GENERALIZED REGULAR GENUS FOR MANIFOLDS WITH BOUNDARY

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We introduce a generalization of the regular genus, a combinatorial invariant of PL manifolds ([10]), which is proved to be strictly related, in dimension three, to generalized Heegaard splittings defined in [12].

1. Introduction.

Throughout this paper we consider only compact, connected, PL-manifolds and PL-maps.

The regular genus of a manifold is an invariant defined by Gagliardi in [7] (for closed manifolds) and [10] (for manifolds with boundary), by using 2-cells embeddings of "edge-coloured" graphs representing the manifold and satisfying some conditions of regularity.

More precisely, in the general case of non-empty boundary, the graphs are required to be "regular with respect to one colour", i.e. they become regular after deleting the edges of one fixed colour.

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In this paper, by introducing the weaker concept of ”regularity with respect to a cyclic permutation”, we extend the definition of the regular genus to a larger class of coloured graphs.

This generalized regular genus is always bounded by the regular one, but it turns out to be generally strictly less than it; this happens for example in the case of $T_g \times \mathbb{D}^1$, (resp. $U_g \times \mathbb{D}^1$), for each $g \geq 1$. In fact we construct coloured graphs representing these manifolds and regularly embedding into the orientable (resp. non orientable) surface with two holes and genus $g$.

Moreover we prove, as in the case of the regular genus, that a punctured 3-sphere (i.e. a 3-sphere with holes) is characterized by having generalized regular genus zero.

For the case of 3-manifolds, it is known (see [2] and [3]) that the regular genus coincides with the classical Heegaard one. This result highly depends on the fact that a coloured graph, regular with respect to a colour and representing a 3-manifold $M$, defines a Heegaard splitting of $M$ (see [3] for details).

Montesinos, in [12], defined a generalization of the concepts of Heegaard splittings and Heegaard genus for orientable 3-manifolds; they coincide with the classical ones in the case of connected boundary. Later the constructions were extended to the non orientable case in [3].

In section 3 we investigate the relationship between coloured graphs representing a 3-manifold and satisfying our ”weaker” condition of regularity and generalized Heegaard splittings of the same manifold; as a consequence we establish an inequality between the generalized Heegaard genus and the generalized regular genus of a 3-manifold with boundary.

2. Coloured graphs and the regular genus of a manifold.

An $(n + 1)$-coloured graph (with boundary) is a pair $(\Gamma, \gamma)$, where $\Gamma = (V(\Gamma), E(\Gamma))$ is a multigraph and $\gamma : E(\Gamma) \rightarrow \Delta_n = \{0, 1, \ldots, n\}$ a map, injective on each pair of adjacent edges of $\Gamma$.

For each $B \subseteq \Delta_n$, we call $B$ residues the connected components of the multigraph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$; we set $i = \Delta_n \setminus \{i\}$ for each $i \in \Delta_n$.

The vertices of $\Gamma$ whose degree is strictly less than $n + 1$ are called boundary vertices; if $(\Gamma, \gamma)$ has no boundary vertices is called without boundary. We denote by $\partial V(\Gamma)$ the set of boundary vertices of $\Gamma$.

If $K$ is an $n$-dimensional homogeneous pseudocomplex, and $V(K)$ its set of vertices, we call coloured n-complex the pair $(K, \xi)$ where $\xi : V(K) \rightarrow \Delta_n$ is a map which is injective on every simplex of $K$.

If $\sigma^h$ is an $h$-simplex of $K$ then the disjoint star $\text{std}(\sigma^h, K)$ of $\sigma^h$ in $K$
is the pseudocomplex obtained by taking the disjoint union of the simplexes of $K$ containing $\sigma^h$ and identifying the $(n - 1)$-simplexes containing $\sigma^h$ together with all their faces.

The disjoint link $\text{lk}(\sigma^h, K)$ of $\sigma^h$ in $K$ is the subcomplex of $\text{std}(\sigma^h, K)$ formed by the simplexes which don’t intersect $\sigma^h$.

