PERFECT ESSENTIAL GRAPHS

M. JAVAD NIKMEHR - A. AZADI - B. SOLEYMANZADEH

Let $R$ be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of $R$. Let $EG(R)$ be a simple undirected graph associated with $R$ whose vertex set is the set of all nonzero zero-divisors of $R$ and and two distinct vertices $x, y$ in this graph are joined by an edge if and only if $\text{Ann}_R(xy)$ is an essential ideal. A perfect graph is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. In this paper, we characterize all Artinian rings whose $EG(R)$ is perfect.

1. Introduction

Usually, after translating of algebraic properties of rings into graph-theoretic language, some problems in ring theory might be more easily solved. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory; for instance see. Moreover, for the most recent study in this field see [2, 5] and [7].

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $Z(R)$, $U(R)$, $\text{Max}(R)$ and $\text{Nil}(R)$, the set of all zero-divisors, the set of units, the set of all maximal ideals of $R$ and the set of all nilpotent

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elements of $R$, respectively. The ring $R$ is said to be reduced if it has no nonzero nilpotent element. For every ideal $I$ of $R$, we denote the annihilator of $I$ by $\text{Ann}_R(I)$. A nonzero ideal $I$ of $R$ is called essential, if it has a nonzero intersection with any nonzero ideal of $R$. Some more definitions about commutative rings can be find in [1, 3, 9]. We use the standard terminology of graphs following [4, 8].

Let $G = (V,E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $G$, we mean the complement graph of $G$. We write $u - v$ to denote an edge with ends $u, v$. A graph $H = (V_0,E_0)$ is called a subgraph of $G$ if $V_0 \subset V$ and $E_0 \subset E$. Moreover, $H$ is called an induced subgraph by $V_0$, denoted by $G[V_0]$, if $V_0 \subset V$ and $E_0 = \{u,v \in E | u,v \in V_0\}$. Also $G$ is called a null graph if it has no edge. A complete graph of $n$ vertices is denoted by $K_n$. An $n$-part graph is one whose vertex set can be partitioned into $n$ subsets, so that no edge has both ends in any one subset. A complete $n$-partite graph is one in which each vertex is jointed to every vertex that is not in the same subset. In a graph $G$, a vertex $x$ is isolated, if no vertices of $G$ is adjacent to $x$. A clique of $G$ is a maximal complete subgraph of $G$ and the number of vertices in the largest clique of $G$, denoted by $\omega(G)$, is called the clique number of $G$. For a graph $G$, let $\chi(G)$ denote the chromatic number of $G$, i.e., the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. Note that for every graph $G$, $\omega(G) \leq \chi(G)$. A graph $G$ is said to be weakly perfect if $\omega(G) = \chi(G)$. A perfect graph $G$ is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

Let $R$ be a commutative ring. In this paper, we consider a simple undirected graph associated with $R \text{ EG}(R)$ (see [6]) whose vertex set equals the set of all nonzero zero-divisors of $R$ and two distinct vertices $x,y$ in this graph are joined by an edge if and only if $\text{Ann}_R(xy)$ is an essential ideal. In this paper, we characterize all Artinian rings whose $\text{EG}(R)$ is perfect.

2. On perfect graph

We begin with the following useful lemma.

**Lemma 2.1.** [6] Let $R$ be a nonreduced ring. Then the following statements hold.

1. For every $x \in \text{Nil}(R)^*$, $x$ is adjacent to all other vertices.
2. $\text{EG}(R)[\text{Nil}(R)^*]$ is a (induced) complete subgraph of $\text{EG}(R)$.

It was shown in [6, Lemma 3.1] that if $x$ is a nilpotent element of $R$, then $\text{Ann}_R(x)$ is an essential ideal of $R$. Here we have the following result:
Lemma 2.2. For every \( x \in \text{Nil}(R) \), \( \text{Ann}_R(x) \) is an essential ideal of \( R \).

To prove our main results we need the following celebrate theorem.

**Theorem 2.3** (see [4] The Strong Perfect Graph Theorem). A graph \( G \) is perfect if and only if neither \( G \) nor \( \overline{G} \) contains an induced odd cycle of length at least 5.

**Corollary 2.4.** Let \( G \) be a graph and \( \{V_1, V_2\} \) be a partition of \( V(G) \). If \( G[V_i] \) is a complete graph, for every \( 1 \leq i \leq 2 \), then \( G \) is a perfect graph.

