation $B(\mathcal{H})$ one can derive many properties of a planar embedding of the hypergraph $\mathcal{H}$ [1, 2, 20]. For example, if the degree of vertex $x \in X$ in $\mathcal{H}$ is denoted by $d_{\mathcal{H}(x)}$ (see [12]) we obtain the generalization of the famous Euler's formula: for any planar embedding of $\mathcal{H} = (X, \mathcal{E})$ with f faces

$$|X| + |\mathcal{E}| - \sum_{E \in \mathcal{E}} |E| + f = |X| + |\mathcal{E}| - \sum_{x \in X} d_{\mathcal{H}}(x) + f = 2.$$

Definition 1. An embedding of a planar hypergraph is called maximal iff every face contains precisely two vertices, or equivalently iff in the corresponding embedding of $B(\mathcal{H})$ every face has length 4.

This maximality is relative in the sense that in every such face one can always insert a new edge of size 2. However if a planar hypergraph $\mathcal{H}$ is not maximal then there is at least one face of size at least 3 and therefore one can insert a new edge of size at least 3 in that face.

If we draw the faces of a maximal planar hypergraph as curves connecting respective vertices, then we obtain a plane graph whose faces correspond to the edges of the initial hypergraph. In this way, we may look at a plane graph as a planar embedding of a maximal hypergraph such that the faces of the graph correspond to the edges of the hypergraph.

Let $\mathcal{H}$ be now a mixed hypergraph. Denote the underlying edge-set of $\mathcal{H}$ by $\mathcal{E} = \mathcal{C} \cup \mathcal{D}$. Observe that if some $\mathcal{C}$-edge coincides as a subset of vertices with some $\mathcal{D}$-edge (i.e. it is a bi-edge) then it appears only once in $\mathcal{E}$. We say that $\mathcal{H}' = (V, \mathcal{E})$ is the *underlying hypergraph* of $\mathcal{H}$.

Definition 2. A mixed hypergraph $\mathcal{H}$ is planar if and only if the underlying hypergraph $\mathcal{H}'$ is planar.

This can be viewed as follows: we can embed $\mathcal{H}'$ in the plane and label all hyperedges with B, C or D appropriately according to whether they are bi-edges, $\mathcal{C}$-edges or $\mathcal{D}$-edges. Note that $\mathcal{C}$-edges of size 2 can be contracted, and bi-edges of size 2 lead to uncolourability, so that in general it suffices to only consider mixed hypergraphs containing neither.

The question of coloring properties of general planar mixed hypergraphs was first raised in [17] (problem 8, p. 43). It is evident that every planar $\mathcal{D}$-hypergraph is colorable, just as every planar $\mathcal{C}$-hypergraph is. The situation changes however if we consider general planar mixed hypergraphs. The smallest non-trivial (reduced) uncolorable planar mixed hypergraph $\mathcal{H}$ is given by $X = \{1, 2, 3\}$, $\mathcal{C} = \{(1, 2, 3)\}$, $\mathcal{D} = \{(1, 2), (2, 3), (1, 3)\}$. It is easy to embed it in the plane with 4 faces (3 containing 2 vertices each and 1 containing 3 vertices). It is not difficult to extend this example to an infinite family of uncolorable planar mixed hypergraphs. The structure of uncolorable planar mixed hypergraphs is unknown. In general, allowing $\mathcal{D}$-edges of size 2 implies that the four color problem is a special case of the theory of planar mixed hypergraphs. Therefore, it is reasonable to study planar mixed hypergraphs without edges of size 2. The first interesting case is that in which $\mathcal{H}$ is a maximal 3-uniform planar bi-hypergraph.

Let us consider the coloring problem for maximal 3-uniform planar bi-hypergraphs $\mathcal{H}$. Since every face of a maximal planar hypergraph is of size 2, we can associate a graph $G(\mathcal{H})$, on the same vertex set, with $\mathcal{H}$: replace every face in $\mathcal{H}$ by an edge in G, so that every edge in $\mathcal{H}$ becomes a face of G. $\mathcal{H}$ is maximal 3-uniform, so that G must be a triangulation in the usual sense. Since every edge of $\mathcal{H}$ is a bi-edge, both $\mathcal{H}$ and G are called *bi-triangulations*.

