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ON AN APPROACH TO STUDYING THE STABILITY OF SHOCK WAVES IN A VISCOUS GAS

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In the article, *a modified initial-boundary value problem* on stability of shock waves in a viscous gas is constructed and studied.

1. Introduction.

As is known, two approaches are used for description of movements with shock waves in various models of continuum mechanics with dissipation. Within the *widely used structural approach*, the shock wave is presented as a narrow transitional zone with continuously varying parameters. Another, also widely spread, approach is based on the assumption that shock waves can be presented as *strong discontinuity surfaces*. For example, in [8], plane shock waves have been studied; and influence of small viscosity on perturbations propagation has been estimated under assumption that the width of the transitional zone is negligibly small. By this assumption, the problem on perturbations propagation has been reduced in [8] to *a linear initial-boundary value problem* with linearized boundary conditions on the shock front as well as in the case of the inviscid gas.

However, it has been shown in [2] by one of the authors, A. M. Blokhin, that such an approach is not admissible for description of shock waves in models of continuum mechanics with dissipation. He has studied well-posedness

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of the above mentioned linear initial-boundary value problem obtained by linearization of the non stationary Navier-Stokes equations and the strong discontinuity equation with respect to a piecewise constant solution. This solution describes the following regime of the viscous gas flow: the supersonic stationary viscous flow ($x < 0$) is separated from the subsonic flow ($x > 0$) by a strong discontinuity surface (a shock wave with the equation $x = 0$). It has been stated in [2] that the shock wave is *unstable*. In order to prove the instability, exponentially growing in time specific solutions to the linear initial-boundary value problem have been constructed.

We note that, from the mathematical point of view, these solutions are, in fact, the Hadamard type examples which show the ill-posedness of this problem. From the physical point of view, existence of such solutions means that the described above stationary regime of the viscous gas flow with a shock wave can not be realized and, consequently, can not be found by *the stabilization method*.

In this article, the so-called *modified initial-boundary value problem* is discussed, for which the stationary regime of the viscous gas flow with a shock wave is asymptotically stable (by Lyapunov) and can be determined (numerically, for example) with the stabilization method.

2. Preliminaries.

We write down a 1-D mathematical model of the viscous non heat conducting gas. This model is derived from the Navier-Stockes equations of the compressible liquid:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + P) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho(e_0 + \frac{u^2}{2})) + \frac{\partial}{\partial x}((\rho(e_0 + \frac{u^2}{2}) + P)u) = 0.$$

Here ρ denotes the density; u, the velocity of the gas; $P = p - \sigma$, the stress; p, the pressure; $\sigma = (\frac{4}{3}\eta + \zeta)\frac{\partial u}{\partial x}$; e_0 is the internal energy; $V = 1/\rho$; η and ζ are the first and second viscosity coefficients (usually, they are functions in ρ and s); s is the mass entropy.

We complete (2.1) with the state equation $e_0 = e_0(\rho, s)$ and the relations $T = \frac{\partial e_0}{\partial s}$ and $p = \rho^2 \frac{\partial e_0}{\partial \rho}$ which follow from the first thermodynamical law. Now we can regard (2.1) as a closed system of viscous conservation laws in (p, u, s).

Reasoning in a standard way (see [4], [7]), we write down jump conditions:

$$[j] = 0, \quad [P] + [u]j = 0, \tag{2.2}$$

$$[e_0 + \frac{u^2}{2}]j + [Pu] = 0.$$

The surface of strong discontinuity is given by the equation $F(t) - x = 0$; $[g] = (g - g_\infty)$ is the jump of g on the discontinuity surface; the subindex ∞ stands for the values ahead of the discontinuity (as $F(t) - x \to +0$); $j = \rho(u - F_t)$; F_t is the velocity of the propagating discontinuity.

Let the strong discontinuity be a stationary shock wave with the equation $x = 0$; as is easily seen, (2.1) has a piecewise constant solution:

$$u = \hat{u}_\infty, \quad \rho = \hat{\rho}_\infty, \quad s = \hat{s}_\infty \quad \text{for} \quad x < 0, \tag{2.3}$$

$$u = \hat{u}, \quad \rho = \hat{\rho}, \quad s = \hat{s} \quad \text{for} \quad x > 0;$$

the constants $\hat{u}_\infty$, $\hat{\rho}_\infty$, $\hat{s}_\infty$, $\hat{u}$, $\hat{\rho}$, $\hat{s}$ are connected by the jump relations (2.2). For the shock wave, $[\hat{\rho}] \neq 0$ and $\hat{j} = \hat{\rho}\hat{u} \neq 0$; so, (2.2) can be rewritten similarly to the Rankine-Hugoniot relations in gas dynamics:

$$\hat{\rho}\hat{u} = \hat{\rho}_\infty \hat{u}_\infty,$$

$$(\hat{u} - \hat{u}_\infty)^2 + (\hat{p} - \hat{p}_\infty)(\hat{V} - \hat{V}_\infty) = 0, \tag{2.4}$$

$$(\hat{e}_0 - \hat{e}_{0\infty}) + \frac{(\hat{p} + \hat{p}_\infty)}{2}(\hat{V} - \hat{V}_\infty) = 0.$$