**Proof.** By Theorem 2.3, it is enough to show that \( G \) and \( \overline{G} \) contain no induced odd cycle of length at least 5. Let \( a_1 - a_2 - \cdots - a_n - a_1 \) be an induced odd cycle of length at least 5 in \( G \). Since \( V = V_1 \cup V_2 \), we have \( |\{a_1, a_2, \ldots, a_n\} \cap V_1| \geq 3 \) or \( |\{a_1, a_2, \ldots, a_n\} \cap V_2| \geq 3 \). This is a contradiction as \( G[V_1] \) and \( G[V_2] \) are complete graphs. Also, since \( G[V_1] \) and \( G[V_2] \) are complete graphs, \( \overline{G} \) is a bipartite graph and thus contains no induced odd cycle of length at least 5.

The following lemmas have a key role in this paper.

**Lemma 2.5.** Let \( n \) be a positive integer and \( R \cong R_1 \times R_2 \times \cdots \times R_n \), where \( R_i \) is a ring, for every \( 1 \leq i \leq n \). If \( \text{EG}(R) \) contains no induced odd cycle of length at least 5, then \( n \leq 4 \).

**Proof.** Suppose that \( n \geq 5 \). Then we can easily get

\[
(1,0,0,1,0,0,\ldots,0) - (0,1,0,0,1,0,\ldots,0) - (1,0,1,0,0,0,\ldots,0)
\]

\[
(0,0,0,1,1,0,\ldots,0) - (0,1,1,0,0,0,\ldots,0) - (1,0,0,1,0,0,\ldots,0)
\]

is a cycle of length 5. Thus Theorem 2.3 lead to a contradiction. So \( n \leq 4 \).

**Lemma 2.6.** Let \( R \cong R_1 \times \cdots \times R_n \), \( a = (x_1, x_2, \ldots, x_n) \) and \( b = (y_1, y_2, \ldots, y_n) \), where \( n \) is a positive integer, every \( R_i \) is an Artinian local ring and \( x_i, y_i \in R_i \), for every \( 1 \leq i \leq n \).

1. \( a - b \) is an edge of \( \text{EG}(R) \) if and only if \( x_iy_i \in \text{Nil}(R_i) \), for all \( 1 \leq i \leq n \).
2. \( a \) is not adjacent to \( b \) in \( \text{EG}(R) \) if and only if \( x_jy_j \in U(R_j) \), for some \( 1 \leq j \leq n \).
3. \( a - b \) is an edge of \( \overline{\text{EG}(R)} \) if and only if \( x_iy_i \in U(R_i) \), for some \( 1 \leq i \leq n \).
4. \( a \) is not adjacent to \( b \) in \( \overline{\text{EG}(R)} \) if and only if \( x_jy_j \in \text{Nil}(R_j) \), for all \( 1 \leq j \leq n \).
Proof. (1) Assume that $x_iy_i \in \text{Nil}(R_i)$, for every $1 \leq i \leq n$. Then $ab \in \text{Nil}(R)$. This, together with Lemma 2.2, implies that $\text{Ann}_R(ab)$ is an essential ideal of $R$. Hence $a$ is adjacent to $b$.

Conversely, let $a - b$ is an edge of $EG(R)$. We claim that $x_iy_i \in \text{Nil}(R_i)$, for every $1 \leq i \leq n$. Assume to the contrary, $x_jy_j \notin \text{Nil}(R_j)$, for some $1 \leq j \leq n$. Since $R_j$ is Artinian local ring, we have $x_jy_j \in \text{U}(R_j)$. Hence $\text{Ann}_R(ab) \cap 0 \times \cdots \times 0 \times R_j \times 0 \times \cdots \times 0 = (0, \ldots, 0)$ and thus $a$ is not adjacent to $b$, a contradiction and so the claim is proved.

(2) Let $a$ is not adjacent to $b$ in $EG(R)$, by Part (1), $x_iy_i \notin \text{Nil}(R_i)$, for some $1 \leq i \leq n$. This together with the fact $R_i$ is Artinian local implies that $x_iy_i \in \text{U}(R_i)$. Converse is clear.

(3) is obtained by Part 2.

(4) is obtained by Part 1.

