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SOME RESULTS ON SIMPLE COMPLETE IDEALS HAVING ONE CHARACTERISTIC PAIR

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Let α be a regular local two-dimensional ring, and let $\mathfrak{m} = (x, y)$ be its maximal ideal. Let $m > n > 1$ be coprime integers, and let $\mathfrak{p}$ be the integral closure of the ideal (x^m, y^n). Then $\mathfrak{p}$ is a simple complete $\mathfrak{m}$-primary ideal, and its value semigroup is generated by m, n. We construct a minimal system of generators $\{z_0, \dots, z_n\}$ of $\mathfrak{p}$, and from this we get a minimal system of generators of $\mathfrak{P}$, the polar ideal of $\mathfrak{p}$, consisting of $n = \theta$ elements. In particular, we show that $\mathfrak{p}$ and $\mathfrak{P}$ are monomial ideals. When $\alpha = \kappa[[\, x, y\,]]$, a ring of formal power series over an algebraically closed field κ of characteristic zero, this implies the following. There exists a non-empty Zariski-open subset U of κ^{n+1} such that for every $\mathbf{u} = (u_0, \dots, u_n) \in U$ the linear combination $f_{\mathbf{u}} := \sum_{i=0}^{n} u_i z_i$ is a general element of $\mathfrak{p}$ and $\partial f_{\mathbf{u}}/\partial y$ is a general element of the polar ideal $\mathfrak{P}$.
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Introduction.

Let C be a curve in affine (x, y)-plane with equation $F(x, y) = 0$. One of the classical methods to study C and its singularities consists in considering the polar curves of C, namely the curves whose equations are linear combinations

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of the partial derivatives of F (see e.g. [7]). This point of view has had modern important developments, see e.g. [2], [3], [4], [5], and also [6].

The notion of polarity extends in a natural way to linear systems, and provides a useful tool for the local study of a linear system at a base point. The natural setting for this study is the theory of complete (i.e. integrally closed) ideals in two-dimensional regular local rings (cf. Zariski [20], appendix 5 and the papers [15], [17] of Lipman) and the notion of polar ideal of a complete ideal introduced by Lipman [16] (see section 4 for the definition).

The aim of this paper is to make some steps toward a concrete understanding of complete ideals and of their polar ideals, addressing a very natural question: how does one construct explicit generators of a simple complete ideal $\mathfrak{p}$ contained in the two-dimensional regular local ring α, and of its polar ideal $\mathfrak{P}$.

We solve the problem for the ideals which arise as the integral closure of (x^m, y^n), where $\{x, y\}$ is a regular system of parameters of α and $m > n > 1$ are coprime integers. These ideals are among the simple complete ideals having just one characteristic pair (see section 1), which are the easiest ones. Also with this restriction things are fairly complicated, as we shall see.

The paper is organized as follows. The first two sections are preliminary. In section 1 we collect some basic definitions and results on complete ideals, while in section 2 we give some general facts on monomial ideals.

The first main result is proved in section 3 (see prop. (3.6)]. Here we construct a numerical function $\sigma_{m,n}$, explicitly computable in terms of m and n, which allows to determine a minimal system of monomial generators of $\mathfrak{p}$ consisting of monomials in x and y. The fact that $\mathfrak{p}$ is a monomial ideal with respect to x and y is a particular case of [14], so our main point is the explicit algorithm.

In section 4 we show first two general results about our ideal $\mathfrak{p}$, which imply, in particular, that $\mathfrak{P}$ is a monomial ideal. Our second main result (cf. (4.4)) gives an explicit construction of a minimal set of monomial generators of $\mathfrak{P}$, obtained by a finer study of the function $\sigma_{m,n}$. As consequences we get some expected facts in the classical case (namely $\alpha = \kappa[[x, y]]$, where κ is an algebraically closed field): for example a "general element" of $\mathfrak{P}$ can be obtained as a polar of a general element of $\mathfrak{p}$.

We want to remark explicitly that our results do not apply to arbitrary simple complete ideals with one characteristic pair. Indeed such an ideal is not, in general, the integral closure of an ideal of the form (x^m, y^n) as above. This follows by the characterization given in remark (3.13).

1. Preliminary results.

1.1. The setting. We use the notation introduced by Lipman (cf. [15], [16], [17]); cf. also [11]). Let K be a field, and denote by $\Omega := \Omega(K)$ the set of all two-dimensional regular local subrings of K having K as field of quotients. The elements of Ω will be called points, and denoted by lower case greek letters $\alpha, \beta, \ldots$. For $\alpha \in \Omega$ we denote by $\mathfrak{m}_\alpha$ the maximal ideal of α, by ord_α the order function of α, and by κ_α the residue field $\alpha/\mathfrak{m}_\alpha$ of α.

(1) Let α be a point, and consider the canonical homomorphism of graded rings

$$\varphi : \mathcal{R}(\mathfrak{m}_\alpha, \alpha) \to \mathrm{gr}_{\mathfrak{m}_\alpha}(\alpha) = \kappa_\alpha[\,\overline{x}, \overline{y}]$$

from the Rees ring $\mathcal{R}(\mathfrak{m}_\alpha, \alpha)$ of α with respect to $\mathfrak{m}_\alpha$ to the associated graded ring $\mathrm{gr}_{\mathfrak{m}_\alpha}(\alpha)$; note that $\mathrm{gr}_{\mathfrak{m}_\alpha}(\alpha)$ is a polynomial ring $\kappa_\alpha[\,\overline{x}, \overline{y}]$ in two indeterminates $\overline{x}, \overline{y}$ over the field κ_α [here $\{x, y\}$ is a system of generators of $\mathfrak{m}_\alpha$, and we have put $\overline{x} := x \bmod \mathfrak{m}_a^2$, $\overline{y} := y \bmod \mathfrak{m}_\alpha^2$]. The set $\mathbb{P}_\alpha$ of homogeneous prime ideals of height one of $\mathrm{gr}_{\mathfrak{m}_\alpha}(\alpha)$ corresponds uniquely to the set of closed point of $\mathrm{Proj}(\mathcal{R}(\mathfrak{m}_\alpha, \alpha))$; the local ring of such a point is called a quadratic transform of α. Let $p \in \mathbb{P}_\alpha$, and let β_p be the quadratic transform of α corresponding to p. Then $\beta_p \in \Omega$ and it dominates α, and $[\beta_p : \alpha] := [\kappa_{\beta_p} : \kappa_\alpha]$ is finite.

Let $\alpha \subset \beta$ be distinct points in Ω. Then there exists a uniquely determined sequence

$$\alpha =: \alpha_0 \subsetneqq \alpha_1 \subsetneqq \cdots \subsetneqq \alpha_n := \beta$$

where, for $i \in \{1, \ldots, n\}$, α_i is a quadratic transform of α_{i-1}. In particular, β dominates α, and $[\beta : \alpha] := [\kappa_\beta : \kappa_\alpha]$ is finite.

(2) Let $\alpha \in \Omega$. The set of non-zero complete [= integrally closed] ideals of α is a semigroup under multiplication, and every complete ideal is, in a unique way, a product of simple complete ideals (an ideal is called simple if it is not the product of two proper ideals) (cf. [20], p. 386, Th. 3). We denote the monoid of complete $\mathfrak{m}_\alpha$-primary ideals of α by $\mathcal{MC}(\alpha)$, and its subset of simple ideals by $\mathcal{MCS}(\alpha)$.

(3) To each non-zero ideal $\mathfrak{a}$ of $\alpha \in \Omega$, it is associated its characteristic ideal $\mathfrak{c}(\mathfrak{a})$: if $\mathrm{ord}_\alpha(\mathfrak{a}) =: s$, then $\mathfrak{c}(\mathfrak{a})$ is generated by the greatest common divisor of all the elements $f \bmod \mathfrak{m}_\alpha^{s+1} \in (\mathrm{gr}_{\mathfrak{m}_\alpha}(\alpha))_s$ where f runs through the set of elements of $\mathfrak{a}$ of order s; it is a homogeneous principal ideal in $\mathrm{gr}_{\mathfrak{m}_\alpha}(\alpha)$. An $\mathfrak{m}_\alpha$-primary complete ideal different from the maximal ideal is simple only if $\mathfrak{c}(\mathfrak{a})$ is a positive power of a prime ideal $p \in \mathbb{P}_\alpha$ (cf. [20], p. 386). Let $p \in \mathbb{P}_\alpha$; the monoid of complete $\mathfrak{m}_\alpha$-primary ideals whose characteristic ideal is a positive power of p will be denoted by $\mathcal{MC}(\alpha, p)$, and its subset of simple ideals by $\mathcal{MCS}(\alpha, p)$.

(4) Let $\mathfrak{p}$ be a simple complete $\mathfrak{m}_\alpha$-primary ideal. Then $\mathfrak{p}$ determines a quadratic sequence

$$\alpha =: \alpha_0 \subsetneqq \alpha_1 \subsetneqq \cdots \subsetneqq \alpha_n =: \beta_{\mathfrak{p}};$$

the extension of the order function of $\beta_{\mathfrak{p}}$ to a valuation $\nu := \nu_{\mathfrak{p}}$ of K is called the valuation defined by $\mathfrak{p}$, and

$$\Gamma_{\mathfrak{p}} := \{\nu(z) \mid z \in \alpha \setminus \{0\}\}$$

is called the semigroup of $\mathfrak{p}$. We call $\mathfrak{p}$ residually rational if $[\beta_{\mathfrak{p}} : \alpha] = 1$; in this case $\Gamma_{\mathfrak{p}} \subset \mathbb{N}_0$ is a subsemigroup of $\mathbb{N}_0$ which can be considered as the value semigroup of a plane irreducible algebroid curve (cf. [11], (8.18) and (8.19)). We say—in accordance with the terminology used in the case of plane irreducible algebroid curves—that $\mathfrak{p}$ *has g characteristic pairs* if $\Gamma_{\mathfrak{p}}$ has a minimal system of generators consisting of $g + 1$ elements.

(5) Let $\alpha \subset \beta$ be points in Ω. To an ideal $\mathfrak{a}$ in α, it is associated its transform $\mathfrak{a}^\beta \subset \beta$ (cf. [16], p. 206–207 for the definition of ideal transform and its properties).

Remark 1.2. In the sequel, we use the following results.

(1) Let α be a point, and let $\mathfrak{a}$ be a complete $\mathfrak{m}_\alpha$-primary ideal. Then we have (by the length formula of Hoskin and Deligne, cf. [15], Th. (3.1), and by a result of Huneke and Sally, cf. [13], Th. 2.1, or [15], Cor. (3.2))

$$\ell_\alpha(\alpha/\mathfrak{a}) = \sum_{\beta \supset \alpha} \frac{1}{2}[\beta : \alpha]\mathrm{ord}_\beta(\mathfrak{a}^\beta)(\mathrm{ord}_\beta(\mathfrak{a}^\beta) + 1),$$

$$\mu(\mathfrak{a}) := \dim_{\kappa_\alpha}(\mathfrak{a}/\mathfrak{m}_\alpha\mathfrak{a}) \;=\; \mathrm{ord}_\alpha(\mathfrak{a}) + 1.$$

(2) Let α be a point, let $p \in \mathbb{P}_\alpha$, and let $\beta := \beta_p$. The map

$$\mathfrak{a} \mapsto \mathfrak{a}^\beta : \mathcal{MC}(\alpha, p) \longrightarrow \mathcal{MC}(\beta)$$

is an isomorphism of monoids, and the inverse map is given by the inverse transform. The restriction induces a bijective map $\mathcal{MCS}(\alpha, p) \longrightarrow \mathcal{MCS}(\beta)$ (cf. [20], p. 388, (A)).