Parameters of the flows ahead of and behind the shock wave satisfy the following inequalities:

$$\hat{u}_\infty > \hat{c}_\infty > 0, \quad \hat{\rho}_\infty > 0, \quad \hat{c} > \hat{u} > 0, \quad \hat{\rho} > 0, \tag{2.5}$$

the sound speeds ahead of and behind the shock wave are

$$\hat{c}_\infty = \sqrt{\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial e_0}{\partial \rho})(\hat{\rho}_\infty, \hat{s}_\infty)}, \quad \hat{c} = \sqrt{\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial e_0}{\partial \rho})(\hat{\rho}, \hat{s})},$$

and $\hat{p}_\infty = \hat{\rho}_\infty^2 \frac{\partial e_0}{\partial \rho}(\hat{\rho}_\infty, \hat{s}_\infty)$, $\hat{p} = \hat{\rho}^2 \frac{\partial e_0}{\partial \rho}(\hat{\rho}, \hat{s})$, $\hat{V}_\infty = 1/\hat{\rho}_\infty$, $\hat{e}_{0\infty} = e_0(\hat{\rho}_\infty, \hat{s}_\infty)$, $\hat{V} = 1/\hat{\rho}$, $\hat{e}_0 = e_0(\hat{\rho}, \hat{s})$. We assume that the state equation $e_0 = e_0(\rho, s)$ meet the requirements for the so-called normal gas (see [6]). This means (see [4], [6]) that the inequalities (2.5) are fulfilled together with $\hat{p} > \hat{p}_\infty$, $\hat{\rho} > \hat{\rho}_\infty$, $\hat{u}_\infty > \hat{u}$, and $\hat{s} > \hat{s}_\infty$.

From the physical point of view, we have the shock wave which separates the supersonic coming stationary flow and the subsonic stationary flow behind the shock wave in a viscous gas.

Next, we linearize (2.1) and jump conditions (2.2) about the piecewise solution (2.3) and obtain the linear mixed problem to find small perturbations of the vector (p, u, s) and a small shift of shock wave front (we denote them by p, u, s, and F again). Without the loss of generality, small perturbations of the entropy s_∞ for $x < 0$ can be equaled to zero. So, we seek the solution to the systems for $x > 0$ and $x < 0$:

$$\begin{cases} M^2 Lu + p_x = 2M^2 u_{xx}, \\ Lp + u_x = 0, \\ Ls = 0; \end{cases} \tag{2.6}$$

$$\begin{cases} M_\infty^2 L_\infty u_\infty + (p_\infty)_x = 2\mu M_\infty^2 (u_\infty)_{xx}, \\ L_\infty p_\infty + (u_\infty)_x = 0; \end{cases} \tag{2.7}$$

which satisfies the boundary conditions at $x = 0$

$$\begin{cases} u + dp - 2\hat{d}u_x = \hat{v}\{u_\infty + d_\infty p_\infty - 2\hat{d}_\infty \mu (u_\infty)_x\}, \\ \nu p + \hat{N}s - 2\hat{n}uu_x = \hat{v}\{\nu_\infty p_\infty + 2\hat{n}u\mu(u_\infty)_x\}, \\ F' = \hat{\mu}\{u + p - u_\infty - p_\infty - \hat{N}s\}. \end{cases} \tag{2.8}$$