In this talk, we will survey further results and open problems in colorings of planar mixed hypergraphs, in particular we will discuss colorings of bi-triangulations [12], [18] and colorings of planar mixed hypergraphs with broken and continuous chromatic spectrum [8], 3].

4. Combinatorics and Philosophy: a point of view.

I believe in some ideas of the German philosopher G. W. F. Hegel (1770-1831) because these ideas "work". Detailed original description of his philosophy can be found in:

1. G. W. F. Hegel. The Science of Logic (1812-1816).
2. G. W. F. Hegel. Encyclopedia of the Philosophical Sciences in Outline (1817).

Born in Stuttgart and educated in Tubingen, Hegel devoted his life wholly to academic pursuits, teaching at Jena, Nuremberg, Heidelberg, and Berlin. His "Wissenschaft der Logik" (The Science of Logic) (1812-1816) attributes the unfolding of concepts of reality in terms of the pattern of dialectical reasoning (thesis antithesis synthesis) that Hegel believed to be the only method of progress in human thought, and his "Die Encyclopadie der philoso-phischen Wissenschaften im Grundrisse" (Encyclopedia of the Philosophical Sciences in Outline) (1817) describes the application of this dialectic to all areas of human knowledge. It is clear that many of his ideas have had roots in the past and have been developed further in different directions.

A it is described in [13] "Hegel's aim was to set forth a philosophical system so comprehensive that it would encompass the ideas of his predecessors and create a conceptual framework in terms of which both the past and future could be philosophically understood. Such an aim would require nothing short of a full account of reality itself. Thus, Hegel conceived the subject matter of philosophy to be reality as a whole. This reality, or the total developmental process of everything that is, he referred to as the Absolute, or Absolute Spirit. According to Hegel, the task of philosophy is to chart the development of Absolute Spirit.

Concerning the rational structure of the Absolute, Hegel, following the ancient Greek philosopher Parmenides, argued that what is rational is real and what is real is rational. This must be understood in terms of Hegel's further claim that the Absolute must ultimately be regarded as pure Thought, or Spirit, or Mind, in the process of self-development (see Idealism). The logic that governs this developmental process is dialectic. The dialectical method involves the notion that movement, or process, or progress, is the result of the conflict of opposites. Traditionally, this dimension of Hegel's thought has been analyzed in terms of the categories of thesis, antithesis, and synthesis. Although Hegel tended to avoid these terms, they are helpful in understanding his concept of the dialectic. The thesis, then, might be an idea or a historical movement. Such an idea or movement contains within itself incompleteness that gives rise to opposition, or an antithesis, a conflicting idea or movement. As a result of the conflict a third point of view arises, a synthesis, which overcomes the conflict by reconciling at a higher level the truth contained in both the thesis and antithesis. This synthesis becomes a new thesis that generates another antithesis, giving rise to a new synthesis, and in such a fashion the process of intellectual or historical development is continually generated. Hegel thought that Absolute Spirit itself (which is to say, the sum total of reality) develops in this dialectical fashion toward an ultimate end or goal.

For Hegel, therefore, reality is understood as the Absolute unfolding dialectically in a process of self-development. As the Absolute undergoes this development, it manifests itself both in nature and in human history. Nature is Absolute Thought or Being objectifying itself in material form. Finite minds and human history are the process of the Absolute manifesting itself in that which is most kin to itself, namely, spirit or consciousness. In The Phenomenology of Mind Hegel traced the stages of this manifestation from the simplest level of consciousness, through self-consciousness, to the advent of reason".

What all these ideas have in common with Combinatorics? First, one could observe that classical coloring theory, 1852-1993, based on the notion of edge, was incomplete and asymmetric. Second, using the notion of the conflict of thesis and anti-thesis, one could define a new kind of coloring - the Mixed Hypergraph Coloring. In this language, the $\mathcal{D}$-edge corresponds to a thesis, the $\mathcal{C}$-edge corresponds to an antithesis, and the mixed hypergraph itself corresponds to a synthesis. One can say that philosophy of Hegel, especially the learning about contradictions, served as a source for the idea of mixed hypergraph.

Contradictions are everywhere and they are universal. The process of natural selection in biology is based on contradictions. Two sexes - males and females interact as opposites. In this sense, the life of populations with sexual reproduction is pure dialectical process. We can find the thesis and antithesis even in DOUBLED spirals of DNA molecules what was discovered much after Hegel.