(3) Let α be a point, and let $\mathfrak{a}$ be a non-zero ideal of α. Then $\mathfrak{a}$ and its integral closure $\overline{\mathfrak{a}}$ in α have the same order, and $\mathfrak{a}$ is simple iff $\overline{\mathfrak{a}}$ is simple (the first part is easy, and for the second part cf. [20], p. 368, Lemma 6, and p. 388, (A)). (For any ideal $\mathfrak{c}$ of a ring S, we denote by $\overline{\mathfrak{c}}$ its integral closure in S.)

(4) Let α be a point, and let β be a point with $\beta \supset \alpha$. Then we have (as is easy to check)

$$\overline{\mathfrak{a}}^\beta = \overline{\mathfrak{a}^\beta}.$$

1.3. Euler polynomials. We define

$$Q_{-1} := 0, \quad Q_0 := 1,$$

and for $i \geq 1$ let the Euler polynomials $Q_i \in \mathbb{Z}[\,T_1, \ldots, T_i\,]$ be defined by

$$Q_i(T_1, \ldots, T_i) = T_1 Q_{i-1}(T_2, \ldots, T_i) + Q_{i-2}(T_3, \ldots, T_i).$$

It is well known that for every $i \in \mathbb{N}$ we have

$$\begin{gathered} Q_i(T_1, \ldots, T_i) Q_{i-2}(T_2, \ldots, T_{i-1}) - \\ - Q_{i-1}(T_1, \ldots, T_{i-1}) Q_{i-1}(T_2, \ldots, T_i) = (-1)^i, \\ Q_i(T_1, \ldots, T_i) = Q_i(T_i, \ldots, T_1), \\ Q_i(T_1, \ldots, T_i) = T_i Q_{i-1}(T_1, \ldots, T_{i-1}) + Q_{i-2}(T_1, \ldots, T_{i-2}), \\ Q_i(T_1, \ldots, T_{i-1}, 1) = Q_{i-1}(T_1, \ldots, T_{i-2}, T_{i-1} + 1). \end{gathered}$$

These results will be used tacitly.

1.4. Euler polynomials and continued fractions. Let n_0, n_1 be natural integers, and let

$$n_0 = s_1 n_1 + n_2, n_1 = s_2 n_2 + n_3, \ldots, n_{k-1} = s_k n_k$$

with integers $n_1 > n_2 > \cdots > n_k \geq 1$ and non-negative integers $s_1, \ldots, s_k$ with $s_k \geq 2$ be the Euclidean algorithm for n_0, n_1; in particular, we have $n_k = \gcd(n_0, n_1)$. The integer k will be denoted also by $k(m, n)$. Then we have the continued fraction expansion

$$\begin{aligned} \frac{n_0}{n_1} &= [s_1, \ldots, s_k] \\ &= s_1 + \cfrac{1}{s_2 + \cfrac{1}{s_3 + \cfrac{1}{\ddots + \cfrac{1}{s_k}}}} \\ &= \frac{Q_k(s_1, \ldots, s_k)}{Q_{k-1}(s_2, \ldots, s_k)}\,. \end{aligned}$$

The integers $Q_k(s_1, \ldots, s_k)$, $Q_{k-1}(s_2, \ldots, s_k)$ are coprime, and for $n_0' := n_0/n_k$, $n_1' := n_1/n_k$ we have

$$n_0' = Q_k(s_1, \ldots, s_k), \ n_1' = Q_{k-1}(s_2, \ldots, s_k).$$

2. Ideals generated by monomials.

2.1 Monomial ideals. Let $R := \kappa[x_1, \dots, x_n]$ be the polynomial ring over the field κ in n indeterminates $x_1, \dots, x_n$, and let $\mathfrak{m}$ be the ideal of R generated by $x_1, \dots, x_n$. The ring of formal power series $\widehat{R} = \kappa[[x_1, \dots, x_n]]$ is the completion of R in the $\mathfrak{m}$-adic topology; let $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$. For every $\delta = (l_1, \dots, l_n) \in \mathbb{N}_0^n$ let $x^\delta = x_1^{l_1} \cdots x_n^{l_n}$.

Let $\mathfrak{a} = (x^{\delta_1}, \dots, x^{\delta_h})$ with $\delta_1, \dots, \delta_h \in \mathbb{N}_0^n$ be a monomial ideal in R. We say that $\delta \in \mathbb{N}_0^n$ is an exponent of $\mathfrak{a}$ if $x^\delta \in \mathfrak{a}$. An element $f \in R$ lies in $\mathfrak{a}$ iff every term of f lies in $\mathfrak{a}$, and $\mathfrak{a}$ admits a unique minimal set of monomial generators (cf. [1], Ex. 1.4.11 and 1.4.13).

Let $\Delta = \Delta(\mathfrak{a})$ be the set of exponents of $\mathfrak{a}$; then $\mathfrak{a}$ is the linear span of the monomials x^δ with $\delta \in \Delta$. For any subset $A \subset \mathbb{R}^n$ we denote by $\mathrm{conv}(A) \subset \mathbb{R}^n$ the convex hull of A. Defining $\overline{\Delta} := \mathrm{conv}(\Delta) \cap \mathbb{N}_0^n$, we have the following: The integral closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is a monomial ideal which has $\{\delta \mid \delta \in \overline{\Delta}\}$ as set of exponents (cf. [8], Ex. 4.23). In particular, since $\mathrm{conv}(\Delta) = \mathrm{conv}(\{\delta_1, \dots, \delta_h\}) + \mathbb{R}_+^n$ (cf. [19], Lemma 4.3), we have $\overline{\Delta} = (\mathrm{conv}(\{\delta_1, \dots, \delta_h\}) + \mathbb{R}_+^n) \cap \mathbb{N}_0^n$.

The following result should be well-known. Its easy proof is left to the reader.

Lemma 2.2. *An $\mathfrak{m}$-primary ideal $\mathfrak{b}$ in R is integrally closed in R iff $\widehat{\mathfrak{b}} = \mathfrak{b}\widehat{R}$ is integrally closed in $\widehat{R}$. Moreover, if $\mathfrak{a}$ is any $\mathfrak{m}$-primary ideal in R, then we have $\overline{\mathfrak{a}\widehat{R}} = \widehat{\overline{\mathfrak{a}}}$.*

Proposition 2.3. *Let $\mathfrak{a}$ be an $\widehat{\mathfrak{m}}$-primary ideal in $\widehat{R}$ which is generated by monomials.*

(1) A power series f lies in $\mathfrak{a}$ iff every term of f lies in $\mathfrak{a}$.
(2) Let $m_1, \dots, m_h$ be a system of monomials in R which generates $\mathfrak{a}$. Then a monomial m lies in $\mathfrak{a}$ iff $m = m_i m'$ for some $i \in \{1, \dots, h\}$ and a monomial m'.
(3) The integral closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is generated by monomials. We have

$$\Delta(\overline{\mathfrak{a}}) = \mathrm{conv}(\Delta(\mathfrak{a})) \cap \mathbb{N}_0^n.$$

Proof. (1) Let $\mathfrak{a}_0$ be the ideal in R generated by the monomials in $\mathfrak{a}$; it is clear that $\mathfrak{a}_0 \subset \mathfrak{m}$, and that $\mathfrak{a} = \widehat{\mathfrak{a}_0} = \mathfrak{a}_0\widehat{R}$. Let $f \in \mathfrak{a}$ be a non-zero element, and let $(g_i)_{i \in \mathbb{N}_0}$ be a Cauchy sequence in $\mathfrak{a}_0$ converging to f. Let ord be the order function in $\widehat{R}$. We write $f = \sum_{j \geq m} f_j$ where f_j is homogeneous of degree j for $j \in \mathbb{N}_0$, $j \geq m$, and $f_m \neq 0$. We choose $i \in \mathbb{N}$ such that

$\operatorname{ord}(f - g_i) \geq m + 1$. Writing g as a sum of homogeneous terms $g_i = \sum_j g_{ij}$ where g_{ij} is homogeneous of degree j, we have $\operatorname{ord}(g_i) = m$ and therefore $g_{im} = f_m$. This implies that all terms of f_m lie in $\mathfrak{a}$. Replacing f by $f - f_m$ and repeating this argument, we get the assertion.

(2) We can write $m = f_1 m_1 + \cdots + f_h m_h$ with power series $f_1, \ldots, f_h$. We write each f_i as a sum of homogeneous terms $f_i = f_{i0} + f_{i1} + \cdots$ where f_{ij} is homogeneous of degree j. Set $d_i := \deg(m_i)$ for $i \in \{1, \ldots, h\}$ and $d := \deg(m)$. Then we find that $m = m_1 f_{1,d-d_1} + \cdots + m_h f_{h,d-d_h}$ (with $f_{i,d-d_i} = 0$ if $d < d_i$), hence the assertion.

(3) Since $\mathfrak{a}$ contains a power of $\widehat{\mathfrak{m}}$, and such a power is generated by monomials, $\mathfrak{a}_0$ contains a power of $\mathfrak{m}$. Since $\mathfrak{a}_0 \subset \mathfrak{m}$, we see that $\mathfrak{a}_0$ is $\mathfrak{m}$-primary. The conclusion follows from (2.2).

Proposition 2.4. *Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary monomial ideal of R. Then $\overline{\mathfrak{a}}\hat{R}$ is the integral closure of $\mathfrak{a}\hat{R}$, and we have $\overline{\mathfrak{a}}\hat{R} \cap R = \overline{\mathfrak{a}}$. In particular, a minimal system of monomial generators of $\overline{\mathfrak{a}}\hat{R}$ is a minimal system of monomial generators of $\overline{\mathfrak{a}}$.*

Proof. Since $\overline{\mathfrak{a}}$ is generated by monomials, we have $\overline{\mathfrak{a}} \subset \mathfrak{m}$; since $\overline{\mathfrak{a}}$ contains a power of $\mathfrak{m}$, we see that $\overline{\mathfrak{a}}$ is an $\mathfrak{m}$-primary ideal, also. Then $\overline{\mathfrak{a}}\hat{R}$ is an integrally closed ideal of $\hat{R}$ by (2.2), hence it is the integral closure of $\mathfrak{a}\hat{R}$, and it is generated by monomials. From this and (2.3) (2) we get the last assertions of the proposition.

The following result in the case of a polynomial ring over a field is well known (cf., e.g., [10], section 3.6, Prop. 15).

Lemma 2.5. *Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. Let $d := \dim(A)$, and let $\{x_1, \ldots, x_d\}$ be a regular system of parameters of A. Let $\mathfrak{a}$ be an ideal which is generated by monomials in $x_1, \ldots, x_d$, and which contains a power of $\mathfrak{m}$. Then $\ell_A(A/\mathfrak{a})$ is equal to the number of monomials in $x_1, \ldots, x_d$ which do not belong to $\mathfrak{a}$.*

Proof. In the sequel, a monomial is always a monomial in $x_1, \ldots, x_d$. We use repeatedly the following fact: If $m_1, \ldots, m_l$ are pairwise distinct monomials of order h, and if $a_1, \ldots, a_l$ are in A, then $\sum_{j=1}^{l} a_j m_j \in \mathfrak{m}^{h+1}$ implies that $a_1, \ldots, a_l \in \mathfrak{m}$.

(1) Let $a \in \mathfrak{a}$ be a non-zero element of order h. Then we can write $a = \sum_{j=1}^{l} a_j m_j$ with $a_1, \ldots, a_l \in A$ and monomials $m_1, \ldots, m_l \in \mathfrak{a}$ which satisfy $\operatorname{ord}_A(m_j) \geq h$ for $j \in \{1, \ldots, l\}$.