Here p,u, s, p_∞, and u_∞ are related to the characteristic parameters $\hat{\rho}\hat{c}^2$, $\hat{u}$, $\hat{s}$, $\hat{\rho}_\infty \hat{c}_\infty^2$, and $\hat{u}_\infty$; the spatial variable x and the time t are related to characteristic length $\hat{l}$ and time $\hat{l}/\hat{u}$. Formulation of the problem (2.6)-(2.8) does not contain a characteristic length; no wonder that the final result does not depend on the choice of the value $\hat{l}$. Next, $L = \tau + \xi$, $L_\infty = \frac{1}{\hat{v}}\tau + \xi$, $\tau = \frac{\partial}{\partial t}$, $\xi = \frac{\partial}{\partial x}$ are differential operators; $\hat{v} = \frac{\hat{u}_\infty}{\hat{u}} > 1$; $M_\infty > 1$, $M < 1$ are the Mach numbers ahead of and behind the shock wave, $d = \frac{1+M^2}{2M^2} + \frac{\beta^2}{2M^2}\hat{L}$, $\beta^2 = 1 - M^2$, $\hat{d} = \frac{1+\hat{L}}{2}$, $d_\infty = \frac{M_\infty^2+1}{2M_\infty^2} + \frac{\beta_\infty}{2M_\infty^2}\hat{L}$; $\beta_\infty^2 = M_\infty^2 - 1$, $\hat{d}_\infty = \frac{1-\hat{L}}{2}$, $\nu = \frac{\beta^2}{M^2}\hat{L}$, $\hat{\nu} = \hat{L}$, $\nu_\infty = \frac{\beta_\infty^2}{M_\infty^2}\hat{L}$; $\hat{\mu} = \frac{\hat{v}}{\hat{v}-1} > 0$, $\hat{L} = \frac{1}{1-\hat{D}}$, $\hat{D} = \frac{2\hat{T}\hat{s}}{\hat{u}^2(\hat{v}-1)\hat{N}}$, $\hat{N} = -\frac{\hat{s}(e_0)_{Vs}(\hat{\rho},\hat{s})}{\hat{V}(e_0)_{VV}(\hat{\rho},\hat{s})}$, $\hat{T} = (e_0)_s(\hat{\rho}, \hat{s})$, $\mu = \frac{r_\infty}{r}$, $r = \frac{4}{3R_1} + \frac{1}{R_2}$, $r_\infty = \frac{4}{3R_{1\infty}} + \frac{1}{R_{2\infty}}$, $R_{1,2,1\infty,2\infty}$ are the Reinolds numbers: $R_1 = \frac{\hat{\rho}\hat{u}\hat{l}}{\hat{\eta}}$, $R_2 = \frac{\hat{\rho}\hat{u}\hat{l}}{\hat{\zeta}}$, and so on; $\hat{\eta} = \eta(\hat{\rho}, \hat{s})$, $\hat{\zeta} = \zeta(\hat{\rho}, \hat{s})$. We note that for the normal gas $(e_0)_V < 0$, $(e_0)_s > 0$, $(e_0)_{VV} > 0$, $(e_0)_{VV}(e_0)_{ss} - (e_0)_{Vs}^2 > 0$.

Remark 2.1. For the polytropic gas with the adiabatic exponent γ the coefficient $\hat{L}$ is (see [2]):

$$\hat{L} = -\frac{\gamma - 1}{\gamma + 1}(1 - \frac{1}{M_\infty^2}), \quad \gamma > 1, \tag{2.9}$$

i.e. $-1 < \hat{L} < 0$.

Remark 2.2. If we separate out a sub-problem on the function s

$$\begin{cases} Ls = 0 \quad \text{for} \quad x > 0, \\ \hat{N}s = 2\hat{\nu}u_x - \nu p + \hat{v}(\nu_\infty p_\infty + 2\hat{\nu}\mu(u_\infty)_x) \quad \text{at} \quad x = 0 \end{cases} \tag{2.10}$$

and the equation on the function $F(t)$

$$F' = \hat{\mu}\{u + p - u_\infty - p_\infty - \hat{N}s\}|_{x=0}, \tag{2.11}$$

the problem (2.6)–(2.8) can be slightly simplified.

The ill-posedness of the problem (2.6)–(2.8) has been proven in [2] by constructing the ill-posedness examples of the Hadamard type. With this purpose, exponentially growing in time special solutions to (2.6)–(2.8) have been found. The revealed instability proves that the stationary regime of the viscous gas flow described in Introduction can not be calculated with the stabilization method. It has also been shown in [2] that the ill-posedness of (2.6)–(2.8) follows from the fact that the number of independent parameters which determine an arbitrary small perturbation of the discontinuity is greater than the number of linearized boundary conditions (2.8) on the discontinuity.

Using the a priori information on the stationary regime, we can derive additional boundary conditions, modify the problem (2.6)–(2.8), and obtain a mixed problem for which the trivial solution becomes *asymptotically stable (by Lyapunov)*.