From now on we shall restrict our attention to the coloured complexes $K$, such that:

- each $(n - 1)$-simplex is a face of exactly two $n$-simplexes of $K$;
- for each simplex $\sigma$ of $K$, $\text{std}(\sigma, K)$ is strongly connected.

Coloured graphs are an useful tool for representing manifolds (see [6] for a survey on this topic), due to the existence of a bijective correspondence between coloured graphs and coloured complexes which triangulate manifolds.

Given a coloured complex $K$, a direct way to see this correspondence is to consider a coloured graph $(\Gamma, \gamma)$ imbedded in $K = K(\Gamma)$ as its dual $1$-skeleton, i.e. the vertices of $\Gamma$ are the barycenters of the $n$-simplexes of $K(\Gamma)$ and the edges of $\Gamma$ are the 1-cells dual of the $(n - 1)$-simplexes of $K(\Gamma)$. Of course the $(n - 1)$-simplex dual to an edge $e$ with $\gamma(e) = i$ has its vertices labelled by $i$. Furthermore, there is a bijective correspondence between the $h$-simplexes $(0 \leq h \leq \text{dim } K(\Gamma))$ of $K(\Gamma)$ and the $(n - h)$-residues of $\Gamma$, in the sense that, if $\sigma^h$ is an $h$-simplex of $K(\Gamma)$, whose vertices are labelled by $\{i_0, \ldots, i_h\}$, there is a unique $(n - h)$-residue $\Xi$ of $\Gamma$ whose edges are coloured by $\Delta_n \setminus \{i_0, \ldots, i_h\}$ and such that $K(\Xi) = \text{lk}(\sigma^h, K)$.

See [6] for a more precise description of the constructions involved.

If $M$ is a manifold (with boundary) of dimension $n$ and $(\Gamma, \gamma)$ a $(n + 1)$-coloured graph (with boundary) such that $|K(\Gamma)| \cong M$, we say that $M$ is represented by $(\Gamma, \gamma)$. In this case $M$ is orientable iff $(\Gamma, \gamma)$ is bipartite.

Let $(\Gamma, \gamma)$ be a $(n + 1)$-coloured graph such that the set of its boundary vertices is $\partial V(\Gamma) = V^{(0)} \cup V^{(1)} \cup \ldots \cup V^{(n)}$ where, for each $i \in \Delta_n$, $V^{(i)}$ is formed by the vertices missing the $i$-coloured edge (of course it can occur that $V^{(i)} = \emptyset$).

We call extended graph associated to $(\Gamma, \gamma)$ the $(n + 1)$-coloured graph $(\Gamma^*, \gamma^*)$ obtained in the following way:

- for each $v \in V^{(i_1)} \cap \ldots \cap V^{(i_h)}$ add to $V(\Gamma)$ the vertices $v_{i_1}, \ldots, v_{i_h}$; we call $V^*$ the set of these new vertices;
- for each $v \in V^{(i_1)} \cap \ldots \cap V^{(i_k)}$ and for each $j = 1, \ldots, h$ add to $E(\Gamma)$ an edge $e_{ij}$ with endpoints $v$ and $v_{i_j}$ and the obvious coloration.
A regular imbedding of \((\Gamma, \gamma)\) into a surface (with boundary) \(F\), is a cellular imbedding of \((\Gamma^*, \gamma^*)\) into \(F\), such that:

(a) the image of a vertex of \(\Gamma^*\) lies on \(\partial F\) iff the vertex belongs to \(V^*\);

(b) the boundary of any region of the imbedding is either the image of a cycle of \((\Gamma^*, \gamma^*)\) (internal region) or the union of the image \(\alpha\) of a path in \((\Gamma^*, \gamma^*)\) and an arc of \(\partial F\), the intersection consisting of the images of two (possibly coincident) vertices belonging to \(V^*\) (boundary region);

(c) there exists a cyclic permutation \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)\) of \(\Delta_n\) such that for each internal region (resp. boundary region), the edges of its boundary (resp. of \(\alpha\)) are alternatively coloured \(\varepsilon_i\) and \(\varepsilon_{i+1}\) (\(i \in \mathbb{Z}_{n+1}\)).