Contradictions determine the development of mathematics. For example, one of the most famous theorem of 20th century, the Goedel's Incompleteness Theorem explicitly states that within any given field of mathematics, there would always be some propositions that couldn't be proven either true or false using the rules and axioms of that mathematical branch itself. This eternal contradiction dooms us to dialectic in our search for truth. It completely fits into the scheme "thesis-antithesis-synthesis".

In a similar way, as many other opposite to each other things, in mixed hypergraph coloring, the $\mathcal{C}$-edges and $\mathcal{D}$-edges interact, and this interaction brings a lot of new properties and qualities.

Dialectical method states that new quality appears as the result of conflict between opposites. The contradictions between the two principles "to be the same" and "to be different" represent the basic contradictions in Combinatorics. The concept of mixed hypergraph provides a proof and shows the way how this method explicitly works.

For example, as a result of contradictions, the following new properties=qualities of colorings have been discovered since the introduction of mixed hypergraphs (in fact, at the very beginning there was a firm general prediction that we will find something NEW OF PRINCIPLE):

- uncolorability
- unique colorability
- C-perfection
- phantom vertices
- redundant edges
- unavoidability of recoloring in greedy algorithm for the maximum number of colors
- gaps in the chromatic spectrum
- new polynomials that are chromatic
- list coloring model without lists

- natural number model

The list is not finished. These concepts are impossible without the conflict between $\mathcal{C}$-edges and $\mathcal{D}$-edges. Fundamental contradictions guarantee that we will discover yet many new unforeseen features of colorings. The mixed hypergraph coloring theory directly contributes to the mathematical comprehension of relationship between the two categories of contemporary philosophy: IDENTITY and DIFFERENCE [15].

Many philosophers, mathematician and not only, studied the relation between these categories.

For example, in ”The Science of Logic”, page 470, (italian translation) Hegel describes a story about Leibnitz: in the parc of a castle, in order to prove that there are no two things that are identical, Leibnitz made ladies to look for two identical leaves among all the leaves on trees.

Another example, Alfred Nobel [14] wrote that all science is built on observations of similarities = identities and differences. He continued: ”A chemical analysis is of course nothing other than this, and even mathematics has no other foundation. History is a picture of past similarities and differences; geography shows the differences in the earth's surface; geology, similarities and differences in the earth's formation, from which we deduce the course of its transformations. Astronomy is the study of similarities and differences between celestial bodies; physics, a study of similarities and differences that arise from the attraction and motive functions of matter. The only exception to this rule is religious doctrine, but even this rests on the similar gullibility of most people. Even metaphysics - if it is not too insane - must find support for its hypotheses in some kind of analogy. One can state, without exaggeration, that the observation of and the search for similarities and differences are the basis of all human knowledge.”

All mathematicians work entirely in this area since every theorem, each formula (=equality) or any inequality contribute to this relation in universal cognition process. There is no more important and used sign in Mathematics other than ”=”. In this sense, any mathematician is a philosopher.

It is interesting to find an interpretation of mixed hypergraph coloring as a philosophical model. For example, as it is described in section 12.10 of [18], this theory explicitly points out on the contradictory, dialectical nature of the concept of natural number. Moreover, it shows that this concept is the MOST SELF-CONTRADICTORY concept that can be thought of using the philosophical categories of identity and difference. When we write ”identity” we mean ”monochromatic”, when we write ”difference” we mean ”polychromatic”.

Another observation is that uncolorability may be treated as "thing-in-itself" ("cosa in s'e") and in this way we find the explicit indication on Kant's philosophy.

One more question concerns the interaction between formal and dialectical logic. By introducing $\mathcal{C}$-edges, we explicitly introduce contradictions in set partitions, but we study them using traditional formal logic. But only a special logic, namely the 'dialectical' logic, admits and theorizes the contradictions, which are resolved and surpassed without to be denied. So, the mixed hypergraph coloring, on one hand, is based on a 'dialectical' logic, on the other hand, it does not refuse the 'formal' logic. In other words, in mixed hypergraph coloring, the dialectics constitutes a 'method' of study, as well as an 'object' of study. Isn't it the first case of such kind?

However, main expectation is that the theory will allow to find new fundamental mathematical, namely, combinatorial relations in genetics on the level of molecular biology. Life is the property of the world to maintain the rest, i.e. to strive to be the same while everything is changing, i.e. is different.