We suggest the following modification of the problem (2.6)-(2.8). We seek solutions to the systems for $x > 0$

$$\begin{cases} M^2 Lu + p_x = 2M^2 u_{xx}, \\ Lp + u_x = 0; \end{cases} \tag{2.12}$$

and for $x < 0$

$$\begin{cases} M_\infty^2 L_\infty u_\infty + (p_\infty)_x = 2\mu M_\infty^2 (u_\infty)_{xx}, \\ L_\infty p_\infty + (u_\infty)_x = 0; \end{cases} \tag{2.13}$$

satisfying the boundary conditions at $x = 0$:

$$\begin{cases} u + dp - 2\hat{d}u_x = \hat{v}(u_\infty + d_\infty p_\infty), \\ \underline{(u_\infty)_x = 0}, \quad \underline{p_x = 0}. \end{cases} \tag{2.14}$$

Here underlined are the additional boundary conditions. It is important to remind once again that we choose additional conditions which are fulfilled on the stationary solution (2.3).

Remark 2.3. The problem (2.10) can be rewritten as follows:

$$\begin{cases} Ls = 0 \quad \text{for} \quad x > 0, \\ \hat{N}s = 2\hat{v}u_x - vp + \hat{v}v_\infty p_\infty \quad \text{at} \quad x = 0. \end{cases} \tag{2.10$'$}$$

Next, without the loss of generality, we can take $u_\infty(t, x) \equiv 0$, $p_\infty(t, x) \equiv 0$ for $x < 0$, $t > 0$. Indeed, we rewrite (2.13) in the form:

$$A^\infty U_t^\infty + B^\infty U_x^\infty = A_1^\infty U_{xx}^\infty, \tag{2.13$'$}$$

where $U^\infty = \begin{pmatrix} u_\infty \\ p_\infty \end{pmatrix}$,

$$A^\infty = \begin{pmatrix} \frac{M_\infty^2}{\hat{v}} & 0 \\ 0 & \frac{1}{\hat{v}} \end{pmatrix}, \quad B^\infty = \begin{pmatrix} M_\infty^2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_1^\infty = \begin{pmatrix} 2\mu M_\infty^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Multiplying (2.13$'$) by $2U^\infty$, after simple calculations we have

$$(U^\infty, A^\infty U^\infty)_t + (U^\infty, B^\infty U^\infty)_x - 2(U^\infty, A_1^\infty U_x^\infty)_x + \\ + 2\,(U_x^\infty, A_1^\infty U_x^\infty) = 0. \tag{2.15}$$

Next, we integrate (2.15) by x (from $-\infty$ to 0), provided that

$$|U^\infty|, |U_x^\infty| \to 0 \quad \text{as} \quad x \to -\infty,$$

account the additional boundary condition $(u_\infty)_x = 0$ at $x = 0$, and arrive at

$$\frac{d}{dt} I^\infty(t) + (U^\infty, B^\infty U^\infty)|_{x=0} + 4\mu M_\infty^2 \int_{R_-^1} (u_\infty(t, x))_x^2 dx = 0, \quad t > 0. \tag{2.16}$$

Here

$$I^{\infty}(t) = \int\limits_{R^1_-} (U^{\infty}, A^{\infty}U^{\infty})dx, \quad R^1_- = \{x | x < 0\}.$$

By $M_\infty > 1$, the matrix B^∞ is positive definite. Therefore, (2.16) implies the inequality

$$I^{\infty}(t) \le I^{\infty}(0) \quad \text{for} \quad t > 0.$$

So, if the initial data for $u_\infty(t, x)$ and $p_\infty(t, x)$ are trivial, then for $t > 0$:

$$u_\infty \equiv 0, \quad p_\infty \equiv 0.$$

By this, without the loss of generality, we can consider the following problem instead of (2.12)–(2.14). We seek the solution to the system for $x > 0$:

$$\text{(2.12)} \qquad \begin{cases} M^2 Lu + p_x = 2M^2 u_{xx}, \\ Lp + u_x = 0, \end{cases}$$

satisfying the boundary conditions at $x = 0$:

$$\text{(2.14}'\text{)} \qquad \begin{cases} u + dp - 2\hat{d}u_x = 0, \\ \underline{p_x = 0}. \end{cases}$$