Lemma 2.7. Let $S_1, S_2, S_3, S_4$ be rings such that $S_1 \cong R_1$, $S_2 \cong R_1 \times R_2$, $S_3 \cong R_1 \times R_2 \times R_3$ and $S_4 \cong R_1 \times R_2 \times R_3 \times R_4$, where $R_i$ is a ring, for every $1 \leq i \leq n$. Then if $EG(S_4)$ is a perfect graph, then $EG(S_3)$, $EG(S_2)$ and $EG(S_1)$ are perfect graphs.

Proof. As $EG(S_4)$ is a perfect graph, it follows that $EG(S_4)[A]$ is a perfect graph, where $A = \{(x_1, x_2, x_3, x_4) \in S_4 | x_4 = 0\}$. It is clear that $EG(S_4)[A] \cong EG(S_3)$. Thus $EG(S_3)$ is a perfect graph. Similarly, $EG(S_2)$ and $EG(S_1)$ are perfect graphs.

We are now in a position to state our main result in this section.

Theorem 2.8. Let $R$ be an Artinian ring. Then $EG(R)$ is a perfect graph if and only if $|\text{Max}(R)| \leq 4$.

Proof. First let $EG(R)$ be a perfect graph. Since $R$ is an Artinian ring, for some positive integer $n$, $R \cong R_1 \times \cdots \times R_n$, where $R_i$ is an Artinian local ring, for every $1 \leq i \leq n$. Now, by Theorem 2.3 and Lemma 2.5, $n \leq 4$. To prove the converse, by Theorem 2.3, it is enough to show that $EG(R)$ and $\overline{EG(R)}$ contains no induced odd cycle of length at least 5. By Lemma 2.7, we need to prove the only case that $n = 4$. So let $R \cong R_1 \times R_2 \times R_3 \times R_4$, where $R_i$ is an Artinian local ring. Indeed, we have the following claims:
Claim 1. $EG(R)$ contains no induced odd cycle of length at least 5. We consider the following partition for vertices of $EG(R)$.

$$A = \{\{(x_1,x_2,x_3,x_4) \mid x_i \in \text{Nil}(R_i) \text{ for all } i\}\setminus\{(0,0,0,0)\}\},$$

$$B = \{(x_1,x_2,x_3,x_4) \mid \text{for some } i, x_i \not\in \text{Nil}(R_i)\}.$$

Thus $A \cap B = \emptyset$ and $V(EG(R)) = A \cup B$. Also we consider the following partition for $B$.

$$B_1 = \{(x,y,z,w) \in B \mid x \in U(R_1)\},$$

$$B_2 = \{(x,y,z,w) \in B \mid x \in \text{Nil}(R_1) \text{ and } y \in U(R_2)\},$$

$$B_3 = \{(x,y,z,w) \in B \mid x \in \text{Nil}(R_1), y \in \text{Nil}(R_2) \text{ and } z \in U(R_3)\},$$

$$B_4 = \{(x,y,z,w) \in B \mid x \in \text{Nil}(R_1), y \in \text{Nil}(R_2), z \in \text{Nil}(R_3) \text{ and } w \in U(R_4)\}.$$

It is easy to see that $B = \bigcup_{i=1}^{4} B_i$ and $B_i \cap B_j = \emptyset$ for every $i \neq j$. Note that we denote elements of $B_j$ by $a_i = (x_i,y_i,z_i,w_i)$ for all $1 \leq i \leq 4$.

Now, assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced odd cycle of length at least 5 in $EG(R)$. We consider the following cases.

Case (1) $\{a_1, \ldots, a_n\} \cap A = \emptyset$. Let $a_i \in \{a_1, \ldots, a_n\} \cap A$, for some $1 \leq i \leq n$. Then by Lemma 2.1, $a_i$ is adjacent to all other vertices, a contradiction. Thus $\{a_1, \ldots, a_n\} \cap A = \emptyset$.

Case (2) $\{a_1, \ldots, a_n\} \cap B_4 = \emptyset$. Assume to the contrary and with no loss of generality, $a_1 = (x_1,y_1,z_1,w_1) \in B_4$. Then

$$a_1 \in \text{Nil}(R_1) \times \text{Nil}(R_2) \times \text{Nil}(R_3) \times U(R_4).$$

Then the fourth components of $a_2$ and $a_n$ must be in $\text{Nil}(R_4)$. Since $x_3x_1, y_1y_3$ and $z_1z_3$ are nilpotent elements and $a_3$ is not adjacent to $a_1$, by part 2 of Lemma 2.6, we conclude that the fourth component of $a_3$ must be in $U(R_4)$. This together with $a_4$ is adjacent to $a_3$ implies that the fourth component of $a_4$ is nilpotent element and so $a_4a_1 \in \text{Nil}(R)$. Therefore by Lemma 2.2, we have $a_4$ is adjacent to $a_1$, which is a contradiction. So the assertion is proved.