This is clear if $h = 0$ or $h = 1$. Assume that $h \geq 2$. Clearly we can write

$$a = \sum_{i \geq 1} \sum_{j=1}^{p_i} a_{ij} m_{ij},$$

with $a_{ij} \in A$, $m_{ij} \in \mathfrak{a}$ pairwise distinct monomials of order i. We have

$$\operatorname{ord}_A\left(\sum_{j=1}^{p_1} a_{1j}m_{1j}\right) = \operatorname{ord}_A\left(a - \sum_{i\geq 2}\sum_{j=1}^{p_i} a_{ij}m_{i,j}\right) \geq 2,$$

hence the elements $a_{11}, \dots, a_{1p_1}$ lie in $\mathfrak{m}$, and therefore we can write

$$a = \sum_{i\geq 2}\sum_{j=1}^{p_i'} a_{ij}'m_{ij}', \quad a_{ij}' \in A,\ m_{ij}' \in \mathfrak{a} \text{ monomials of order } i\ .$$

It is clear that, continuing in this way, we get a representation of a as stated.

(2) Let $M(\mathfrak{a})$ be the set of monomials which are not contained in $\mathfrak{a}$. This set is finite, since $\mathfrak{a}$ contains a power of $\mathfrak{m}$.

Let $m \in M(\mathfrak{a})$ be a monomial of largest degree, and define $\mathfrak{b} := \mathfrak{a} + Am$. We have $\mathfrak{m}\mathfrak{b} \subset \mathfrak{a}$ (since $m\mathfrak{m} \subset \mathfrak{a}$ by the choice of m), and therefore $\mathfrak{b}/\mathfrak{a}$ is a k-vector space generated by the image of m in $\mathfrak{b}/\mathfrak{a}$, hence $\mathfrak{b}/\mathfrak{a}$ is a simple A-module. We show that $m' \notin \mathfrak{b}$ for every $m' \in M(\mathfrak{a})$, $m' \neq m$.

In fact, assume that $m' \in \mathfrak{b}$; let h be the order of m'. Then we have $m' = a + bm$ with $a \in \mathfrak{a}$ and $b \in A$, and we have $\operatorname{ord}_A(m' - bm) \geq h$ by the choice of m. By (1) we have

$$m' - bm = \sum_{j=1}^{l} a_j m_j,$$

with $a_j \in A$, $m_j \in \mathfrak{a}$ pairwise distinct monomials of order $\geq h$. We may assume that $m_1, \dots, m_{l'}$ have order h, and that the other monomials have larger order. This implies that

$$m' - bm - \sum_{j=1}^{l'} a_j m_j \in \mathfrak{m}^{h+1},$$

whence $1 \in \mathfrak{m}$, a contradiction.

(3) We prove the lemma by induction on $\#(M(\mathfrak{a}))$. The assertion clearly holds if $M(\mathfrak{a}) = \emptyset$, since in this case $\mathfrak{a} = A$. Let $n \in \mathbb{N}$, and assume that the assertion holds for all ideals $\mathfrak{b}$ in A generated by monomials which contain a power of $\mathfrak{m}$, and which satisfy $\#(M(\mathfrak{b})) < n$. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal in A which is generated by monomials and with $\#(M(\mathfrak{a})) = n$, and let $m \in M(\mathfrak{a})$ be a monomial of largest degree; we define $\mathfrak{b} := \mathfrak{a} + Am$. On the one hand, we have $m \notin M(\mathfrak{b})$ and $M(\mathfrak{b}) \cup \{m\} = M(\mathfrak{a})$ by (2), hence

$\#(M(\mathfrak{b})) + 1 = \#(M(\mathfrak{a}))$. On the other hand, $\mathfrak{b}/\mathfrak{a}$ is a simple A-module by (2); this implies that $\ell_A(A/\mathfrak{b}) + 1 = \ell_A(A/\mathfrak{a})$. By induction, we have $\ell_A(A/\mathfrak{b}) = \#(M(\mathfrak{b}))$, hence we have $\ell_A(A/\mathfrak{a}) = \#(M(\mathfrak{a}))$.

3. A particular class of simple complete ideals with only one characteristic pair.

In this section we describe the simple complete ideals we want to deal with, and we will show how to construct minimal set of generators for them.

3.1. A particular class of ideals. Let m, n be coprime natural integers with $m > n$.

(1) Let $n_0 := m$, $n_1 := n$, and let

$$n_0 = s_1 n_1 + n_2,\ n_1 = s_2 n_2 + n_3, \dots, n_{k-1} = s_k n_k$$

with $n_1 > n_2 > \cdots > n_k = 1$ be the Euclidean algorithm for the integers m, n (note that $k \geq 1$ and that $n_{k-1} = s_k$).

(2) Let α be a point, and let $\{x, y\}$ be a regular system of parameters for α. We define

$$z_0 := y,\ z_1 := x,\ z_{i+1} := \frac{z_{i-1}}{z_i^{s_i}} \quad \text{for } i \in \{1, \dots, k-1\}.$$

Furthermore, let $x_0 := x$, $y_0 := y$ and put $s_0 := 0$. We have $s_1 \geq 1$. We define for $i \in \{1, \dots, k-1\}$, $j \in \{1, \dots, s_i\}$ and $i = k$, $j = \{1, \dots, s_k - 1\}$

$$x_{s_1+\cdots+s_{i-1}+j} := z_i, \quad y_{s_1+\cdots+s_{i-1}+j} := \frac{z_{i-1}}{z_i^j}.$$

Let $t := s_1 + \cdots + s_k$. Now, we consider the sequence of quadratic transforms

$$\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_{s_1} \subset \alpha_{s_1+1} \subset \cdots \subset \alpha_{s_1+s_2} \subset \cdots \subset \alpha_{t-1};$$

it is easy to check that, for $i \in \{1, \dots, k-1\}$, $j \in \{1, \dots, s_i\}$, and for $i = k$, $j \in \{1, \dots, s_k - 1\}$, $\{x_i, y_i\}$ is a regular system of parameters in α_i.

(2) Let $\mathfrak{a} := (x^m, y^n)$. For $i \in \{1, \dots, k-1\}$ let $j \in \{1, \dots, s_i\}$, and for $i = k$ let $j \in \{1, \dots, s_k - 1\}$, and define $l := s_1 + \cdots + s_{i-1} + j$. The transform of $\mathfrak{a}$ in α_l is the ideal generated by $x_l^{n_i - jn_{i+1}}$ and $y_l^{n_{i+1}}$. Now let $i \in \{1, \dots, k\}$ and $j \in \{0, \dots, s_i - 1\}$, and set $l := s_1 + \cdots + s_{i-1} + j$. Then we have $\mathrm{ord}_{\alpha_l}(\mathfrak{a}^{\alpha_l}) = n_{i+1}$. Moreover, we have $\mathfrak{c}(\mathfrak{a}^{\alpha_l}) = (\overline{y}_l)$ in the associated graded ring of α_l.

Remark 3.2. In [14] it is shown that the integral closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is a monomial ideal. Our main concern in the sequel of this section is to construct *explicitly* a set of monomial generators of $\overline{\mathfrak{a}}$ (cf. prop. (3.6)). We begin with a result which allows us to apply (2.5).

Proposition 3.3. *With notation as in (3.1), we have*

$$(*) \qquad \ell_\alpha(\alpha/\overline{\mathfrak{a}}) = \sum_{i=1}^{k} s_i \frac{n_i(n_i+1)}{2} = \frac{mn+m+n-1}{2},$$

and for every $l \in \{1, \dots, t-1\}$ the inverse transform of the integral closure of $\mathfrak{a}^{\alpha_l}$ is the integral closure of $\mathfrak{a}^{\alpha_{l-1}}$. In particular, $\overline{\mathfrak{a}}$ is a simple complete $\mathfrak{m}_\alpha$-primary ideal and its transform in α_{t-1} is the maximal ideal of α_{t-1}.

Proof. (by induction on k): For $k = 1$ we have $n = 1$, $m = s_1 n$, and therefore (cf. (1.2), (1) and (3)) $\ell_\alpha(\alpha/\overline{\mathfrak{a}}) = s_1$. On the other hand, we have $\mathfrak{a} = (x^m, y)$, and therefore we have $\ell_\alpha(\alpha/\mathfrak{a}) = s_1$. Therefore we have $\mathfrak{a} = \overline{\mathfrak{a}}$, hence $\mathfrak{a}$ is simple (since $\mathrm{ord}_\alpha(\mathfrak{a}) = 1$) and complete.

Now we consider the case $k \geq 2$. It is clear that $\alpha_0, \dots, \alpha_{t-1}$ are the only points β with $\mathfrak{a}^\beta \neq \beta$, and since $\mathfrak{a}^{\alpha_{s_1+\dots+s_{k-1}}}$ is simple and complete [by the case $k = 1$], we get by recursion, and by (1.2) and (3.1), that for every $l \in \{1, \dots, t-1\}$ the inverse transform of the integral closure of $\mathfrak{a}^{\alpha_l}$ is the integral closure of $\mathfrak{a}^{\alpha_{l-1}}$. In particular, we get that $\mathfrak{a}$ is simple, and that the transform of $\overline{\mathfrak{a}}$ in α_{t-1} is the maximal ideal of α_{t-1}. The formula in $(*)$ now follows immediately from the last results in (3.1) and from (1.2) (1).

In the following, we keep the notation introduced in (3.1), and we construct a minimal system of generators of $\overline{\mathfrak{a}}$. We begin our construction for a set of minimal generators of $\overline{\mathfrak{a}}$ by considering the easy cases $k = 1$ and $k = 2$.

3.4. The case $k = 1$. In this case we already know that $\mathfrak{a}$ is simple and complete (cf. the first part of the proof of (3.3)); clearly $\{x^m, y\}$ is a minimal system of generators of $\mathfrak{a}$.

3.5. The case $k = 2$. We have $n_0 = s_1 n_1 + 1$, $n_1 = s_2 n_2$ and $s_2 = n_1$ and $n_2 = 1$. We show:

$$(*) \qquad \{x^m, y^n, x^{s_1(n_1-j)+1} y^j \quad \text{for } j \in \{1, \dots, n_1 - 1\}\}$$

is a minimal system of generators of $\overline{\mathfrak{a}}$.

Proof. Let $j \in \{1, \dots, n_1 - 1\}$; since $(x^{s_1(n_1-j)+1}y^j)^{n_1} = x^j x^{(s_1 n_1+1)(n_1-j)} y^{n_1 j}$, the element $x^{s_1(n_1-j)+1}y^j$ is integral over $\mathfrak{a}$. Let $\mathfrak{a}_1$ be the ideal generated by the elements in $(*)$. Then we have $\mathfrak{a}_1 \subset \overline{\mathfrak{a}}$. None of the monomials $x^c y^d$ where c, $d \in \{0, \dots, n_1 - 1\}$ and $c + d \leq n_1 - 1$, lies in $\mathfrak{a}_1$. We count the other monomials $x^a y^b$ which do not lie in $\mathfrak{a}_1$. It is easy to check that for $d \in \{0, \dots, s_1 - 1\}$ there are exactly n monomials of degree $n_1 + d$ which do not lie in $\mathfrak{a}_1$, while for $c \in \{1, \dots, n_1 - 1\}$ and $d \in \{0, \dots, s_1 - 2\}$ there are exactly $n_1 - c$ monomials of degree $n_1 + cs_1 - (c-1) + d$ which do not lie in $\mathfrak{a}_1$, and all monomials of degree larger than $n_0 - 1$ lie in $\mathfrak{a}_1$. Therefore we have

$$\begin{aligned}\ell_\alpha(\alpha/\mathfrak{a}_1) &= \frac{n_1(n_1+1)}{2} + s_1 n_1 + (s_1 - 1)\sum_{j=1}^{n_1-1}(n_1 - j) \\ &= s_1 \frac{n_1(n_1+1)}{2} + s_2.\end{aligned}$$

This means (cf. (1.2) and (3.3)) that $\mathfrak{a}_1 = \overline{\mathfrak{a}}$, and that the set of $n+1$ elements in $(*)$ is a set of generators of $\overline{\mathfrak{a}}$ which is minimal by (1.2) (1) .

