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PERFECT ESSENTIAL GRAPHS

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Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. Let $EG(R)$ be a simple undirected graph associated with R whose vertex set is the set of all nonzero zero-divisors of R and and two distinct vertices x, y in this graph are joined by an edge if and only if $\mathrm{Ann}_R(xy)$ is an essential ideal. A perfect graph is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. In this paper, we characterize all Artinian rings whose $EG(R)$ is perfect.

1. Introduction

Usually, after translating of algebraic properties of rings into graph-theoretic language, some problems in ring theory might be more easily solved. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory; for instance see. Moreover, for the most recent study in this field see [2, 5] and [7].

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $Z(R)$, $\mathrm{U}(R)$, Max(R) and Nil(R), the set of all zero-divisors, the set of units, the set of all maximal ideals of R and the set of all nilpotent

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elements of R, respectively. The ring R is said to be *reduced* if it has no nonzero nilpotent element. For every ideal I of R, we denote the *annihilator* of I by $\mathrm{Ann}_R(I)$. A nonzero ideal I of R is called *essential*, if it has a nonzero intersection with any nonzero ideal of R. Some more definitions about commutative rings can be find in [1, 3, 9]. We use the standard terminology of graphs following [4, 8].

Let $G = (V,E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By G, we mean the complement graph of G. We write $u-v$ to denote an edge with ends u,v. A graph $H = (V_0,E_0)$ is called a *subgraph* of G if $V_0 \subset V$ and $E_0 \subset E$. Moreover, H is called an *induced subgraph* by V_0, denoted by $G[V_0]$, if $V_0 \subset V$ and $E_0 = \{u,v \in E | u,v \in V_0\}$. Also G is called a *null graph* if it has no edge. A complete graph of n vertices is denoted by K_n. An n-part graph is one whose vertex set can be partitioned into n subsets, so that no edge has both ends in any one subset. A complete n-partite graph is one in which each vertex is jointed to every vertex that is not in the same subset. In a graph G, a vertex x is *isolated*, if no vertices of G is adjacent to x. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number of G. For a graph G, let $\chi(G)$ denote the *chromatic* number of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Note that for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect* graph G is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

Let R be a commutative ring. In this paper, we consider a simple undirected graph associated with R $EG(R)$ (see [6]) whose vertex set equals the set of all nonzero zero-divisors of R and two distinct vertices x,y in this graph are joined by an edge if and only if $\mathrm{Ann}_R(xy)$ is an essential ideal. In this paper, we characterize all Artinian rings whose $EG(R)$ is perfect.

2. On perfect graph

We begin with the following useful lemma.

Lemma 2.1. *[6] Let R be a nonreduced ring. Then the following statements hold.*

(1) For every $x \in \mathrm{Nil}(R)^*$, x is adjacent to all other vertices.

(2) $EG(R)[\mathrm{Nil}(R)^*]$ is a (induced) complete subgraph of $EG(R)$.

It was shown in [6, Lemma 3.1] that if x is a nilpotent element of R, then $\mathrm{Ann}_R(x)$ is an essential ideal of R. Here we have the following result:

Lemma 2.2. *For every $x \in \mathrm{Nil}(R)$, $\mathrm{Ann}_R(x)$ is an essential ideal of R.*

To prove our main results we need the following celebrate theorem.

Theorem 2.3 (see [4] The Strong Perfect Graph Theorem)**.** *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least* 5.

Corollary 2.4. *Let G be a graph and $\{V_1, V_2\}$ be a partition of $V(G)$. If $G[V_i]$ is a complete graph, for every $1 \leq i \leq 2$, then G is a perfect graph.*

Proof. By Theorem 2.3, it is enough to show that G and $\overline{G}$ contain no induced odd cycle of length at least 5. Let $a_1 - a_2 - \cdots - a_n - a_1$ be an induced odd cycle of length at least 5 in G. Since $V = V_1 \cup V_2$, we have $|\{a_1, a_2, \ldots, a_n\} \cap V_1| \geq 3$ or $|\{a_1, a_2, \ldots, a_n\} \cap V_2| \geq 3$. This is a contradiction as $G[V_1]$ and $G[V_2]$ are complete graphs. Also, since $G[V_1]$ and $G[V_2]$ are complete graphs, $\overline{G}$ is a bipartite graph and thus contains no induced odd cycle of length at least 5. □

The following lemmas have a key role in this paper.