Case (3) $\{a_1, \ldots, a_n\} \cap B_1 = \emptyset$. Assume to the contrary and with no loss of generality, $a_1 = (x_1,y_1,z_1,w_1) \in B_1$. It is easy to see that for every $1 \leq i \leq 4$, there is no adjacency between two vertices of $B_i$. This together with Case (2) implies that $a_n$ and $a_2$ are in $B_2 \cup B_3$. We consider the following three subcases.

subcase (1) $\{a_n, a_2\} \subset B_3$. We can let

$$\{a_n, a_2\} \subset \text{Nil}(R_1) \times \text{Nil}(R_2) \times U(R_3) \times R_4.$$
Then the third components of $a_1$ and $a_3$ must be in $\text{Nil}(R_3)$. Also since $a_n$ is not adjacent to $a_3$, by part 2 of Lemma 2.6, the fourth components of $a_n$ and $a_3$ must be in $U(R_4)$. This yields

$$a_1 \in U(R_1) \times R_2 \times \text{Nil}(R_3) \times R_4,$$
$$a_n \in \text{Nil}(R_1) \times \text{Nil}(R_2) \times U(R_3) \times U(R_4),$$
$$a_3 \in R_1 \times R_2 \times \text{Nil}(R_3) \times U(R_4).$$

Then the fourth components of $a_1$ and $a_2$ must be in $\text{Nil}(R_4)$. Hence we find that

$$a_1 \in U(R_1) \times R_2 \times \text{Nil}(R_3) \times \text{Nil}(R_4),$$
$$a_2 \in \text{Nil}(R_1) \times \text{Nil}(R_2) \times U(R_3) \times \text{Nil}(R_4).$$

Since $a_2$ is not adjacent to $a_4$, the third components of $a_4$ must be in $U(R_3)$. This implies that $a_4$ is not adjacent to $a_n$ and so $n \geq 7$. It is easy to see that the third component of $a_5$ must be in $\text{Nil}(R_3)$ and so $a_5a_2 \in \text{Nil}(R)$. This implies that $a_5 - a_2$, a contradiction. So in this case the assertion is proved.

Subcase (2) \{a_n, a_2\} $\subset B_2$. By a similar way as used in Subcase 1, we get a contradiction.

Subcase (3). $a_n \in B_2$ and $a_2 \in B_3$. By a similar way as used in Subcase (1), we get a contradiction. Thus \{a_1, \ldots, a_n\} $\cap B_1 = \emptyset$.

By the above cases \{a_1, \ldots, a_n\} $\subseteq B_2 \cup B_3$, but this is contradicts the fact $\overline{EG(R)}[B_2 \cup B_3]$ is a bipartite graph, and thus $\overline{EG(R)}$ contains no induced odd cycle of length at least 5.

In Claim 2, $A$, $B$ and $B_i$ are sets that mentioned in Claim 1.

Claim 2. $\overline{EG(R)}$ contains no induced odd cycle of length at least 5. We show that $\overline{EG(R)}$ contains no induced odd cycle at least 5. Assume to the contrary,

$$a_1 - a_2 - \cdots - a_n - a_1$$

is an induced odd cycle of length at least 5 in $\overline{EG(R)}$. It is clear that $\overline{EG(R)}[A]$ is a null graph and so \{a_1, \ldots, a_n\}$ \cap A = \emptyset$. First we show that

$$\{a_1, \ldots, a_n\} \cap B_4 = \emptyset.$$

Assume to the contrary and with no loss of generality, $a_1 = (x_1, y_1, z_1, w_1) \in B_4$. Then $a_1 \in \text{Nil}(R_1) \times \text{Nil}(R_2) \times \text{Nil}(R_3) \times U(R_4)$. This together with part 3 of Lemma 2.6 implies that the forth components of $a_2$ and $a_n$ must be in $U(R_4)$ and so we have