This system of generators can also be written as

$$\{x^{m-\sigma_{m,n}(j)}y^j \mid j \in \{0, \dots, n\}\}$$

where $\sigma_{mn,} : \{0, \dots, n\} \to \{0, \dots, m\}$ is the strictly increasing function

$$\sigma_{m,n}(j) = \begin{cases} s_1 j & \text{for every } j \in \{0, \dots, n-1\}, \\ m & \text{for } j = n; \end{cases}$$

in particular, we have

$$\sigma_{m,n}(0) = 0,\ \sigma_{m,n}(1) = s_1,\ \sigma_{m,n}(n-1) = m - (1 + s_1),\ \sigma_{m,n}(n) = m.$$

Now we give our first main result.

Proposition 3.6. *We assume that $k \geq 2$. There exists a strictly increasing function*

$$\sigma_{m,n} : \{0, \dots, n\} \to \{0, \dots, m\}$$

with $m - \sigma_{m,n}(j) \geq n - j$ for $j \in \{0, \dots, n\}$ and

$$\sigma_{m,n}(0) = 0,\ \sigma_{m,n}(1) = s_1,\ \sigma_{m,n}(n-1) = m - (1 + s_1),\ \sigma_{m,n}(n) = m$$

such that the integral closure of $\mathfrak{a} = (x^m, y^n)$ has the set

$$(*) \qquad \{x^{m-\sigma_{m,n}(j)}y^j \mid j \in \{0, \dots, n\}\}$$

consisting of $n+1$ *monomials as a minimal system of generators. Moreover, we have*

$$\sigma_{m,n}(n)n - \sum_{j=1}^{n-1} \sigma_{m,n}(j) = \sum_{l=1}^{k} s_l \frac{n_l(n_l+1)}{2}.$$

For $k \geq 3$ *we have the following recursion formula: For every* $j \in \{0, \dots, n_1 - 1\}$ *there exists a uniquely determined* $\lambda(j) \in \{1, \dots, n_2\}$ *such that* $n_1 - \sigma_{n_1,n_2}(n_2 - \lambda(j) + 1) \leq j \leq n_1 - \sigma_{n_1,n_2}(n_2 - \lambda(j)) - 1$*, and we have*

$$\sigma_{m,n}(j) = \begin{cases} s_1 j + \lambda(j) - 1 & \text{if } j \in \{n_1 - \sigma_{n_1,n_2}(n_2 - \lambda(j) + 1), \dots, \\ & \qquad n_1 - \sigma_{n_1,n_2}(n_2 - \lambda(j)) - 1\}, \\ s_1 n_1 + n_2 & \text{if } j = n_1. \end{cases}$$

Proof. (1) Let $m > n$ be positive integers, let $\sigma : \{0, \dots, n\} \to \{0, \dots, m\}$ be a strictly increasing function with $\sigma(0) = 0$ and $\sigma(n) = m$, and with $m - \sigma(j) \geq n - j$ for $j \in \{0, \dots, n\}$, and let $\mathfrak{a}_1$ be the ideal generated by the set $\{x^{m-\sigma(j)} y^j \mid j \in \{0, \dots, n\}\}$; this set is a minimal system of generators of $\mathfrak{a}_1$. We have $\operatorname{ord}_\alpha(\mathfrak{a}_1) = n$ since $m - \sigma(j) + j \geq n$ for $j \in \{0, \dots, n\}$. We determine $\ell_\alpha(\alpha/\mathfrak{a}_1)$ by counting the monomials $x^a y^b$ which do not lie in $\mathfrak{a}_1$. Let a, b be nonnegative integers with $a + b \geq m$. If $b \geq n$, then $x^a y^b$ lies in $\mathfrak{a}_1$, and if $b < n$, then we have $a \geq m - b > m - \sigma(b)$, and again $x^a y^b$ lies in $\mathfrak{a}_1$. The monomials

$$\{x^i y^j \mid j \in \{0, \dots, n-1\}, i \in \{0, \dots, m - \sigma(j) - 1\}\}$$

are all the monomials which do not lie in $\mathfrak{a}_1$, and therefore we get (cf. (2.5))

$$\ell_\alpha(\alpha/\mathfrak{a}_1) = \sum_{j=0}^{n-1} (m - \sigma(j)) = mn - \sum_{j=1}^{n-1} \sigma(j) = \sigma(n)n - \sum_{j=1}^{n-1} \sigma(j).$$

(2) Now we prove the proposition by induction on k and s_1. If $k = 2$, then the result follows from (3.5) and the fact that $m - s_1 j = s_1(n - j) \geq n - j$ for $j \in \{0, \dots, n\}$. We assume that $k \geq 3$, and that the statements of the proposition hold for $k - 1$. Let $\beta := \alpha_{s_1 - 1}$, $\gamma := \alpha_{s_1}$, $\mathfrak{b} := \mathfrak{a}^\beta$, $\mathfrak{c} := \mathfrak{a}^\gamma$. Let $u := x$, $v := y/x^{s_1 - 1}$; then $\{u, v\}$ is a regular system of parameters of β, and $\{u, v/u\}$ is a regular system of parameters of γ. We have

$$\mathfrak{b} = (u^{n_1+n_2}, v^{n_1}), \; \mathfrak{c} = (u^{n_2}, (v/u)^{n_1}).$$

Let $\sigma := \sigma_{n_1,n_2}$. By induction, the integral closure $\bar{\mathfrak{c}}$ of $\mathfrak{c}$ has

$$\{(v/u)^{n_1 - \sigma(i)} u^i \mid i \in \{0, \dots, n_2\}\}$$

as a minimal system of generators, and $\bar{\mathfrak{c}}$ is simple; the inverse transform of $\bar{\mathfrak{c}}$ is the ideal $u^{n_1}\bar{\mathfrak{c}} \cap \beta = \overline{\mathfrak{b}}$. We consider the elements

$$(v/u)^{n_1-\sigma(i)+j}u^i, \quad i \in \{1,\dots,n_2\},\ j \in \{0,\dots,\sigma(i)-\sigma(i-1)-1\}.$$

They lie in $\bar{\mathfrak{c}}$, and therefore the elements

$$(*) \quad u^{i+\sigma(i)-j}v^{n_1-\sigma(i)+j}, \quad i \in \{1,\dots,n_2\},\ j \in \{0,\dots,\sigma(i)-\sigma(i-1)-1\},$$

lie in $\overline{\mathfrak{b}}$. For every $j \in \{0,\dots,n_1-1\}$ there exists a unique $i \in \{1,\dots,n_2\}$ such that $j \in \{n_1-\sigma(n_2-i+1),\dots,n_1-\sigma(n_2-i)-1\}$. We include the element v^{n_1} in the set $(*)$, and write the elements of this set as

$$(**) \qquad u^{n_1+n_2-\tau_1(j)}v^j \quad \text{for } j \in \{0,\dots,n_1\}$$

where we have defined

$$\tau_1(j) := \begin{cases} j+i-1 & \text{if } i \in \{1,\dots,n_2\},\ j \in \{n_1-\sigma(n_2-i+1),\dots, \\ & \quad n_1-\sigma(n_2-i)-1\}, \\ n_1+n_2 & \text{if } j = n_1. \end{cases}$$

It is easy to check that $\tau_1 : \{0,\dots,n_1\} \to \{0,\dots,n_1+n_2\}$ is strictly increasing, that $n_1+n_2-\tau_1(j) \geq n_1-j$ for $j \in \{0,\dots,n_1\}$, and that

$$\tau_1(0)=0,\ \tau_1(1)=1,\ \tau_1(n_1-1)=n_1+n_2-2.$$

Let $\mathfrak{b}_1$ be the ideal generated by the elements in $(**)$; it is clear that $(**)$ is a minimal set of generators of $\mathfrak{b}_1$, and that $\mathfrak{b}_1 \subset \overline{\mathfrak{b}}$.

We calculate $\ell_\beta(\beta/\mathfrak{b}_1)$ by using (1): We have

$$\ell_\beta(\beta/\mathfrak{b}_1) = \sum_{j=0}^{n_1-1}(n_1+n_2-\tau_1(j)).$$

We get

$$\begin{aligned}\sum_{j=0}^{n_1-1}\tau_1(j) &= \sum_{i=1}^{n_2}\ \sum_{j=n_1-\sigma(n_2-i+1)}^{n_1-\sigma(n_2-i)-1}(i-j+1) \\ &= \sum_{i=1}^{n_2}(i-1)(\sigma(n_2-i+1)-\sigma(n_2-i))\end{aligned}$$

$$+\frac{1}{2}(n_1-\sigma(n_2-i))(n_1-\sigma(n_2-i)-1)$$
$$-\frac{1}{2}(n_1-\sigma(n_2-i+1))(n_1-\sigma(n_2-i+1)-1)$$
$$=\sum_{j=1}^{n_2-1}\sigma(j)+\frac{1}{2}n_1(n_1-1).$$

Therefore we have

$$\ell_\beta(\beta/\mathfrak{b}_1) = n_1(n_1+n_2)-\frac{n_1(n_1-1)}{2}-\sum_{j=0}^{n_2-1}\sigma(j)$$
$$= \frac{n_1(n_1+1)}{2}+\sigma(n_2)n_2-\sum_{j=1}^{n_2-1}\sigma(j).$$

Since, by induction and (1),

$$\sigma(n_2)n_2-\sum_{j=1}^{n_2-1}\sigma(j)=\sum_{l=2}^{k}s_l\frac{n_l(n_l+1)}{2},$$

we see that

$$\ell_\beta(\beta/\mathfrak{b}_1)=\frac{n_1(n_1+1)}{2}+\sum_{l=2}^{k}s_l\frac{n_l(n_l+1)}{2}.$$

From (3.3) we obtain $\ell_\beta(\beta/\mathfrak{b}_1)=\ell_\beta(\beta/\overline{\mathfrak{b}})$, and therefore we have $\mathfrak{b}_1=\overline{\mathfrak{b}}$. Note that we have also shown (using (1)) that

$$(*)\qquad \tau_1(n)n-\sum_{j=1}^{n-1}\tau_1(j)=\frac{n_1(n_1+1)}{2}+\sum_{l=2}^{k}s_l\frac{n_l(n_l+1)}{2}.$$

Thus, we have proved the proposition for k and $s_1=1$ (cf. (1)).

For every integer $s\geq 1$ let $\tau_s:\{0,\dots,n\}\to\{0,\dots,sn+n_2\}$ be the function defined by

$$\tau_s(j):=\tau_1(j)+(s-1)j \quad \text{for } j\in\{0,\dots,n\};$$

note that τ_s is strictly increasing, that $sn+n_2-\tau_s(j)\geq n-j$ for $j\in\{0,\dots,n\}$, and that

$$\tau_s(0)=0,\ \tau_s(1)=s,\ \tau_s(n-1)=sn+n_2-(1+s),\ \tau_s(n)=sn+n_2.$$

Let $s \geq 1$ be an integer, set $\widetilde{m} = sn + n_2$, and assume that the integral closure of the ideal $(x^{\widetilde{m}}, y^n)$ is generated by the set $\{x^{\widetilde{m}-\tau_s(j)}y^j \mid j \in \{0, \ldots, n\}\}$. We consider the ideal $\mathfrak{a}' := (x^{\widetilde{m}'}, y^n)$ where $\widetilde{m}' := \widetilde{m} + n = (s+1)n + n_2$ and its quadratic transform $(x^{\widetilde{m}}, y^n)$. Since we know by induction a system of generators of the integral closure of $(x^{\widetilde{m}}, y^n)$, we see that the integral closure of $\mathfrak{a}'$ contains the elements $x^{\widetilde{m}'-\tau_{s+1}(j)}y^j$ for $j \in \{0, \ldots, n\}$; let $\mathfrak{a}_1$ be the ideal generated by these elements. By (1) we have

$$\begin{aligned}
\ell_\alpha(\alpha/\mathfrak{a}_1) &= \tau_{s+1}(n)n - \sum_{j=1}^{n-1} \tau_{s+1}(j) \\
&= s\frac{n(n+1)}{2} + \tau_1(n)n - \sum_{j=1}^{n-1} \tau_1(j) \\
&\overset{\text{cf. }(*)}{=} s\frac{n(n+1)}{2} + \frac{n(n+1)}{2} + \sum_{l=2}^{k} s_l \frac{n_l(n_l+1)}{2} \\
&= (s+1)\frac{n(n+1)}{2} + \sum_{l=2}^{k} s_l \frac{n_l(n_l+1)}{2},
\end{aligned}$$

hence we have (cf. (3.3)) $\ell_\alpha(\alpha/\mathfrak{a}_1) = \ell_\alpha(\alpha/\overline{\mathfrak{a}'})$, and therefore we obtain $\mathfrak{a}_1 = \overline{\mathfrak{a}'}$. Thus, we have shown that the integral closure of $\mathfrak{a}' = (x^{\widetilde{m}'}, y^n)$ is generated by the set of monomials $\{x^{\widetilde{m}'-\tau_{s+1}(j)}y^j \mid j \in \{0, \ldots, n\}\}$.