Our HEREDITY is based on comparison and, being DISCRETE and FINITE, roughly speaking, begins with "having at least two the SAME features"="having at least two vertices of the SAME color". Thus the meaning of all our activity is a function of what is combinatorially recorded in our chromosomes. New questions arise and new conclusions are expected as the result of investigations in mixed hypergraph coloring.

In this way, the idea of mixed hypergraph coloring, generated by the philosophy of Hegel, will be back to philosophy.

REFERENCES

[1] C. Berge, *Graphs and Hypergraphs,* North Holland, 1973.

[2] C. Berge, *Hypergraphs: Combinatorics of Finite Sets,* North Holland, 1989.

[3] Z. Dvorak - D. Kral, *On Planar Mixed Hypergraphs,* Electronic Journal on Combinatorics, 8 (1), N. R35, 2001.

[4] P. Erdös - A. Hajnal, *On chromatic number of graphs and set systems,* Acta Math. Acad. Sci. Hungar., 17 (1966), pp. 61–99.

[5] L. Gionfriddo, *Voloshin's colorings of P3-designs,* Discrete Mathematics, 275 (2004), pp. 137–149.

[6] T. Jiang - D. Mubayi - V. Voloshin - Zs. Tuza - D.B. West, *Chromatic spectrum is broken,* 6th Twente Workshop on Graphs and Combinatorial Optimization, 26-28

May, 1999, H. J. Broersma, U. Faigle, J. L. Hurink (eds.). University of Twente, May, (1999), pp. 231–234.

[7] T. Jiang - D. Mubayi - V. Voloshin Zs. Tuza - D. B. West, *The chromatic spectrum of mixed hypergraphs,* Graphs and Combinatorics, 18 (2002), pp. 309–318.

[8] D. Kobler - A. Kundgen, *Gaps in the chromatic spectrum of face-constrained plane graphs,* Electron. J. Combin., 8 (1) (2001), N3.

[9] D. Kral', *On feasible sets of Mixed Hypergraphs, On Feasible Sets of Mixed Hypergraphs,* preprint, KAM-DIMATIA Series 2001-540.

[10] D. Kral' - J. Kratochvil - H.-J. Voss, *Mixed hypergraphs with bounded degree: edge-coloring of mixed multigraphs,* Theoretical Computer Science, Volume 295, Issues 1-3 , 24 February 2003, pp. 263–278.

[11] D. Kral - J. Kratochvil - A. Proskurowski - H.-J. Voss, *Coloring mixed hypertrees,* Proceedings 26th Workshop on Graph-Theoretic oncepts in Computer Science, Lecture Notes in Computer Science vol. 1928, pp. 279-289, Springer-Verlag, 2000.

[12] A. Kundgen - E. Mendelsohn - V. Voloshin, *Coloring of planar mixed hypergraphs,* Electron. J. Combin., 7 (2000), R60.

[13] Microsoft, *Microsoft Encarta Encyclopedia 2003,* 1993-2002 Microsoft Corporation. All rights reserved.

[14] Nob, *http://www.nobel.se/nobel/alfred-nobel/biographical/frangsmyr/:,* Life and Philosophy of Alfred Nobel.

[15] Thomasson - Amie - "Categories", *The Stanford Encyclopedia of Philosophy (Summer 2004 Edition),* Edward N. Zalta (ed.), URL = "http://plato.stanford.edu/archives/sum2004/entries/categories/".

[16] V.I. Voloshin, *The mixed hypergraphs,* Computer Science J. Moldova, 1 (1993), pp. 45–52.

[17] V.I. Voloshin, *On the upper chromatic number of a hypergraph,* Australasian J. Comb., 11 (1995), pp. 25–45.

[18] V. Voloshin, *Coloring Mixed Hypergraphs: Theory, Algorithms and Applications,* Fields Institute Monograph, American Mathematical Society, 2002.

[19] V. Voloshin - H. Zhou, *Pseudo-chordal mixed hypergraphs,* Discrete Math., 202 (1999), pp. 239–248.

[20] A.A. Zykov, *Hypergraphs (in Russian),* Uspekhi Mat. Nauk, 29 (1974), pp. 89–154.

Troy University,
Troy, AL 36082, USA.
e-mail: vvoloshin@troy.edu