The problem (2.10′) transforms into the problem:

$$\text{(2.10}''\text{)} \qquad \begin{cases} Ls = 0 \quad \text{for} \quad x > 0, \\ \hat{N}s = 2\hat{\nu}u_x - \nu p \quad \text{at} \quad x = 0, \end{cases}$$

while (2.11) takes the form:

$$\text{(2.11}'\text{)} \qquad F' = \hat{\mu}\{u + p - \hat{N}s\}|_{x=0}.$$

In what follows we use the so-called *auxiliary problem*. To derive it, we follow [2] and introduce the potential $\varphi = \varphi(t, x)$:

$$u = \varphi_x, \quad p = 2M^2\varphi_{xx} - M^2 L\varphi.$$

Then the first equation in (2.12) is obviously fulfilled and the second one implies the equation on φ for $t > 0$, $x > 0$

$$\text{(2.17)} \qquad \{M^2 L^2 \varphi - \varphi_{xx} - 2M^2 L\varphi_{xx}\} = 0.$$

The following boundary conditions are valid at $x = 0$ (see (2.14′)):

$$\text{(2.18)} \qquad \begin{cases} \varphi_t = \hat{a}\varphi_x + \hat{b}\varphi_{xx}, \\ L\varphi_x = 2\varphi_{xxx}, \end{cases}$$

where $\hat{a} = \frac{\beta^2(1-\hat{L})}{2dM^2}$, $\hat{b} = \frac{1-\hat{L}}{d}$.

Remark 2.4. Now we rewrite the second boundary condition in (2.18). We consider (2.17) on the boundary $x = 0$ and derive φ_{xxx} at $x = 0$. Then we substitute the obtained expression into the second boundary condition, account the first boundary condition differentiated by t, and finally obtain

$$\varphi_{xt} = d\varphi_{xx} + d_1\varphi_{xxt} \quad \text{at} \quad x = 0,$$

where $d_1 = 1 + \hat{L}$, i.e. the boundary conditions (2.18) at $x = 0$ turns into:

$$\begin{cases} \varphi_t = \hat{a}\varphi_x + \hat{b}\varphi_{xx}, \\ \varphi_{xt} = d\varphi_{xx} + d_1\varphi_{xxt}. \end{cases} \tag{2.17$'$}$$

We call (2.16), (2.17′) by *the auxiliary problem*.

3. Asymptotical stability of the trivial solution to the problem (2.12), (2.14′).

Is is convenient to rewrite (2.17) first in the form

$$\{M^2\tilde{L}^2 - \xi^2 - M^2\varsigma^2\}\varphi = 0 \tag{3.1}$$

and then, using some special operators L_1, L_2 (see [1]), as follows:

$$\{M^2L_1^2 - L_2^2 - \frac{M^2}{\beta^2}\varsigma^2\}\varphi = 0. \tag{3.1$'$}$$

Here $\varsigma = \frac{\partial^2}{\partial x^2} = \xi^2$ (the operators τ, ξ are given above), $\tilde{L} = L_1 + L_2 = \tilde{T} + \xi$, $\tilde{T} = \tau - \varsigma$, $L_1 = \frac{1}{\beta^2}\tilde{T}$, $L_2 = \xi - M^2L_1$, i.e. $\xi = L_2 + M^2L_1$, $\tau = \beta^2L_1 + \varsigma$.

We also form the vectors

$$Y = \begin{pmatrix} ML_1\varphi \\ L_2\varphi \\ \frac{M}{\beta}\varsigma\varphi \end{pmatrix}, X = \begin{pmatrix} \tau\varphi \\ \xi\varphi \\ \varsigma\varphi \end{pmatrix} = \begin{pmatrix} \Lambda \\ \varsigma\varphi \end{pmatrix}, \Lambda = \begin{pmatrix} \tau\varphi \\ \xi\varphi \end{pmatrix}.$$

Then

$$Y = \mathcal{N}X, \tag{3.2}$$

where

$$\mathcal{N} = \frac{M}{\beta^2}\begin{pmatrix} 1 & 0 & -1 \\ -M & \frac{\beta^2}{M} & M \\ 0 & 0 & \beta \end{pmatrix}, \quad \text{moreover} \ \det\mathcal{N} = \beta \neq 0.$$

We follow ideas in [1] and replace the equation (2.17) by a system in Y. It suffices to notice that if the potential $\varphi = \varphi(t, x)$ satisfies (2.17), then the vector Y satisfies the system

$$ME \cdot L_1 Y + Q \cdot L_2 Y + \frac{M}{\beta} R \cdot \varsigma Y = 0 \tag{3.3}$$

or

$$D_1 \cdot \tau Y + Q \cdot \xi Y = H \cdot \varsigma Y. \tag{3.3$'$}$$

Here

$$E = \begin{pmatrix} 1 & -m_1 & -n_1 \\ -m_1 & 1 & 0 \\ -n_1 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} m_1 & -1 & 0 \\ -1 & m_1 & n_1 \\ 0 & n_1 & -m_1 \end{pmatrix},$$

$$R = \begin{pmatrix} n_1 & 0 & -1 \\ 0 & -n_1 & m_1 \\ -1 & m_1 & n_1 \end{pmatrix}, \quad D_1 = \frac{M}{\beta^2}(E - MQ),$$

$$H = \frac{M}{\beta^2}(E - MQ - \beta R),$$

m_1 and n_1 are real constants.