Now we define $\sigma_{m,n}$ as in the proposition; $\sigma_{m,n} : \{0, \ldots, n\} \to \{0, \ldots, m\}$ is a function satisfying the assertions of the proposition.

This ends the proof of the proposition.

Remark 3.7. It is useful to define a σ-function also in the following case. Let $n_0 := s_1 n_1$ with $s_1 \geq 2$ and $n_1 := 1$. We define $\sigma_{n_0,n_1} : \{0, n_1\} \to \{0, n_0\}$ by

$$\sigma_{n_0,n_1}(j) := \begin{cases} 0 & \text{if } j = 0, \\ n_0 & \text{if } j = n_1. \end{cases}$$

Then the complete ideal $\mathfrak{p}_{s_1-1}$ is generated by the two elements

$$x^{n_0-\sigma_{n_0,n_1}(0)}y^{\sigma_{n_0,n_1}(0)} = x^{n_0}, \; x^{n_0-\sigma_{n_0,n_1}(n_1)}y^{\sigma_{n_0,n_1}(n_1)} = y.$$

Starting with this particular σ-function, we can use the recursive construction given in (3.6) to get all σ-functions.

Example 3.8. For $k = 3$ we have

$$\sigma_{m,n}(j) = \begin{cases} 0 & \text{for } j = 0, \\ \lceil j/s_2 \rceil + s_1 j - 1 & \text{for } j \in \{1, \dots, n-1\}, \\ m & \text{for } j = n, \end{cases}$$

and for $k = 4$ we have

$$\sigma_{m,n}(j) = \begin{cases} s_1 j + (p[j]s_3 + q[j]) - 1 & \text{for } j \in \{0, \dots, n-1\}, \\ m & \text{for } j = n \end{cases}$$

where for $j \in \{0, \dots, n-1\}$ we have defined

$$p[j] := \left\lfloor \frac{j}{1 + s_2 s_3} \right\rfloor$$

and

$$q[j] := \begin{cases} 1 & \text{if } j = (1 + s_2 s_3)p[j], \\ \left\lceil \frac{j-(1+s_2s_3)p[j]}{s_2} \right\rceil & \text{otherwise.} \end{cases}$$

Remark 3.9. Let κ be a field, and let $\mathfrak{a}$ be the ideal in the polynomial ring $\kappa[x, y]$ generated by x^m and y^n. Then the integral closure of $\mathfrak{a}$ has $\{x^{m-\sigma_{m,n}(j)}, y^j \mid j \in \{0, \dots, n\}\}$ as a minimal set of monomial generators. Moreover, $\Delta(\overline{\mathfrak{a}})$ consists of all points $(m - \sigma_{m,n}(j) + r, j + s)$ where $j \in \{0, \dots, n\}$ and $r, s \in \mathbb{N}_0$. This follows immediately from (2.4).

Now we want to study in some more detail the function σ. This will provide a better understanding of this function and will be a useful tool for the following section.

3.10. Further results on the function σ. Let $k \in \mathbb{N}$, and let $s_1, \dots, s_k, s_{k+1}$ be natural integers with $s_{k+1} \geq 2$. We define

$$n_i := Q_{k+1-i}(s_1, \dots, s_{k+1}) \quad \text{for } i \in \{0, \dots, k+1\}, \quad \sigma_k := \sigma_{n_0, n_1}.$$

The Euclidean algorithm for n_0, n_1 gives

$$n_{i-1} = s_i n_i + n_{i+1} \quad \text{for } i \in \{1, \dots, k\}, n_k = s_{k+1} n_{k+1} \text{ with } n_{k+1} = 1.$$

(1) Assume that $s_k \geq 2$. We define

$$n'_i := Q_{k-i}(s_{i+1}, \dots, s_k) \quad \text{for } i \in \{0, \dots, k\}, \quad \sigma'_k := \sigma_{n'_0, n'_1}.$$

The Euclidean algorithm for n'_0, n'_1 gives

$$n'_{i-1} = s_i n'_i + n'_{i+1} \quad \text{for } i \in \{1, \dots, k-1\},\ n'_{k-1} = s_k n'_k \text{ with } n'_k = 1.$$

Note that $n_0 > n'_0$, $n_1 > n'_1$. We show: *For k odd we have*

$$\sigma_k(j) = \sigma'_k(j) \quad \text{for } j \in \{0, \dots, n'_1\},$$

$$n_0 - \sigma_k(n_1 - j) = \begin{cases} n'_0 - \sigma'_k(n'_1 - j) & \text{for } j \in \{0, \dots, n'_1 - 1\}, \\ n'_0 + 1 & \text{for } j = n'_1, \end{cases}$$

and for k even we have

$$\sigma_k(j) = \begin{cases} \sigma'_k(j) & \text{for } j \in \{0, \dots, n'_1 - 1\}, \\ n'_0 - 1 & \text{for } j = n'_1, \end{cases}$$

$$n_0 - \sigma_k(n_1 - j) = n'_0 - \sigma'_k(n'_1 - j) \quad \text{for } j \in \{0, \dots, n'_1\}.$$

Proof. The case $k = 1$ is trivial (cf. (3.6)). Let $k \geq 2$ and assume that the result holds for the integers $1, 2, \dots, k-1$. We define

$$\sigma_* := \sigma_{n_1, n_2}, \quad \sigma'_* := \sigma_{n'_1, n'_2}.$$

First, let k be even. Let $j \in \{0, \dots, n'_1 - 1\}$; we choose $i \in \{1, \dots, n'_2\}$ with

$$n'_1 - \sigma'_*(n'_2 - i + 1) \leq j \leq n'_1 - \sigma'_*(n'_2 - i) - 1.$$

By induction, since $k - 1$ is odd, we have

$$n_1 - \sigma_*(n_2 - l) = n'_1 - \sigma'_*(n'_2 - l) \quad \text{for } l \in \{0, \dots, n'_2 - 1\},$$

$$n_1 - \sigma_*(n_2 - n'_2) = n'_1 + 1.$$

Therefore we get

$$n_1 - \sigma_*(n_2 - i + 1) \leq j \leq n_1 - \sigma_*(n_2 - i) - 1 \quad \text{if } i \in \{1, \dots, n'_2 - 1\},$$

and if $i = n'_2$, then we have for all j with $n'_1 - \sigma'_*(1) \leq j \leq n'_1 - 1$ the estimate

$$n_1 - \sigma_*(n_2 - n'_2 + 1) \leq j \leq n_1 - \sigma_*(n_2 - n'_2) - 2,$$

hence, by the recursion formula of (3.6),

$$\sigma_*(j) = s_1 j + i - 1 = \sigma'_*(j) \quad \text{for } j \in \{0, \dots, n'_1 - 1\}.$$

For $j = n_1'$ we obtain, since $n_1' = n_1 - \sigma_*(n_2 - n_2') - 1$,

$$\sigma_k(n_1') = s_1 n_1' + n_2' - 1 = n_0' - 1.$$

Now we calculate $\sigma_k(n_1 - j)$ and $\sigma_k'(n_1' - j)$ for $j \in \{0, \dots, n_1'\}$. First, let $j \in \{1, \dots, n_1'\}$; then we have $0 \leq n_1' - j \leq n_1' - 1$. We choose $i' \in \{1, \dots, n_2'\}$ with

$$n_1' - \sigma_*'(n_2' - i' + 1) \leq n_1' - j \leq n_1' - \sigma_*'(n_2' - i') - 1;$$

then we have $\sigma_*'(n_2' - i') + 1 \leq j \leq \sigma_*'(n_2' - i' + 1)$. Now we choose $i \in \{1, \dots, n_2\}$ with

$$n_1 - \sigma_*(n_2 - i + 1) \leq n_1 - j \leq n_1 - \sigma_*(n_2 - i) - 1;$$

then we have $\sigma_*(n_2 - i) + 1 \leq j \leq \sigma_*(n_2 - i + 1)$. Since σ_* is strictly increasing and coincides with σ_*' in the range $\{0, \dots, n_2'\}$ by induction, we obtain $n_2 - i = n_2' - i'$. Now we get

$$\begin{aligned} n_0 - \sigma_k(n_1 - j) &= s_1 n_1 + n_2 - (s_1(n_1 - j) + i - 1) = (n_2 - i) + s_1 j + 1 \\ &= (n_2' - i') + s_1 j + 1 = s_1 n_1' + n_2' - (s_1(n_1' - j) + i' - 1) \\ &= n_0' - \sigma_k'(n_1' - j). \end{aligned}$$

Since $0 = n_0 - \sigma_k(n_1) = n_0' - \sigma_*'(n_1')$, we have settled the case of even k.

Now we consider the case that k is odd. Just as above we get by induction

$$\sigma_k(j) = \sigma_k'(j) \quad \text{for } j \in \{0, \dots, n_1' - 1\};$$

for $j = n_1'$ we have $n_1' = n_1 - \sigma_*(n_2 - n_2')$, hence

$$\sigma_k(n_1') = s_1 n_1' + n_2' = n_0'.$$

Furthermore, again by induction, we find as above

$$n_0 - \sigma_k(n_1 - j) = n_0' - \sigma_k'(n_1' - j) \quad \text{for } j \in \{0, \dots, n_1' - 1\},$$

and we have $\sigma_*(n_2') = n_1' - 1$, hence $n_1 - n_1' = n_1 - \sigma_*(n_2') - 1$, and therefore

$$\sigma_k(n_1 - n_1') = s_1(n_1 - n_1') + n_2 - n_2' - 1,$$

hence

$$n_0 - \sigma_k(n_1 - n_1') = s_1 n_1 + n_2 - (s_1(n_1 - n_1') + n_2 - n_2' - 1) = n_0' + 1.$$

(2) Assume that $k \geq 2$ and $s_k = 1$, and define

$$n'_i = Q_{k-i-1}(s_{i+1}, \ldots, s_{k-1} + 1) \quad \text{for } i \in \{0, \ldots, k-1\}, \quad \sigma'_k := \sigma_{n'_0, n'_1}.$$

Just as in (1) we can show that *we have the same relations between σ_k and σ'_k as in* (1).

(3) Assume that $s_{k+1} = 2$, and define

$$n''_i = Q_{k-i}(s_{i+1}, \ldots, s_k + 1) \quad \text{for } i \in \{0, \ldots, k\}, \quad \sigma''_k := \sigma_{n''_0, n''_1}.$$

Using similar arguments as above we can show: *For k odd we have*

$$\sigma_k(j) = \begin{cases} \sigma''_k(j) & \text{for } j \in \{0, \ldots, n''_1 - 1\}, \\ n''_0 - 1 & \text{for } j = n''_1, \end{cases}$$

$$n_0 - \sigma_k(n_1 - j) = n''_0 - \sigma''_k(n''_1 - j) \quad \text{for } j \in \{0, \ldots, n''_1\},$$

and for k even we have

$$\sigma_k(j) = \sigma''_k(j) \quad \text{for } j \in \{0, \ldots, n''_1\},$$

$$n_0 - \sigma_k(n_1 - j) = \begin{cases} n''_0 - \sigma''_k(n''_1 - j) & \text{for } j \in \{0, \ldots, n''_1 - 1\}, \\ n''_0 + 1 & \text{for } j = n''_1. \end{cases}$$

(4) Assume that $s_{k+1} \geq 3$, and define

$$n''_i := Q_{k+1-i}(s_{i+1}, \ldots, s_{k+1} - 1) \quad \text{for } i \in \{0, \ldots, k+1\}, \quad \sigma''_k := \sigma_{n''_0, n''_1}.$$

Then we can show that we have *the same relations between σ_k and σ''_k as in* (3).