From now on, to avoid long notations, we write \(\Gamma\) for a \((n + 1)\)-coloured graph instead of \((\Gamma, \gamma)\).

For each \(i, j \in \Delta_n\), let us denote by \(\hat{g}_{ij}(\Gamma)\) the number of cycles of \(\Gamma_{i,j}\), by \(p(\Gamma)\) (resp. \(q(\Gamma)\)) the number of vertices (resp. of edges) of \(\Gamma\).

Given a cyclic permutation \(\varepsilon\) of \(\Delta_n\), a \((n + 1)\)-coloured graph \(\Gamma\) is regular with respect to \(\varepsilon\), if for each \(i \in \mathbb{Z}_{n+1}\), \(v \in V^{(\varepsilon_i)}\) and \(w \in V^{(\varepsilon_{i+1})}\), \(v\) and \(w\) don’t belong to the same connected component of \(\Gamma_{[\varepsilon_i,\varepsilon_{i+1},\varepsilon_{i+2}]}\).

In particular, since it can be \(v = w\), each vertex of \(\Gamma\) can’t miss two colours which are consecutive in \(\varepsilon\).

**Remark 1.** Note that, if there exists \(i \in \Delta_n\) such that \(V^{(\varepsilon)} = \emptyset\), for each \(j \neq i\) (i.e. \(\Gamma\) is regular with respect to the colour \(i\) in the sense of [10]), then \(\Gamma\) is regular with respect to any cyclic permutation of \(\Delta_n\).

For each \(i \in \Delta_n\), let us denote by \(\hat{g}_{ii}(\Gamma)\) the number of closed walks in \(\Gamma\) defined by starting from a vertex belonging to \(V^{(\varepsilon_i)}\), following first the \(\varepsilon_{i+1}\)-coloured edge and going on by the following rules:

- if we arrive in a vertex \(w\) by a \(\varepsilon_{i+1}\)- (resp. \(\varepsilon_{i-1}\)-) coloured edge, then we follow the \(\varepsilon_{i-1}\)- (resp. \(\varepsilon_{i+1}\)-) coloured edge whether \(w \in V^{(\varepsilon_i)}\) or \(w \notin V^{(\varepsilon_i)}\);

- if we arrive in a vertex by a \(\varepsilon_i\)-coloured edge \(e\), then we follow the \(\varepsilon_{i+1}\)- or the \(\varepsilon_{i-1}\)-coloured edge whether the edge we met before \(e\) is \(\varepsilon_{i+1}\)- or the \(\varepsilon_{i-1}\)-coloured.

**Proposition 1.** Given a \((n + 1)\)-coloured bipartite (resp. non bipartite) graph \(\Gamma\), and a cyclic permutation \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)\) of \(\Delta_n\) such that \(\Gamma\) is regular with respect to \(\varepsilon\), there exists a regular embedding of \(\Gamma^*\) into the orientable (resp.
non orientable) surface with boundary $F_\varepsilon$ with Euler characteristic:
\[
\chi(F_\varepsilon) = \sum_{i \in \mathbb{Z}_{n+1}} \hat{g}_{\varepsilon_i; \varepsilon_{i+1}}(\Gamma) - q(\Gamma) + p(\Gamma)
\]
and hole number:
\[
\lambda_{\varepsilon}(F_\varepsilon) = \sum_{i \in \mathbb{Z}_{n+1}} g_{\varepsilon_i}(\Gamma)
\]

**Proof.** Let us write $\varepsilon_{i_1, \ldots, i_k}$ for the cyclic permutation of $\Delta_{n-1}$ obtained from $\varepsilon$ by deleting $\varepsilon_{i_1}, \ldots, \varepsilon_{i_k}$.

We shall prove first the orientable case.