As is known (see [1]), the matrices E, Q, R can be presented as follows:

$$\begin{cases} E = T_0^* \{I_2 \otimes \mathcal{H}\} T_0, \\ Q = T_0^* \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathcal{H} \right\} T_0, \\ R = T_0^* \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathcal{H} \right\} T_0, \end{cases} \tag{3.4}$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 1 + n_1 & m_1 \\ m_1 & 1 - n_1 \end{pmatrix},$$

moreover, $\mathcal{H} > 0$ if $1 - m_1^2 - n_1^2 > 0$ (below we restrict ourselves with the case $m_1 = n_1 = 0$, so, this condition is certainly fulfilled);

$I_2 \otimes \mathcal{H}$ is the Kronecker product of the matrices I_2 and $\mathcal{H}$ and so on (see [5] on the Kronecker matrix product).

In a view of (3.4), we easily find that

$$\begin{cases} D_1 = \frac{M}{\beta^2} T_0^* \left\{ \begin{pmatrix} 1 & -M \\ -M & 1 \end{pmatrix} \otimes \mathcal{H} \right\} T_0, \\ H = \frac{M}{\beta^2} T_0^* \left\{ \begin{pmatrix} 1-\beta & -M \\ -M & 1+\beta \end{pmatrix} \otimes \mathcal{H} \right\} T_0, \end{cases} \tag{3.5}$$

moreover, $D_1 > 0$ and $H \geq 0$.

We multiply (3.3′) by the vector $2Y$ and, after simple calculations, obtain

$$(Y, D_1 Y)_t + (Y, QY)_x - 2(Y, HY_x)_x + 2(Y_x, HY_x) = 0. \tag{3.6}$$

Then we integrate (3.6) by x (from 0 to $+\infty$), provided that

$$|Y|, |Y_x| \to 0 \quad \text{as} \quad x \to +\infty,$$

and have

$$\frac{d}{dt}\{\int\limits_{R^1_+} (Y, D_1 Y)dx\} + \{2(Y, HY_x) - (Y, QY)\}|_{x=0} + \tag{3.7}$$

$$+2\int\limits_{R^1_+} (Y_x, HY_x)dx = 0, \quad R^1_+ = \{x | x > 0\}.$$

Let

$$J(t) = \int\limits_{R^1_+} (Y, D_1 Y)dx.$$

Then, by (3.2),

$$J(t) = \int\limits_{R^1_+} (X, DX)dx = \int\limits_{R^1_+} (V, \tilde{D}V)dx.$$

Here

$$D = \mathcal{N}^* D_1 \mathcal{N},$$

$$V = \begin{pmatrix} U \\ u_x \end{pmatrix}, \ U = \begin{pmatrix} u \\ p \end{pmatrix}, \ X = \mathcal{N}_1 V,$$

$$\mathcal{N}_1 = \begin{pmatrix} -1 & -\frac{1}{M^2} & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{moreover,} \quad \det \mathcal{N}_1 = \frac{1}{M^2} \neq 0,$$

$$\tilde{D} = \mathcal{N}_0^* D_1 \mathcal{N}_0, \ \mathcal{N}_0 = \mathcal{N}\mathcal{N}_1 = \frac{M}{\beta^2} \begin{pmatrix} -1 & -\frac{1}{M^2} & 1 \\ \frac{1}{M} & \frac{1}{M} & -M \\ 0 & 0 & \beta \end{pmatrix}.$$

Obviously, $\tilde{D} > 0$. Next,

$$(Y_x, HY_x) = (\xi X, H_1 \xi X),$$

where

$$H_1 = \mathcal{N}^* H \mathcal{N} = \frac{M^3}{\beta^4} \begin{pmatrix} & & 0 \\ & h & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

implies

$$(Y_x, HY_x) = \frac{M^3}{\beta^4} (\xi \Lambda, h \xi \Lambda). \tag{3.8}$$

Here

$$h = \mathcal{P}^* \mathcal{H} \mathcal{P}, \tag{3.9}$$

$$\mathcal{P} = \begin{pmatrix} -\sqrt{\frac{1-\beta}{2}} & \frac{\beta}{M}\sqrt{\frac{1+\beta}{2}} \\ -\sqrt{\frac{1+\beta}{2}} & -\frac{\beta}{M}\sqrt{\frac{1-\beta}{2}} \end{pmatrix}, \quad \text{i.e.} \quad h > 0.$$