3.11. Subadditivity. Let $k \in \mathbb{N}$, $k \geq 2$, and let $s_1, \ldots, s_k$ be natural integers with $s_k \geq 2$; let

$$n_i := Q_{k-i}(s_{i+1}, \ldots, s_k) \quad \text{for } i \in \{0, \ldots, k\}, \quad \sigma_k := \sigma_{n_0, n_1}.$$

Then we have

$$\sigma_k(j) + \sigma_k(l) \leq \sigma_k(j + l) \quad \textit{for } j, l \in \{0, \ldots, n_1\} \text{ with } j + l \leq n_1.$$

Proof. Let $\mathfrak{p}$ be the integral closure of the ideal in α generated by x^{n_0}, y^{n_1}. The properties of the function σ_k are independent of the ring α we are working with; therefore we may assume that $\alpha = \kappa[[x, y]]$, a ring of formal power series over a field κ. The convex hull in $\mathbb{R}^2$ of the set $\{(n_0, 0), (0, n_1)\}$ is the line through these points; therefore $\Delta(\mathfrak{p})$ is the set of all points $(s, t) \in \mathbb{N}_0^2$ with $n_1 s + n_0 t - n_0 n_1 \geq 0$ (cf. (2.3) (2)). Let $j, l \in \{0, \ldots, n_1\}$ with $j + l \leq n_1$. We have $n_0 j \geq n_1 \sigma_k(j)$, $n_0 l \geq n_1 \sigma_k(l)$, hence $n_0(j + l) \geq n_1(\sigma_k(j) + \sigma_k(l))$, which implies that $x^{n_0 - (\sigma_k(j) + \sigma_k(l))} y^{j+l} \in \mathfrak{p}$, and therefore we have $\sigma_k(j + l) \geq \sigma_k(j) + \sigma_k(l)$ (cf. (3.6)).

In the following proposition we characterize when $\mathfrak{p}$ is the integral closure of (x^m, y^n).

Proposition 3.12. *Let $\mathfrak{p}$ be a simple complete $\mathfrak{m}_\alpha$-primary ideal in α which is residually rational. We assume that the semigroup $\Gamma_\mathfrak{p}$ of $\mathfrak{p}$ is generated by two coprime natural integers m, n with $m > n$. Let $\nu := \nu_\mathfrak{p}$ be the valuation of K defined by $\mathfrak{p}$. Then we have $\nu(\mathfrak{p}) = mn + p$ with $p \in \mathbb{N}_0$. There exists a regular system of parameters in α such that $\nu(x) = n$, $\nu(y) = m$ and that $\mathfrak{p}$ is the integral closure of the ideal (x^m, y^n) iff $p = 0$.*

Proof. In the sequel, we are using the Hamburger-Noether algorithm, cf. [11], sections 7.5 and 7.6 for details. Let $\alpha =: \alpha_0 \subset \cdots \subset \alpha_h$ be the quadratic sequence defined by $\mathfrak{p}$. We choose a regular system of parameters $\{x, y\}$ of α with $\nu(y) > \nu(x)$. Let $p_1 := \nu(y)$, $c_1 := \nu(x)$; with $\eta_0 := y =: y_0$, $\eta_1 := x =: x_0$ let

$$\nu(\eta_{i-1}) = s_i \nu(\eta_i) + \nu(\eta_{i+1}) \quad \text{for } i \in \{1, \ldots, k\}$$

be the Euclidean algorithm for p_1 and c_1. Let $m_1 := s_1 + \cdots + s_k$. Then $x_{m_1-1} := \eta_k$, $y_{m_1-1} := \eta_{k-1}/\eta_k^{s_k-1}$ is a regular system of parameters in α_{m_1-1}. If $m_1 - 1 = h$—equivalently, if $\nu(x_{m_1-1}) = 1$ and the image of y_{m_1-1}/x_{m_1-1} in the residue field κ_ν of ν is transcendental over κ_α—then we have $\nu(x_{m_1-1}) = \nu(y_{m_1-1}) = 1$ and $p_1 = m$, $c_1 = n$ by [11], Th. (9.18), since the semigroup $\Gamma_\mathfrak{p}$ is generated by m and n, hence $\nu(\mathfrak{p}) = mn$ (cf. [11], Cor. (7.12)). In the other case there exists a uniquely determined unit $a \in \alpha_{m_1-1}$ such that setting

$$x_{m_1} := \eta_k, \; y_{m_1} := \frac{\eta_{k-1} - a\eta_k^{s_k}}{\eta_k^{s_k}},$$

$\{x_{m_1}, y_{m_1}\}$ is a regular system of parameters in α_{m_1}, and we have $p_2 := \nu(y_{m_1}) \geq 1$, $c_2 := \nu(x_{m_1}) = 1$ since $\Gamma_\mathfrak{p}$ is generated by m and n. We have $l = 2$ in the Hamburger-Noether tableau (cf. [11], section 7, for notation), and by the corollary cited above we get $\nu(\mathfrak{p}) = p_1 c_1 + p_2 c_2 = mn + p_2$.

We consider the case $\nu(\mathfrak{p}) = mn$, i.e., the case $\alpha_{m_1-1} = \beta_{\mathfrak{p}}$. For every $j \in \{0, \ldots, n\}$ we have $(m - \sigma_{m,n}(j))n + jm - mn = jm - \sigma_{m,n}(j)n \geq 0$ (cf. the proof of (3.11)); since $\mathfrak{p}$ is a ν-ideal, this implies that the integral closure $\mathfrak{p}'$ of the ideal (x^m, y^n) lies in $\mathfrak{p}$. It is easy to check that also the transform of $\mathfrak{p}'$ in $\beta_{\mathfrak{p}}$ is equal to the maximal ideal of $\beta_{\mathfrak{p}}$. Therefore the simple complete $\mathfrak{m}_\alpha$-primary ideals $\mathfrak{p}'$ and $\mathfrak{p}$ have the same transform in $\beta_{\mathfrak{p}}$, hence we have $\mathfrak{p}' = \mathfrak{p}$.

Conversely, if $\{x, y\}$ is a regular system of parameters in α with $\nu(x) = n$, $\nu(y) = m$, and $\mathfrak{p}$ is the integral closure of the ideal (x^m, y^n), then clearly $\mathfrak{p}^{\alpha_{m_1-1}}$ is the maximal ideal of α_{m_1-1}, hence $m_1 - 1 = h$ and $\nu(\mathfrak{p}) = mn$.

Remark 3.13. Using the notation introduced in the proof of (3.12), let p be a positive integer. If $p = 1$, then we define a quadratic transform α_{m_1} of α_{m_1-1} by $x_{m_1} := x_{m_1-1}$, $y_{m_1} := (y_{m_1-1} - x_{m_1-1})/x_{m_1-1}$, and if $p > 1$, then we define a sequence $\alpha_{m_1} \subset \cdots \subset \alpha_{m_1+p-1}$ of quadratic transforms recursively by $x_{m_1+i} := x_{m_1}$, $y_{m_1+i} := y_{m_1+i-1}/x_{m_1}$ for $i \in \{1, \ldots, p-1\}$; note that $\{x_{m_1+p-1}, y_{m_1+p-1}\}$ is a regular system of parameters in α_{m_1+p-1}. Let ν be the valuation of K defined by the order function of α_{m_1+p-1}. Then we have $\nu(x_{m_1+p-1}) = 1$, $\nu(y_{m_1+p-1}) = p$, and for the simple complete $\mathfrak{m}$-primary ideal $\mathfrak{p}$ in α corresponding to the maximal ideal of α_{m_1+p-1} we have $\Gamma_{\mathfrak{p}} = m\mathbb{N}_0 + n\mathbb{N}_0$ and $\nu(\mathfrak{p}) = mn + p$. In particular, there are simple complete ideals $\mathfrak{p}$ with $\Gamma_{\mathfrak{p}} = m\mathbb{N}_0 + n\mathbb{N}_0$ which are not of the form $\mathfrak{p} = \overline{\mathfrak{a}}$ with $\mathfrak{a} = (x^m, y^n)$.

4. The polar ideal.

In this section we want to give explicit generators for the polar ideal of $\bar{\mathfrak{a}}$. We begin by stating two facts which we get when applying the results of our paper [11].

4.1. Some further results. Let $\mathfrak{p}$ be the integral closure of the ideal (x^m, y^n), and let $\nu := \nu_{\mathfrak{p}}$. For the following results cf. [11], Nr. (7.5). In the sequence of quadratic transforms

$$\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_{s_1} \subset \alpha_{s_1+1} \subset \cdots \subset \alpha_{s_1+s_2} \subset \cdots \subset \alpha_{t-1}$$

the only non-trivial proximity relations are

$$\alpha_{s_1-1} \prec \alpha_{s_1+s_2}, \alpha_{s_1+s_2-1} \prec \alpha_{s_1+s_2+s_3}, \ldots, \alpha_{s_1+s_2\cdots+s_{k-2}-1} \prec \alpha_{s_1+s_2+\cdots+s_{k-1}}$$

and

$$\alpha_{s_1+s_2+\cdots+s_{k-1}-1} \prec \alpha_{s_1+s_2+\cdots+s_k-1}.$$

In particular, if $k = 1$, there are no non-trivial proximity relations. Note that ν is the extension of the order function $\mathrm{ord}_{\alpha_{t-1}}$ to K. Let

$$\mathfrak{m}_\alpha =: \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_{t-1} := \mathfrak{p}$$

be the sequence of simple complete ideals of α determined by $\mathfrak{p}$; these ideals are the only simple ν-ideals in α, and every ν-ideal in α is a product of these ideals. Let $i \in \{1, \ldots, k\}$ and $j \in \{0, \ldots, s_{i-1}\}$; then by [11], (7.5) (6) (b), $\mathfrak{p}_{s_1+\cdots+s_{i-1}+j}$ is the integral closure in α of the ideal generated by $x^{Q_i(s_1,\ldots,s_{i-1},j+1)}$ $y^{Q_{i-1}(s_2,\ldots,s_{i-1},j+1)}$. In particular, *every* $\mathfrak{m}_\alpha$*-primary* ν*-ideal in* α *is a monomial ideal with respect to* $\{x, y\}$.

4.2. Generating sequence. Let $\mathfrak{p}$ be as in (4.1). For $i \in \{1, \ldots, t\}$ let $\{x_i, y_i\}$ be the regular system of parameters for α_i defined in [11], (7.5), starting with $x_0 := x$, $y_0 := y$. Cf. By [11], Lemma (9.6), $\mathfrak{p}$ is an s-ideal. The points in $\Theta(\alpha, \beta_\mathfrak{p})$ (cf. [11], (3.3) (4) for the definition of this set) are the two points γ, γ' where γ has $\{x_y, y_t/x_t\}$ as a regular system of parameters and satisfies $\alpha_{t-s_k} \prec \gamma$ and where γ' has $\{y_t, x_y/y_t\}$ as a regular system of parameters and satisfies $\alpha_{t-2} \prec \gamma'$. Define $f := x^m - y^n$; it is easy to check that $(f\alpha)^{\alpha_t} = (x_t - y_t)\alpha_t$; therefore f is a general element of $\mathfrak{p}$ (cf. [11], definition (3.11)). The ideal $\mathfrak{p}_{s_1}$ is generated by x^{s_1+1} and y; since $\Theta(\alpha, \alpha_{s_1})$ consists only of the point α_{s_1+1}, we see that y is a general element of $\mathfrak{p}_{s_1}$.