Lemma 2.5. *Let n be a positive integer and $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a ring, for every $1 \leq i \leq n$. If $EG(R)$ contains no induced odd cycle of length at least* 5*, then $n \leq 4$.*

Proof. Suppose that $n \geq 5$. Then we can easily get

$$(1,0,0,1,0,0,\ldots,0) - (0,1,0,0,1,0,\ldots,0) - (1,0,1,0,0,0,\ldots,0) -$$

$$(0,0,0,1,1,0,\ldots,0) - (0,1,1,0,0,0,\ldots,0) - (1,0,0,1,0,0,\ldots,0)$$

is a cycle of length 5. Thus Theorem 2.3 lead to a contradiction. So $n \leq 4$. □

Lemma 2.6. *Let $R \cong R_1 \times \cdots \times R_n$, $a = (x_1, x_2, \ldots, x_n)$ and $b = (y_1, y_2, \ldots, y_n)$, where n is a positive integer, every R_i is an Artinian local ring and $x_i, y_i \in R_i$, for every $1 \leq i \leq n$.*

(1) $a - b$ is an edge of $EG(R)$ if and only if $x_i y_i \in \mathrm{Nil}(R_i)$, for all $1 \leq i \leq n$.

(2) a is not adjacent to b in $EG(R)$ if and only if $x_j y_j \in \mathrm{U}(R_j)$, for some $1 \leq j \leq n$.

(3) $a - b$ is an edge of $\overline{EG(R)}$ if and only if $x_i y_i \in \mathrm{U}(R_i)$, for some $1 \leq i \leq n$.

(4) a is not adjacent to b in $\overline{EG(R)}$ if and only if $x_j y_j \in \mathrm{Nil}(R_j)$, for all $1 \leq j \leq n$.

Proof. (1) Assume that $x_iy_i \in \text{Nil}(R_i)$, for every $1 \le i \le n$. Then $ab \in \text{Nil}(R)$. This, together with Lemma 2.2, implies that $\text{Ann}_R(ab)$ is an essential ideal of R. Hence a is adjacent to b.

Conversely, let $a - b$ is an edge of $EG(R)$. We claim that $x_iy_i \in \text{Nil}(R_i)$, for every $1 \le i \le n$. Assume to the contrary, $x_jy_j \notin \text{Nil}(R_j)$, for some $1 \le j \le n$. Since R_j is Artinian local ring, we have $x_jy_j \in \text{U}(R_j)$. Hence $\text{Ann}_R(ab) \cap 0 \times \cdots \times 0 \times R_j \times 0 \times \cdots \times 0 = (0, \ldots, 0)$ and thus a is not adjacent to b, a contradiction and so the claim is proved.

(2) Let a is not adjacent to b in $EG(R)$, by Part (1), $x_iy_i \notin \text{Nil}(R_i)$, for some $1 \le i \le n$. This together with the fact R_i is Artinian local implies that $x_iy_i \in \text{U}(R_i)$. Converse is clear.

(3) is obtained by Part 2.

(4) is obtained by Part 1.

□

Lemma 2.7. *Let S_1, S_2, S_3, S_4 be rings such that $S_1 \cong R_1$, $S_2 \cong R_1 \times R_2$, $S_3 \cong R_1 \times R_2 \times R_3$ and $S_4 \cong R_1 \times R_2 \times R_3 \times R_4$, where R_i is a ring, for every $1 \le i \le n$. Then if $EG(S_4)$ is a perfect graph, then $EG(S_3)$, $EG(S_2)$ and $EG(S_1)$ are perfect graphs.*

Proof. As $EG(S_4)$ is a perfect graph, it follows that $EG(S_4)[A]$ is a perfect graph, where $A = \{(x_1, x_2, x_3, x_4) \in S_4 |\ x_4 = 0\}$. It is clear that $EG(S_4)[A] \cong EG(S_3)$. Thus $EG(S_3)$ is a perfect graph. Similarly, $EG(S_2)$ and $EG(S_1)$ are perfect graphs □

We are now in a position to state our main result in this section.