Now we consider the aggregate $\{2(Y, HY_x) - (Y, QY)\}$ at $x = 0$. Accounting (3.9), we have

$$2(Y, HY_x) - (Y, QY) = 2\frac{M^3}{\beta^4}(\Lambda, h\xi\Lambda) + 2ML_1\varphi L_2\varphi = \tag{3.10}$$

$$= 2\frac{M^3}{\beta^4}\{\hat{a}\varphi_x\varphi_{tx} + \hat{b}\varphi_{xx}\varphi_{tx} + \frac{\beta^2}{M^2}\varphi_x\varphi_{xx}\} +$$

$$+ 2ML_1\varphi L_2\varphi = 2\frac{M^3}{\beta^4}\{\hat{a}\varphi_x\varphi_{xt} + \hat{b}d_1\varphi_{xx}\varphi_{xxt}$$

$$+\hat{b}d\varphi_{xx}^2 + \frac{\beta^2}{M^2}\varphi_x\varphi_{xx}\} + 2ML_1\varphi L_2\varphi.$$

We rewrite the first boundary condition in (2.17′) in terms of the operators L_1, L_2:

$$L_1\varphi = \kappa L_2\varphi + \kappa_0\varsigma\varphi, \tag{3.11}$$

where

$$\kappa = \frac{1-\hat{L}}{1+\hat{L}}, \quad \kappa_0 = \frac{M^2\kappa - 1}{\beta^2}.$$

In a view of (3.11), the expression (3.10) turns into

$$2(Y, HY_x) - (Y, QY) = 2\frac{M^3}{\beta^4}\{\hat{a}\varphi_x\varphi_{xt} + \hat{b}d_1\varphi_{xx}\varphi_{xxt} + \tag{3.10′}$$

$$+ (\hat{b}d + \beta^2\kappa_0)(\varsigma\varphi)^2 + \frac{\beta^2}{M^2}(M^2\kappa + 1)L_2\varphi\varsigma\varphi\} +$$

$$+ 2M\kappa(L_2\varphi)^2 + 2M\kappa_0 L_2\varphi\varsigma\varphi =$$

$$= \{2\frac{M^3}{\beta^4}\hat{a}\varphi_x\varphi_{xt} + 2\frac{M^3}{\beta^4}\hat{b}d_1\varphi_{xx}\varphi_{xxt}\} +$$

$$+ 2\frac{M^3}{\beta^4}(M^2\kappa - \hat{L})(\varsigma\varphi)^2 + \frac{4M^3}{\beta^2}\kappa L_2\varphi\varsigma\varphi + 2M\kappa(L_2\varphi)^2.$$

It is easy to check that the quadratic form in the variables $\varsigma\varphi$, $L_2\varphi$ in the right-hand side of (3.10′) is positive-definite if $\hat{L} < 0$ (see *Remark 2.1*). At $x = 0$ by (3.11) we have

$$\xi\varphi = L_2\varphi + M^2L_1\varphi = (M^2\kappa + 1)L_2\varphi + M^2\kappa_0\varsigma\varphi.$$

In fact, we have already used this relation while deriving (3.10′). So, this quadratic form can be rewritten in terms of $\xi\varphi$ and $\varsigma\varphi$; and (3.10′) turns into

$$2(Y, HY_x) - (Y, QY) = \frac{\partial}{\partial t}\{\frac{M^3\hat{a}}{\beta^4}\varphi_x^2 + \frac{M^3\hat{b}d_1}{\beta^4}\varphi_{xx}^2\} + \tag{3.10″}$$

$$+ 2M\kappa(M^2\kappa + 1)^2(\varphi_x + \frac{2M^2}{\beta^2}\varphi_{xx})^2 - 2\frac{M^3}{\beta^4}\hat{L}\varphi_{xx}^2.$$

Accounting (3.8) and (3.10″), we derive from (3.7)

$$(3.7') \qquad \frac{d}{dt}\tilde{J}(t) + \{\frac{2M\kappa}{(M^2\kappa+1)^2}(\varphi_x + \frac{2M^2}{\beta^2}\varphi_{xx})^2 - 2\frac{M^3}{\beta^4}\hat{L}\varphi_{xx}^2\}_{x=0} + $$
$$+ 2\frac{M^3}{\beta^4}\int\limits_{R_+^1}(\xi\Lambda, h\xi\Lambda)dx = 0.$$

Here $\tilde{J}(t) = J(t) + \frac{M^3}{\beta^4}(\hat{a}\varphi_x^2 + \hat{b}d_1\varphi_{xx}^2)|_{x=0}$, $\xi\Lambda = \begin{pmatrix}\tau\xi\varphi\\ \varsigma\varphi\end{pmatrix} = \begin{pmatrix}u_t\\ u_x\end{pmatrix}$.