We can define a 2-cell embedding of $\Gamma$ into a closed surface $S_\varepsilon$ by means of a rotation system $\Phi$ (see [14]) on $\Gamma$ as follows:

Let $B, N$ be the two bipartition classes of $\Gamma$, for each $v \in V(\Gamma)$ let us set

\[
\Phi_v = \begin{cases}
\varepsilon_{i_1, \ldots, i_k} & \text{if } v \in V^{(\varepsilon_{i_1})} \cup \ldots \cup V^{(\varepsilon_{i_k})} \\
\varepsilon & \text{otherwise}
\end{cases}
\]

As a consequence of the condition of regularity on $\Gamma$, the 2-cells of the regular immersion of $\Gamma$, defined by the above rotation system, can only be of two types: either the cell is bounded by edges coloured alternatively $\varepsilon_i$ and $\varepsilon_{i+1}$ ($i \in \mathbb{Z}_{n+1}$), or it is bounded by edges coloured $\varepsilon_{i-1}, \varepsilon_i$ and $\varepsilon_{i+1}$.

In the first case the boundary of the cell contains no vertices belonging to $V^{(\varepsilon_i)}$, in the other case it contains vertices belonging to $V^{(\varepsilon_i)}$, but, by the regularity conditions, not to $V^{(\varepsilon_{i+1})}$.

Let us call $A^1_{i_j}, \ldots, A^r_{i_j}$ the cells whose boundary contains vertices of $V^{(\varepsilon_i)}$.

Obviously $r_i = \hat{g}_{\varepsilon_i}(\Gamma)$. For each $i \in \Delta_{n}$ and $j = 1, \ldots, r_i$, let us consider a disk $D^1_{i_j}$ in the interior of $A^1_{i_j}$. We can add to $\Gamma$ the vertices $v^*$ on the boundary of $D^1_{i_j}$ and the "missing" $\varepsilon_i$-coloured edges (in a suitable order) in the interior of $A^1_{i_j}$, thus obtaining a regular embedding of $\Gamma^*$ into the surface $F_\varepsilon$ obtained by deleting from $S_\varepsilon$ the interiors of the disks $D^1_{i_j}$.

The formulas for the Euler characteristic and hole number are straightforward.

If $\Gamma$ is not bipartite we use, instead of a rotation system, a generalized embedding scheme (see [13]) $(\phi, \lambda)$ associated to $\varepsilon$, where $\phi$ is the rotation system defined for each $v \in V(\Gamma)$ as

\[
\phi_v = \begin{cases}
\varepsilon_{i_1, \ldots, i_k} & \text{if } v \in V^{(\varepsilon_{i_1})} \cup \ldots \cup V^{(\varepsilon_{i_k})} \\
\varepsilon & \text{otherwise}
\end{cases}
\]
and \( \lambda : E(\Gamma) \longrightarrow \mathbb{Z}_2 \) is defined \( \lambda(e) = 1 \) for each \( e \in E(\Gamma) \).

The (bipartite) derived \((n+1)\)-coloured graph \( \Gamma^\lambda \) has vertices \( V(\Gamma) \times \{0, 1\} \) and for each \( v, w \in V(\Gamma) \), \( i, j \in \mathbb{Z}_2 \), \( k \in \Delta_n \) the vertices \((v, i)\) and \((w, j)\) are \( k \)-adjacent in \( \Gamma^\lambda \) iff \( v \) and \( w \) are \( k \)-adjacent in \( \Gamma \) and \( i + j = 1 \).

Note that \( \Gamma^\lambda \) is regular with respect to \( \varepsilon \), since \( \Gamma \) is.

Moreover \( \phi \) induces a rotation system \( \phi^\lambda \) on \( \Gamma^\lambda \) as \( \phi^\lambda_{(v, 0)} = \phi_v \) and \( \phi^\lambda_{(v, 1)} = \phi_v^{-1} \) (see [10]).