Theorem 2.8. *Let R be an Artinian ring. Then $EG(R)$ is a perfect graph if and only if $|\text{Max(R)}| \le 4$.*

Proof. First let $EG(R)$ be a perfect graph. Since R is an Artinian ring, for some positive integer n, $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \le i \le n$. Now, by Theorem 2.3 and Lemma 2.5, $n \le 4$. To prove the converse, By Theorem 2.3, it is enough to show that $EG(R)$ and $\overline{EG(R)}$ contains no induced odd cycle of length at least 5. By Lemma 2.7, we need to prove the only case that $n = 4$. So let $R \cong R_1 \times R_2 \times R_3 \times R_4$, where R_i is an Artinian local ring. Indeed, we have the following claims:

Claim 1. $EG(R)$ contains no induced odd cycle of length at least 5. We consider the following partition for vertices of $EG(R)$.

$$A = \{\{(x_1,x_2,x_3,x_4) \mid x_i \in \mathrm{Nil}(R_i) \text{ for all } i\} \setminus \{(0,0,0,0)\}\}, \qquad \square$$

$$B = \{(x_1,x_2,x_3,x_4) \mid \text{ for some } i,\ x_i \notin \mathrm{Nil}(R_i)\}.$$

Thus $A \cap B = \varnothing$ and $V(EG(R)) = A \cup B$. Also we consider the following partition for B.

$$B_1 = \{(x,y,z,w) \in B \mid x \in \mathrm{U}(R_1)\},$$

$$B_2 = \{(x,y,z,w) \in B \mid x \in \mathrm{Nil}(R_1) \text{ and } y \in \mathrm{U}(R_2)\},$$

$$B_3 = \{(x,y,z,w) \in B \mid x \in \mathrm{Nil}(R_1),\ y \in \mathrm{Nil}(R_2) \text{ and } z \in \mathrm{U}(R_3)\},$$

$$B_4 = \{(x,y,z,w) \in B \mid x \in \mathrm{Nil}(R_1),\ y \in \mathrm{Nil}(R_2),\ z \in \mathrm{Nil}(R_3) \text{ and } w \in \mathrm{U}(R_4)\}.$$

It is easy to see that $B = \cup_{i=1}^{4} B_i$ and $B_i \cap B_j = \varnothing$ for every $i \neq j$. Note that we denote elements of B_j by $a_i = (x_i, y_i, z_i, w_i)$ for all $1 \leq j \leq 4$.

Now, assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced odd cycle of length at least 5 in $EG(R)$. We consider the following cases.

Case (1) $\{a_1, \ldots, a_n\} \cap A = \varnothing$. Let $a_i \in \{a_1, \ldots, a_n\} \cap A$, for some $1 \leq i \leq n$. Then by Lemma 2.1, a_i is adjacent to all other vertices, a contradiction. Thus $\{a_1, \ldots, a_n\} \cap A = \varnothing$.

Case (2) $\{a_1, \ldots, a_n\} \cap B_4 = \varnothing$. Assume to the contrary and with no loss of generality, $a_1 = (x_1, y_1, z_1, w_1) \in B_4$. Then

$$a_1 \in \mathrm{Nil(R_1)} \times \mathrm{Nil(R_2)} \times \mathrm{Nil(R_3)} \times \mathrm{U(R_4)}.$$

Then the fourth components of a_2 and a_n must be in $\mathrm{Nil(R_4)}$. Since x_3x_1, y_1y_3 and z_1z_3 are nilpotent elements and a_3 is not adjacent to a_1, by part 2 of Lemma 2.6, we conclude that the fourth component of a_3 must be in $\mathrm{U}(R_4)$. This together with a_4 is adjacent to a_3 implies that the fourth component of a_4 is nilpotent element and so $a_4a_1 \in \mathrm{Nil(R)}$. Therefore by Lemma 2.2, we have a_4 is adjacent to a_1, which is a contradiction. So the assertion is proved.