$$a_n \in R_1 \times R_2 \times R_3 \times U(R_4),$$
$$a_2 \in R_1 \times R_2 \times R_3 \times U(R_4).$$
It is easy to see that $a_2$ is adjacent to $a_n$, a contradiction, and so

$$\{a_1, \ldots, a_n\} \cap B_4 = \emptyset.$$

Finally to complete the proof, we prove that $\{a_1, \ldots, a_n\} \cap B_3 = \emptyset$. To get a contradiction, let $a_1 = (x_1, y_1, z_1, w_1) \in B_3$. Then

$$a_1 \in \text{Nil}(R_1) \times \text{Nil}(R_2) \times U(R_3) \times R_4.$$ 

Since $a_1 = a_n$, $a_1 = a_2$ and $a_2$ is not adjacent to $a_n$, we consider the following two cases.

Case (1)

$$a_1 \in \text{Nil}(R_1) \times \text{Nil}(R_2) \times U(R_3) \times U(R_4),$$

$$a_2 \in R_1 \times R_2 \times U(R_3) \times \text{Nil}(R_4),$$

$$a_n \in R_1 \times R_2 \times \text{Nil}(R_3) \times U(R_4).$$

Since $a_3$ is not adjacent to $a_1$, the third and the fourth components $a_3$ must be nilpotent. On the other hand, $a_3$ is adjacent to $a_2$. This implies that $x_3x_2 \in U(R_1)$ or $y_3y_3 \in U(R_2)$.

First suppose that $x_3x_2 \in U(R_1)$. Now, we know that

$$a_3 \in U(R_1) \times R_2 \times \text{Nil}(R_3) \times \text{Nil}(R_4),$$

$$a_2 \in U(R_1) \times R_2 \times U(R_3) \times \text{Nil}(R_4).$$

This together with that $a_3$ is adjacent to $a_4$ implies that $x_3x_4 \in U(R_1)$ or $y_3y_4 \in U(R_2)$. If $x_3x_4 \in U(R_1)$, then we have $x_2x_4 \in U(R_1)$. Therefore $a_4$ is adjacent to $a_2$, which is a contradiction. Thus we conclude that $y_3y_4 \in U(R_2)$. This yields

$$a_3 \in U(R_1) \times U(R_2) \times \text{Nil}(R_3) \times \text{Nil}(R_4),$$

$$a_4 \in \text{Nil}(R_1) \times U(R_2) \times R_3 \times R_4.$$ 

Since $a_4$ is not adjacent to $a_1$, we have

$$a_4 \in \text{Nil}(R_1) \times U(R_2) \times \text{Nil}(R_3) \times \text{Nil}(R_4).$$

Thus $a_4$ is not adjacent to $a_n$ and so $n \geq 7$. On the other hand, since $a_4 - a_5$, the second components of $a_5$ must be unit and so $a_5$ is adjacent to $a_2$, which is a contradiction.

So, suppose that $y_2y_3 \in U(R_2)$. Similarly, we get a contradiction. Thus in this case the assertion is proved.

Case (2)

$$a_1 \in \text{Nil}(R_1) \times \text{Nil}(R_2) \times U(R_3) \times U(R_4),$$
\[ a_2 \in R_1 \times R_2 \times \text{Nil}(R_3) \times U(R_4), \]
\[ a_n \in R_1 \times R_2 \times U(R_3) \times \text{Nil}(R_4). \]

By similar argument that of case (1), we get a contradiction. This means that \( \{a_1, \ldots, a_n\} \subseteq B_2 \cup B_1. \) Clearly, \( \overline{EG(R)}[B_1], \overline{EG(R)}[B_2] \) are complete, and thus by Corollary 2.4, \( \overline{EG(R)}[B_1 \cup B_2] \) is a perfect graph, a contradiction. Hence \( \overline{EG(R)} \) contain no induced odd cycle of length at least 5. Therefore by Claim 1, Claim 2 and Theorem 2.3, \( EG(R) \) is a perfect graph.

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REFERENCES

M. JAVAD NIKMEHR
Faculty of Mathematics
K. N. Toosi University of Technology
e-mail: nikmehr@kntu.ac.ir

A. AZADI
Faculty of Mathematics
K. N. Toosi University of Technology
e-mail: abdoreza.azadi@email.kntu.ac.ir

B. SOLEYMANZADEH
Faculty of Mathematics
K. N. Toosi University of Technology
e-mail: b.soleymanzadeh@mail.kntu.ac.ir