Let \( \iota_{\varepsilon} \) (resp. \( \iota_{\varepsilon}^\lambda \)) be the regular embedding of \( \Gamma \) (resp. of \( \Gamma^\lambda \)) into the non-orientable (resp. orientable) closed surface \( S_{\varepsilon} \) (resp. \( S_{\varepsilon}^\lambda \)) associated to \( (\phi, \lambda) \) (resp. to \( \phi^\lambda \)).

An easy calculation shows that the number of 2-cells of \( \iota_{\varepsilon}^\lambda \) is double of the number of 2-cells of \( \iota_{\varepsilon} \), therefore \( \chi(S_{\varepsilon}^\lambda) = 2\chi(S_{\varepsilon}) \) and we can use the same arguments as in the orientable case to obtain the formulas for the surface with boundary \( F_{\varepsilon} \).

Let us define \( \chi_{\varepsilon}(\Gamma) = \chi(F_{\varepsilon}) \), \( \lambda_{\varepsilon}(\Gamma) = \lambda(F_{\varepsilon}) \) and

\[
\rho_{\varepsilon}(\Gamma) = \begin{cases} 
1 - \frac{\chi_{\varepsilon}(\Gamma) + \lambda_{\varepsilon}(\Gamma)}{2} & \text{if } \Gamma \text{ is bipartite} \\
\frac{\chi_{\varepsilon}(\Gamma) - \lambda_{\varepsilon}(\Gamma)}{2} & \text{if } \Gamma \text{ is not bipartite}.
\end{cases}
\]

The \textit{generalized regular genus} \( \varrho(\Gamma) \) of \( \Gamma \) is the minimum \( \rho_{\varepsilon}(\Gamma) \) among all cyclic permutations \( \varepsilon \) of \( \Delta_n \) such that \( \Gamma \) is regular with respect to \( \varepsilon \).

Given a \( n \)-manifold \( M \) the \textit{generalized regular genus} of \( M \) is the non-negative integer \( \mathfrak{b}(M) \) defined as the minimum \( \varrho(\Gamma) \) among all \((n + 1)\)-coloured graphs \( \Gamma \) representing \( M \) and regular with respect to at least one cyclic permutation \( \varepsilon \) of \( \Delta_n \).

Given a \( n \)-manifold \( M \), we denote by \( \mathfrak{b}(M) \) the regular genus of \( M \) ([10]).

As a direct consequence of the above definition, Remark 1 and the definition of regular genus, we have:

\textbf{Proposition 2.} For each \( n \)-manifold \( M \),

\[ \mathfrak{b}(M) \leq \mathfrak{b}(M). \]

Now we are going to prove that the generalized regular genus is generally strictly less than the regular one.

In [11] a 4-coloured graph is shown which represents \( T_1 \times \mathbb{D}^1 \) and regularly embeds into the bordered surface of genus 1, while the regular genus is known to be 2 (see [10]).

In the following, for each \( g \geq 1 \) (resp. \( h \geq 1 \)) we shall construct a bipartite (resp. non bipartite) 4-coloured graph \( \Gamma_g \) (resp. \( \Gamma_h \)) representing \( T_g \times \mathbb{D}^1 \),
where \( T_g \) is the closed orientable surface of genus \( g \) (resp. \( U_h \times \mathbb{D}^1 \), where \( U_h \) is the closed non orientable surface of genus \( h \)) and regularly embedding into the orientable (resp. non orientable) surface with two holes and genus \( g \) (resp. \( h \)). In both cases the graph is such that \( \partial V = V^{(2)} \cup V^{(3)} \) and \( V^{(2)} \cap V^{(3)} = \emptyset \).