Case (3) $\{a_1, \ldots, a_n\} \cap B_1 = \varnothing$. Assume to the contrary and with no loss of generality, $a_1 = (x_1, y_1, z_1, w_1) \in B_1$. It is easy to see that for every $1 \leq i \leq 4$, there is no adjacency between two vertices of B_i. This together with Case (2) implies that a_n and a_2 are in $B_2 \cup B_3$. We consider the following three subcases.

subcase (1) $\{a_n, a_2\} \subset B_3$. We can let

$$\{a_n, a_2\} \subset \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times R_4.$$

Then the third components of a_1 and a_3 must be in $\mathrm{Nil}(R_3)$. Also since a_n is not adjacent to a_3, by part 2 of Lemma 2.6, the fourth components of a_n and a_3 must be in $\mathrm{U}(R_4)$. This yields

$$a_1 \in \mathrm{U}(R_1) \times R_2 \times \mathrm{Nil}(R_3) \times R_4,$$

$$a_n \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{U}(R_4),$$

$$a_3 \in R_1 \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4).$$

Then the fourth components of a_1 and a_2 must be in $\mathrm{Nil}(R_4)$. Hence we find that

$$a_1 \in \mathrm{U}(R_1) \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{Nil}(R_4),$$

$$a_2 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4).$$

Since a_2 is not adjacent to a_4, the third components of a_4 must be in $\mathrm{U}(R_3)$. This implies that a_4 is not adjacent to a_n and so $n \geq 7$. It is easy to see that the third component of a_5 must be in $\mathrm{Nil}(R_3)$ and so $a_5a_2 \in \mathrm{Nil}(R)$. This implies that $a_5 - a_2$, a contradiction. So in this case the assertion is proved.

Subcase (2) $\{a_n, a_2\} \subset B_2$. By a similar way as used in Subcase 1, we get a contradiction.

Subcase (3). $a_n \in B_2$ and $a_2 \in B_3$. By a similar way as used in Subcase (1), we get a contradiction. Thus $\{a_1, \ldots, a_n\} \cap B_1 = \varnothing$.

By the above cases $\{a_1, \ldots, a_n\} \subseteq B_2 \cup B_3$, but this is contradicts the fact $EG(R)[B_2 \cup B_3]$ is a bipartite graph, and thus $EG(R)$ contains no induced odd cycle of length at least 5.

In Claim 2, A, B and B_i are sets that mentioned in Claim 1.

Claim 2. $\overline{EG(R)}$ contains no induced odd cycle of length at least 5. We show that $\overline{EG(R)}$ contains no induced odd cycle at least 5. Assume to the contrary,

$$a_1 - a_2 - \cdots - a_n - a_1$$

is an induced odd cycle of length at least 5 in $\overline{EG(R)}$. It is clear that $\overline{EG(R)}[A]$ is a null graph and so $\{a_1, \ldots, a_n\} \cap A = \varnothing$. First we show that

$$\{a_1, \ldots, a_n\} \cap B_4 = \varnothing.$$

Assume to the contrary and with no loss of generality, $a_1 = (x_1, y_1, z_1, w_1) \in B_4$. Then $a_1 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4)$. This together with part 3 of Lemma 2.6 implies that the forth components of a_2 and a_n must be in $\mathrm{U}(R_4)$ and so we have

$$a_n \in R_1 \times R_2 \times R_3 \times \mathrm{U}(R_4),$$

$$a_2 \in R_1 \times R_2 \times R_3 \times \mathrm{U}(R_4).$$

It is easy to see that a_2 is adjacent to a_n, a contradiction, and so

$$\{a_1,\dots,a_n\}\cap B_4=\varnothing.$$

Finally to complete the proof, we prove that $\{a_1,\dots,a_n\}\cap B_3=\varnothing$. To get a contradiction, let $a_1=(x_1,y_1,z_1,w_1)\in B_3$. Then

$$a_1\in \mathrm{Nil}(R_1)\times \mathrm{Nil}(R_2)\times \mathrm{U}(R_3)\times R_4.$$