Let $\overline{\alpha}_i := \alpha_i/(f\alpha)^{\alpha_i}$ for $i \in \{0, \ldots, t\}$. The ring $\overline{\alpha}_t$ is a discrete valuation ring; let $\overline{\nu}$ be the valuation of the quotient field of $\overline{\alpha}_t$ defined by $\overline{\alpha}_t$. Define $\overline{x} := x \bmod(f)$, $\overline{y} := y \bmod(f)$. Since

$$Q_{k-1}(s_2, \ldots, s_{k-1}, s_k - 1) + Q_{k-2}(s_2, \ldots, s_{k-1}) = n,$$

$$Q_k(s_1, \ldots, s_{k-1}, s_k - 1) + Q_{k-1}(s_1, \ldots, s_{k-2}) = m,$$

we get (cf. [11], (7.5)(5)(a)) $\overline{x} = \overline{x}_t^n$, $\overline{y} = \overline{x}_t^m$, and therefore

$$\nu(\mathfrak{p}_0) = \nu(x) = \overline{\nu}(\overline{x}) = n, \quad \nu(\mathfrak{p}_{s_1}) = \nu(y) = \overline{\nu}(\overline{y}) = m.$$

This implies: *The sequence* (x, y) *is a generating sequence for* ν (cf. [11], Th. (9.9)) .

4.3. The polar ideal of $\mathfrak{p}$. Let $\mathfrak{P}$ be the polar ideal of $\mathfrak{p}$ (the polar ideal $\mathfrak{Q}_\mathfrak{q}$ of a simple complete $\mathfrak{m}_\alpha$-primary ideal $\mathfrak{q}$ is defined in [16], section 5: it is the smallest among those $\mathfrak{m}_\alpha$-primary $\nu_\mathfrak{q}$-ideals $\mathfrak{Q}$ satisfying $\mathrm{ord}_\alpha(\mathfrak{Q}) = \mathrm{ord}_\alpha(\mathfrak{q}) - 1$)). Thus, we have $\mathrm{ord}_\alpha(\mathfrak{P}) = \mathrm{ord}_\alpha(\mathfrak{p}) - 1$, and $\mathfrak{P}$ is a ν-ideal.

Therefore, by the remark at the end of (4.1), $\mathfrak{P}$ is a monomial ideal with respect to $\{x, y\}$. In the following, we construct a minimal system of monomial generators of $\mathfrak{P}$ (cf. (4.4) below).

We define

$$b_l := \operatorname{ord}_{\alpha_l}(\mathfrak{P}^{\alpha_l}) \quad \text{for } l \in \{0, \dots, t-1\}.$$

For $i \in \{1, \dots, k-1\}$ and $j \in \{0, \dots, s_i - 1\}$ and for $i = k,\ j \in \{0, \dots, s_k - 2\}$ we have (cf. (3.1) and [16], Th. (5.2))

$$b_{s_0+s_1+\cdots+s_{i-1}+j} = \begin{cases} n_i - 1 & \text{for } i \text{ odd,} \\ n_i & \text{for } i \text{ even,} \end{cases}$$

and

$$b_{s_1+\cdots+s_k-1} = 0$$

[note that $n_k = 1$]. Let

$$\mathfrak{P} = \prod_{l=0}^{t-1} \mathfrak{p}_l^{c_l}$$

be the factorization of the ν-ideal $\mathfrak{P}$. Then we have (cf. [16], Remark (A))

$$c_l = b_l - \sum_{\alpha_\lambda \succ \alpha_l} b_\lambda \quad \text{for } l \in \{0, \dots, t-1\}.$$

This means, as one easily checks, that

$$\begin{aligned} c_{s_1+\cdots+s_i-1} &= s_{i+1} && \text{for } i \in \{1, \dots, k-1\} \text{ and } i \text{ even,} \\ c_{s_1+\cdots+s_k-2} &= 1 && \text{if } k \text{ is even,} \\ c_l &= 0 && \text{for all other } l \in \{0, \dots, t-1\}. \end{aligned}$$

In particular, we have

$$\mathfrak{P} = \begin{cases} \alpha & \text{if } k = 1, \\ \mathfrak{p}_{s_1+s_2-2} & \text{if } k = 2, \end{cases}$$

and if $k \geq 3$, we have

$$\mathfrak{P} = \begin{cases} \mathfrak{p}_{s_1+s_2-1}^{s_3} \cdots \mathfrak{p}_{s_1+\cdots+s_{k-1}-1}^{s_k} & \text{if } k \text{ is odd,} \\ \mathfrak{p}_{s_1+s_2-1}^{s_3} \cdots \mathfrak{p}_{s_1+\cdots+s_{k-2}-1}^{s_{k-1}} \mathfrak{p}_{s_1+\cdots+s_k-2} & \text{if } k \text{ is even.} \end{cases}$$

Proposition 4.4. *Assume that $k := k(m,n) \geq 2$, let $\mathfrak{p}$ be the integral closure of the ideal generated by x^m, y^n, and let $\mathfrak{P}$ be the polar ideal of $\mathfrak{p}$. Then $\mathfrak{P}$ is minimally generated by*

$$\{x^{m-\sigma_{m,n}(j+1)}y^j \mid j \in \{0,\dots,n-1\}\}.$$

In particular, x^{m-s_1} is the lowest power of x in $\mathfrak{P}$, and y^{n-1} is the lowest power of y is $\mathfrak{P}$.

Proof. We prove the proposition by induction on k.

(1) The case $k = 2$ is already settled; we consider the case $k = 3$. In this case we have $\mathfrak{P} := \mathfrak{p}_{s_1+s_2-1}^{s_3}$. The ideal $\mathfrak{p}_{s_1+s_2-1}$ has order s_2 and is the integral closure of the ideal generated by $x^{s_1s_2+1}$, y^{s_2}, hence is generated by the s_2+1 elements

$$x^{s_1s_2+1-s_1j}y^j \quad \text{for } j \in \{0,\dots,s_2-1\},\ y^{s_2}.$$

The ideal $\mathfrak{p}_{s_1+s_2-1}^{s_3}$ has order s_2s_3, and is generated by the monomials

$$(*) \qquad (x^{s_1s_2+1})^{i_0}(x^{s_1s_2+1-s_1}y)^{i_1}\cdots(x^{s_1s_2+1-s_1(s_2-1)}y^{s_2-1})^{i_{s_2-1}}y^{s_2i_{s_2}}$$

where $(i_0,\dots,i_{s_2}) \in \mathbb{N}_0^{s_2+1}$ and $i_0+\dots+i_{s_2} = s_3$. Let $j \in \{1,\dots,s_2s_3-1\}$ and $(i_0,\dots,i_{s_2}) \in \mathbb{N}_0^{s_2+1}$ with

$$(**) \qquad i_0+\dots+i_{s_2} = s_3,\ 1\cdot i_1+2\cdot i_2+\dots+s_2\cdot i_{s_2} = j.$$

The monomial corresponding to this choice of $(i_0,\dots,i_{s_2})$ is

$$x^{(s_1s_2+1)s_3-i_{s_2}-s_1j}y^j$$

since

$$\sum_{l=0}^{s_2-1}(s_1s_2+1-ls_1)i_l = (s_1s_2+1)s_3-i_{s_2}-s_1j.$$

Claim: For every $j \in \{1,\dots,s_2s_3-1\}$ there exists $(i_0,\dots,i_{s_2}) \in \mathbb{N}_0^{s_2+1}$ satisfying $(**)$.

Proof. We write

$$j = qs_2+r \quad \text{with } r \in \{0,\dots,s_2-1\}.$$

Then we have $0 \leq q < s_3$. In case $r = 0$ we choose

$$i_0 := s_3-q,\ i_1 = \dots = i_{s_2-1} := 0,\ i_{s_2} := q,$$

and in case $r \neq 0$ we choose

$$\begin{aligned}&i_0 := s_3 - (1+q), i_1 = \cdots = i_{s_r - 1} := 0,\\ &i_{s_r} := 1, i_{s_{r+1}} = \cdots = i_{s_2 - 1} := 0, i_{s_2} := q.\end{aligned}$$

For every (s_2+1)-tuple $(i_0, \ldots, i_{s_2}) \in \mathbb{N}_0^{s_2+1}$ satisfying $(**)$ we have $i_{s_2} \leq j/s_2$. This implies the following: Given $j \in \{1, \ldots, s_2 s_3 - 1\}$, the smallest nonnegative integer a such that $x^a y^j$ lies in $\mathfrak{p}_{s_1+s_2-1}^{s_3}$ is

$$a := (s_1 s_2 + 1)s_3 - \left\lfloor \frac{j}{s_2} \right\rfloor - s_1 j.$$

This shows that

$$\left\{ x^{(s_1 s_2+1)s_3 - \lfloor j/s_2 \rfloor - s_1 j} y^j \mid j \in \{0, \ldots, s_2 s_3 - 1\}, y^{s_2 s_3} \right\}$$

is a minimal system of generators of $\mathfrak{P}$. To finish this case, it is enough to show that

$$m - \sigma_{m,n}(j+1) = (s_1 s_2 + 1)s_3 - \left\lfloor \frac{j}{s_2} \right\rfloor - s_1 j \quad \text{for } j \in \{0, \ldots, n-1\}$$

[note that $m = (s_1 s_2 + 1)s_3 + s_1$, $n = s_2 s_1 + 1$], equivalently, that

$$\left\lfloor \frac{j}{s_2} \right\rfloor = \left\lceil \frac{j+1}{s_2} \right\rceil - 1 \quad \text{for } j \in \{0, \ldots, s_2 s_3\};$$

clearly this equation holds.

Let $k \geq 3$, and assume that the proposition holds for all coprime natural integers m, n with $m > n$ and $k(m,n) < k$. Let m, n be coprime integers with $m > n$, and let

$$n_0 = s_1 n_1 + n_2,\ n_2 = s_2 n_2 + n_3, \ldots, n_k = s_{k+1} n_{k+1}$$

be the Euclidean algorithm for $n_0 := m$, $n_1 := n$. We have

$$n_i = Q_{k+1-i}(s_{i+1}, \ldots, s_{k+1}) \quad \text{for } i \in \{0, \ldots, k+1\},$$

and we define $\sigma_k := \sigma_{m,n}$. Let $\mathfrak{p}$ be the integral closure of the ideal (x^{n_0}, y^{n_1}), and let $\mathfrak{P} := \mathfrak{P}_\mathfrak{p}$ be the polar ideal of $\mathfrak{p}$. We determine a minimal system of generators of $\mathfrak{P}$.

(2) We consider the case that k is odd. We have

$$\mathfrak{P} = \mathfrak{p}_{s_1+s_2-1}^{s_3} \cdots \mathfrak{p}_{s+1+\cdots+s_{k-1}-1}^{s_k} \mathfrak{p}_{s_1+\cdots+s_{k+1}-2}.$$

(a) Assume that $s_k \geq 2$, and define n'_i for $i \in \{0, \ldots, k\}$ and σ'_k as in (3.10) (1). Let $\mathfrak{p}'$ be the integral closure of the ideal $(x^{n'_0}, y^{n'_1})$; the polar ideal $\mathfrak{P}'$ of $\mathfrak{p}'$ is

$$\mathfrak{P}' = \mathfrak{p}_{s_1+s_s-1}^{s_3} \cdots \mathfrak{p}_{s_1+\cdots+s_{k-1}-1}^{s_k}.$$

By induction, it is generated by the set $\{x^{n'_0-\sigma'_k(j+1)}y^j \mid j \in \{0, \ldots, n'_1 - 1\}\}$.