We differentiate the second equation in (2.12) by x and reduce it to the form

$$Lp_x + \frac{1}{2M^2}p_x + \frac{1}{2}Lu = 0.$$

Hence

$$(3.12) \qquad \frac{d}{dt}(\int\limits_{R_+^1} p_x^2 dx) + \int\limits_{R_+^1}(p_x^2/M^2 + Lu\cdot p_x)dx = 0.$$

With (3.7′) and (3.12) in hands, we finally arrive at the desired relation

$$(3.13) \quad \frac{dW(t)}{dt} + \{\frac{2M\kappa}{(M^2\kappa+1)^2}(u(t,0) + \frac{2M^2}{\beta^2}u_x(t,0))^2 - 2\frac{M^3}{\beta^4}\hat{L}u_x^2(t,0)\} + $$
$$+ \int\limits_{R_+^1}\left\{2\frac{M^3}{\beta^4}\left(\begin{pmatrix}u_t\\ u_x\end{pmatrix}, h\begin{pmatrix}u_t\\ u_x\end{pmatrix}\right) + \varepsilon\frac{p_x^2}{M^2} + \varepsilon Lu\cdot p_x\right\}dx = 0,$$

where $W(t) = J(t) + \frac{M^3}{\beta^4}(\hat{a}u^2(t,0) + \hat{b}d_1u_x^2(t,0)) + \varepsilon\int\limits_{R_+^1} p_x^2 dx$,

$$J(t) = \int\limits_{R_+^1}\left(\begin{pmatrix}U\\ u_x\end{pmatrix}, \tilde{D}\begin{pmatrix}U\\ u_x\end{pmatrix}\right)dx,\ U = \begin{pmatrix}u\\ p\end{pmatrix},$$

and $\hat{L} < 0$ (see *Remark 2.1*), $\varepsilon > 0$ is a constant such that the quadratic form under the integral sign in the last summand in (3.13) is positive definite.

It follows from (3.13) that

$$\frac{dW(t)}{dt} \le 0,$$

i.e.

$$W(t) \leq W(0) \quad \text{for any } t > 0. \tag{3.14}$$

The a priori estimation (3.14) implies the Lyapunov stability of the trivial solution to (2.12), (2.14′). Actually, without the loss of generality, we can suppose that the function $W(t)$ is strictly decreasing, i.e.

$$\frac{dW(t)}{dt} < 0.$$

Indeed, if there exists a point $t = t_* < \infty$ such that $W'(t_*) = 0$ (we take the very first point), then, by (3.13), we obtain

$$u(t_*, 0) = u_x(t_*, 0) = 0,$$

$$u_x(t_*, x) \equiv 0, \quad p_x(t_*, x) \equiv 0,$$

i.e. $u(t_*, x) \equiv 0$, $p(t_*, x) \equiv 0$.

Next, by (3.14),

$$u(t, x) \equiv 0, \quad p(t, x) \equiv 0 \quad \text{for any} \quad t > t_*.$$

So, generally speaking, the positive function $W(t)$ is monotone decreasing and has no asymptotes except for $W \equiv 0$ in the class of considered functions u, p, i.e. $W(t) \to +0$ as $t \to +\infty$. The last statement means that, because of the structure of $W(t)$),

$$||U(t)||_{W_2^1(R_+^1)} \to 0, \quad |u(t, 0)| \to 0, \quad |u_x(t, 0)| \to 0$$

as $t \to +\infty$.

Remark 3.1. We note that (2.14′) implies

$$|p(t, 0)| \to 0 \quad \text{as} \quad t \to +\infty.$$

By *Remark 2.2* and (2.11′), velocity of the moving shock front tends to zero, i.e.

$$|F'| \to 0 \quad \text{as} \quad t \to +\infty.$$

4. Concluding remarks.

In the article, simple additional conditions which allow to prove the asymptotical stability (by Lyapunov) for the stationary regime of the viscous gas flow with a shock wave are suggested. Another variant of additional conditions is discussed in [3].

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