Since a_1-a_n, a_1-a_2 and a_2 is not adjacent to a_n, we consider the following two cases.
Case (1)

$$a_1\in \mathrm{Nil}(R_1)\times \mathrm{Nil}(R_2)\times \mathrm{U}(R_3)\times \mathrm{U}(R_4),$$
$$a_2\in R_1\times R_2\times \mathrm{U}(R_3)\times \mathrm{Nil}(R_4),$$
$$a_n\in R_1\times R_2\times \mathrm{Nil}(R_3)\times \mathrm{U}(R_4).$$

Since a_3 is not adjacent to a_1, the third and the fourth components a_3 must be nilpotent. On the other hand, a_3 is adjacent to a_2. This implies that $x_3x_2\in \mathrm{U}(R_1)$ or $y_2y_3\in \mathrm{U}(R_2)$.
First suppose that $x_3x_2\in \mathrm{U}(R_1)$. Now, we know that

$$a_3\in \mathrm{U}(R_1)\times R_2\times \mathrm{Nil}(R_3)\times \mathrm{Nil}(R_4),$$
$$a_2\in \mathrm{U}(R_1)\times R_2\times \mathrm{U}(R_3)\times \mathrm{Nil}(R_4).$$

This together with that a_3 is adjacent to a_4 implies that $x_3x_4\in \mathrm{U}(R_1)$ or $y_3y_4\in \mathrm{U}(R_2)$. If $x_3x_4\in \mathrm{U}(R_1)$, then we have $x_2x_4\in \mathrm{U}(R_1)$. Therefore a_4 is adjacent to a_2, which is a contradiction. Thus we conclude that $y_3y_4\in \mathrm{U}(R_2)$. This yields

$$a_3\in \mathrm{U}(R_1)\times \mathrm{U}(R_2)\times \mathrm{Nil}(R_3)\times \mathrm{Nil}(R_4),$$

$$a_4\in \mathrm{Nil}(R_1)\times \mathrm{U}(R_2)\times R_3\times R_4.$$

Since a_4 is not adjacent to a_1, we have

$$a_4\in \mathrm{Nil}(R_1)\times \mathrm{U}(R_2)\times \mathrm{Nil}(R_3)\times \mathrm{Nil}(R_4).$$

Thus a_4 is not adjacent to a_n and so $n\geq 7$. On the other hand, since a_4-a_5, the second components of a_5 must be unit and so a_5 is adjacent to a_2, which is a contradiction.

So, suppose that $y_2y_3\in \mathrm{U}(R_2)$. Similarly, we get a contradiction. Thus in this case the assertion is proved.

Case (2)

$$a_1\in \mathrm{Nil}(R_1)\times \mathrm{Nil}(R_2)\times \mathrm{U}(R_3)\times \mathrm{U}(R_4),$$

$$a_2 \in R_1 \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4),$$

$$a_n \in R_1 \times R_2 \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4).$$

By similar argument that of case (1), we get a contradiction.
This means that $\{a_1,\ldots,a_n\} \subseteq B_2 \cup B_1$. Clearly, $\overline{EG(R)}[B_1]$, $\overline{EG(R)}[B_2]$ are complete, and thus by Corollary 2.4, $\overline{EG(R)}[B_1 \cup B_2]$ is a perfect graph, a contradiction. Hence $\overline{EG(R)}$ contain no induced odd cycle of length at least 5. Therefore by Claim 1, Claim 2 and Theorem 2.3, $EG(R)$ is a perfect graph.

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