(a1) Assume that $s_{k+1} = 2$. Then we define n''_i for $i \in \{0, \ldots, k\}$, σ''_k as in (3.10) (3). We have

$$(\dagger) \qquad n_i = n'_i + n''_i \quad \text{for } i \in \{0, \ldots, k\}.$$

The ideal $\mathfrak{p}_{s_1+\cdots+s_{k+1}-2}$ is the integral closure of the ideal generated by $x^{n''_0}, y^{n''_1}$, hence $\mathfrak{p}_{s_1+\cdots+s_{k+1}-2}$ is generated by the set (cf. (3.6)) $\{x^{n''_0-\sigma''_k(j)}y^j \mid j \in \{0, \ldots, n''_1\}\}$. Since $\mathfrak{P} = \mathfrak{P}'\mathfrak{p}_{s_1+\cdots+s_{k+1}-2}$, the ideal $\mathfrak{P}$ is generated by $\{x^{n'_0+n''_0-\tau(l)}y^l \mid l \in \{0, \ldots, n'_1 + n''_1 - 1\}\}$ where we have defined for $l \in \{0, \ldots, n_1 - 1\}$ the integer $\tau(l)$ by

$$\tau(l) := \max\big\{\sigma'_k(l'+1) + \sigma''_k(l'') \mid l' + l'' = l\,,$$
$$l' \in \{0, \ldots, n'_1 - 1\},\ l'' \in \{0, \ldots, n''_1\}\big\}.$$

We have to prove that

$$(\dagger\dagger) \qquad \sigma_{k+1}(l+1) = \tau(l) \quad \text{for } l \in \{0, \ldots, n_1 - 1\}.$$

For $l = 0$ we have $\sigma_{k+1}(0) = s_1$, $\tau(0) = \sigma'_k(1) = s_1$. For $l = n_1 - 1$ we have $\sigma_{k+1}(n_1) = n_0$. The only integers l', l'' in the range above with $l' + l'' = n_1 - 1$ are $l' = n'_1 - 1$, $l'' = n''_1$, and we get

$$\sigma'_k(n'_1) + \sigma''_k(n''_1) = n'_0 + n''_0 = n_0.$$

Therefore we have to prove $(\dagger\dagger)$ only for $l \in \{1, \ldots, n_1 - 2\}$.

We show that

$$\sigma_k(n'_1 + j) = n'_0 + \sigma_k(j) \quad \textit{for } j \in \{0, \ldots, n''_1\}.$$

In the following, we use the notation introduced in (3.10). The claim certainly is true for $j = n''_1$. Let $j \in \{0, \dots, n''_1 - 1\}$, and choose $i \in \{1, \dots, n_2\}$ with

$$n_1 - \sigma_*(n_2 - i + 1) \le n'_1 + j \le n_1 - \sigma_*(n_2 - i) - 1.$$

Since $n_1 - \sigma_*(n_2 - n'_2) = n'_1 + 1$ (cf. (3.10) (2) and note that k is odd), we have $i \ge n'_2 + 1$, and from (3.10) (3) and (†) above we get

$$n''_1 - \sigma''_*(n''_2 - (i - n'_2) + 1) \le j \le n''_1 - \sigma''_*(n''_2 - (i - n'_2)) - 1.$$

Therefore we have

$$\sigma_k(j) = \sigma''_k(j) = s_1 j + i - n_2 - 1,$$

$$n'_0 + \sigma_k(j) = s_1(n_1 + j) + i - 1 = \sigma_k(n'_1 + j).$$

Similarly, one can show that

$$n''_0 + \sigma_k(j) = \sigma_k(n''_1 + j) \quad \text{for } j \in \{0, \dots, n'_1\}.$$

For $l \in \{0, \dots, n'_1 - 1\}$ we choose $l' := l$, $l'' := 0$. Then we have $\sigma'_k(l' + 1) = \sigma_k(l + 1)$ (cf. (3.10) (2)). For $l \in \{n'_1 - 1, \dots, n_1 - 2\}$ we choose $l' := n'_1 - 1$, $l'' := l - (n'_1 - 1)$. Then we have

$$\sigma_k(l + 1) = \sigma_k(n'_1 + l'') = \sigma'_k(n'_1) + \sigma''_k(l'').$$

Likewise, if $l'' := n''_2$, $l \in \{n''_2, \dots, n_1 - 2\}$ and $l' := l - l''$, we obtain

$$\sigma_k(l + 1) = \sigma'_k(l' + 1) + \sigma''_k(n''_1).$$

From (3.10), (1) and (3), we get the result.

(a2) Assume that $s_{k+1} \ge 3$. Then we define n''_i for $i \in \{0, \dots, k\}$, σ''_k as in (3.10) (4). We have

$$n_i = n'_i + n''_i \quad \text{for } i \in \{0, \dots, k\}.$$

The ideal $\mathfrak{p}_{s_1+\dots+s_{k+1}-2}$ is the integral closure of the ideal generated by $x^{n''_0}$, $y^{n''_1}$, hence $\mathfrak{p}_{s_1+\dots+s_{k+1}-2}$ is generated by the set (cf. (3.6)) $\{x^{n''_0 - \sigma''_k(j)} y^j \mid j \in \{0, \dots, n''_1\}\}$. We have $\mathfrak{P} = \mathfrak{P}'\mathfrak{p}_{s_1+\dots+s_{k+1}-2}$. Just as in (a1), using (3.10), (1) and (4), we can prove the assertion.

(b) Now we assume that $s_k = 1$. We define n'_i for $i \in \{0, \dots, k-1\}$ and σ'_k as in (3.10) (2). Let $\mathfrak{p}'$ be the integral closure of the ideal generated by $(x^{n'_0}, y^{n'_1})$. Then

$$\mathfrak{P}' = \mathfrak{p}^{s_3}_{s_1+s_2-1} \cdots \mathfrak{p}^{s_{k-2}}_{s_1+\cdots+s_{k-3}-1} \mathfrak{p}_{s_1+\cdots+s_{k-1}-1}$$

is the polar ideal of $\mathfrak{p}'$, and

$$\mathfrak{P} = \mathfrak{P}' \mathfrak{p}_{s_1+\cdots+s_{k+1}-2}$$

is the polar ideal of $\mathfrak{p}$. We distinguish the two cases $s_{k+1} = 2$ and $s_{k+1} \geq 3$, and with the help of (3.10) (2) and (3.10) (3) resp. (3.10) (4) we can argue as above in the case (a).

(3) We consider the case that k is even. We have

$$\mathfrak{P} = \mathfrak{p}^{s_3}_{s_1+s_2-1} \cdots \mathfrak{p}^{s_{k-1}}_{s_1+\cdots+s_{k-2}-1} \mathfrak{p}^{s_{k+1}}_{s_1+\cdots+s_k-1}.$$

(a) The case $s_k \geq 2$: We define n'_i for $i \in \{0, \dots, k\}$ and σ'_k as in (3.10) (1); let $\mathfrak{p}' := \mathfrak{p}_{s_1+\cdots+s_k-1}$, the integral closure of the ideal generated by $(x^{n'_0}, y^{n'_1})$. The polar ideal $\mathfrak{P}'$ of $\mathfrak{p}'$ is

$$\mathfrak{P}' = \mathfrak{p}^{s_3}_{s_1+s_2-1} \cdots \mathfrak{p}^{s_{k-1}}_{s_1+\cdots+s_{k-2}-1} \mathfrak{p}_{s_1+\cdots+s_k-2}$$

and we have

$$\mathfrak{P} = \mathfrak{P}' \mathfrak{p}^{s_{k+1}}_{s_1+\cdots+s_k-1}.$$

We handle this case by induction on s_{k+1}, using similar methods as above.

(b) The case $s_k = 1$: Here again, we use similar methods as above.

4.5. The polar curve. Let κ be an algebraically closed field of characteristic zero, and let $\alpha = \kappa[[x, y]]$ be the ring of formal power series over κ in two indeterminates x, y. Furthermore, we assume that $k \geq 2$.

For every $\mathbf{u} = (u_0, \dots, u_n) \in \kappa^{n+1}$ we define

$$f_{\mathbf{u}} := \sum_{j=0}^{n} u_j x^{m-\sigma_{m,n}(j)} y^j;$$

by [11], (3.27), there exists a non-empty Zariski-open subset $U \subset \kappa^{n+1}$ such that $f_{\mathbf{u}}$ is a general element of $\mathfrak{p}$ for every $\mathbf{u} \in U$. It is easy to check that $\mathbf{u} = (u_0, \dots, u_n) \in U$ implies that $u_0 \neq 0$ and $u_n \neq 0$.

For every $\mathbf{v} = (v_0, \dots, v_{n-1})$ we define

$$g_{\mathbf{v}} = \sum_{j=0}^{n-1} v_j x^{m-\sigma_{m,n}(j+1)} y^j;$$

there exists a non-empty Zariski-open subset $V \subset \kappa^n$ such that $g_{\mathbf{v}}$ is a general element of the polar ideal $\mathfrak{P}$ of $\mathfrak{p}$ for every $\mathbf{v} \in V$. Therefore, we have the following:

Corollary 4.6. *Let m, n be coprime positive integers with $m > n$, and let $\mathfrak{p}$ be the integral closure of the ideal generated by x^m, y^n in $\kappa[[x, y]]$. There exists a non-empty open subset U of κ^{n+1} such that $f_{\mathbf{u}}$ is a general element of $\mathfrak{p}$ and $\partial f_{\mathbf{u}}/\partial y$ is a general element of the polar ideal $\mathfrak{P}$ of $\mathfrak{p}$.*

This result can also be rephrased in the following way: "A polar of a general element of $\mathfrak{p}$ is a general element of the polar ideal $\mathfrak{P}$ of $\mathfrak{p}$" (cf. [6], section 6.6).

Remark 4.7. We keep the hypotheses of (4.5). Since $\sigma_{m,n}$ is integer-valued and strictly increasing, we have

$$\sigma_{m,n}(j+1) \geq \sigma_{m,n}(j) + 1 \quad \text{for } j \in \{0, \ldots, n-1\},$$

and therefore we obtain for $b, c \in \kappa$ not both zero and $\mathbf{u} = (u_0, \ldots, u_n) \in \kappa^{n+1}$ with $u_0 \neq 0$, $u_n \neq 0$

$$\begin{aligned} g_{b,c,\mathbf{u}} &:= b\frac{\partial f_{\mathbf{u}}}{\partial x} + c\frac{\partial f_{\mathbf{u}}}{\partial y} \\ &= \sum_{j=0}^{n-1} \Bigg(\big(b(m - \sigma_{m,n}(j))u_j x^{\sigma_{m,n}(j+1)-\sigma_{m,n}(j)-1} \\ &\quad + c u_{j+1}(j+1)\big) x^{m-\sigma_{m,n}(j+1)} y^j \Bigg). \end{aligned}$$

Clearly $g_{b,c,\mathbf{u}}$ is an element of $\mathfrak{P}$, and defines also a polar curve of $f_{\mathbf{u}}$ (note that $m - \sigma_{m,n}(j) \neq 0$ for $j \in \{0, \ldots, n-1\}$). Therefore we have the following result: A general polar of a general element $f_{\mathbf{u}}$ of $\mathfrak{p}$ is an element of the polar ideal of $\mathfrak{p}$.

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Added in Proof: Let $\alpha \in \Omega$ have infinite residue field, let $\mathfrak{a}$ be a complete $\mathfrak{m}_\alpha$-primary ideal, and let $\mathfrak{b}$ be a minimal reduction of $\mathfrak{a}$. Then $\mathfrak{a}$ is the integral closure of $\mathfrak{b}$, and $\mathfrak{b}$ is generated by a regular sequence (f, g), i.e., $\mathfrak{b} = f\alpha + g\alpha$. By the main theorem of [14], $\mathfrak{a}$ is generated by monomials in f and g. Can one construct explicitly a system of monomial generators of $\mathfrak{a}$?

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