The graphs are as follows:

- \( \Gamma_g \) (resp. \( \Gamma_h \)) has \( 6(2g + 1) \) (resp. \( 6(h + 1) \)) vertices labeled as \( A_1, \ldots, A_{2(2g+1)}, a_1, \ldots, a_{2(2g+1)}, B_1, \ldots, B_{2(2g+1)} \) (resp. \( A_1, \ldots, A_{2(h+1)}, a_1, \ldots, a_{2(h+1)}, B_1, \ldots, B_{2(h+1)} \))
- for each \( i = 1, \ldots, 2(2g + 1) \) (resp. each \( i = 1, \ldots, 2(h + 1) \) \( A_i \) \( V^{(2)} \) and \( B_i \) \( V^{(3)} \)
- the 0- , 1- and 2-adjacency are drawn in Figure 1 for the orientable case; the non orientable is analogous;

![Diagram](image)

- the 3-adjacency are:
  for each \( i = 1, \ldots, g \) , \( A_{2i} \) with \( A_{4g-2i+3} \) , \( A_{2i-1} \) with \( A_{4g-2i+2} \) and \( A_{2g+1} \) with \( A_{2(2g+1)} \) (resp. for each \( i = 1, \ldots, h \) , \( A_i \) with \( A_{2h-i+2} \) and \( A_h+1 \) with \( A_{2(h+1)} \)) The 3-adjacency of the \( a_i \)’s are analogous.

We claim that \( \Gamma_g \) represents \( T_g \times \mathbb{D}^1 \) (resp. \( \Gamma_h \) represents \( U_h \times \mathbb{D}^1 \)). In fact the above construction comes from an easy generalization of the one in [8] for \( T_1 \times \mathbb{D}^1 \) and \( U_1 \times \mathbb{D}^1 \), together with a permutation of the colours on one of the boundary components.

Let \( \varepsilon = (0132) \), then for each \( g \geq 1 \) (resp. \( h \geq 1 \) ), \( \Gamma_g \) (resp. \( \Gamma_h \)) is regular with respect to \( \varepsilon \) and it is easy to see that:
\[ g_{01} = g_{02} = g_{23} = 2g + 1, \quad g_{03} = g_{12} = g_{13} = 1 \]

(resp. \( g_{01} = g_{02} = g_{13} = h + 1, \quad g_{03} = g_{12} = g_{23} = 1 \)).

Since \( \chi_e(\Gamma_g) = -2g \) (resp. \( \chi_e(\Gamma_h) = -h \)) and the number of holes is 2 both in the orientable and the non orientable case, we have \( \varrho_e(\Gamma_g) = g \) (resp. \( \varrho_e(\Gamma_h) = h \)).

Therefore \( \overline{g}(T_g \times \mathbb{D}^1) \leq g < \overline{g}(T_h \times \mathbb{D}^1) = 2g \) and \( \overline{g}(U_h \times \mathbb{D}^1) \leq h < \overline{g}(U_h \times \mathbb{D}^1) = 2h \) (see [1]); actually the first are equalities, since we can establish the following theorem:

**Theorem 3.** \( \overline{g}(T_g \times \mathbb{D}^1) = \overline{g}(U_g \times \mathbb{D}^1) = g \)

Before proving the theorem let us fix some notations.

Let \( \varepsilon = (\alpha\alpha'\beta\beta') \) be a cyclic permutation of \( \Delta_3 \) and \( \Gamma \) a 4-coloured graph representing a 3-manifold \( M \) and regular with respect to \( \varepsilon \). We denote by \( \partial_i K(\Gamma) \) \( (i = 1, \ldots, r) \) the \( i \)-th boundary component of \( K(\Gamma) \) and by \( V_i(\Gamma) \) the subset of \( \partial V(\Gamma) \) formed by those vertices whose dual 3-simplices have a face on \( \partial_i K(\Gamma) \).

Note that, since \( \Gamma \) is regular with respect to \( \varepsilon \), then for each \( i = 1, \ldots, r \), \( V_i(\Gamma) \subseteq V^{(\alpha)}(\Gamma) \cup V^{(\beta)}(\Gamma) \) or \( V_i(\Gamma) \subseteq V^{(\alpha)}(\Gamma) \cup V^{(\beta)}(\Gamma) \).

The proof of Theorem 3 requires two lemmas.

**Lemma 4.** Given a 3-manifold with \( r \) boundary components \( M \), a cyclic permutation \( \varepsilon \) of \( \Delta_3 \) and a 4-coloured graph \( \Gamma \) representing \( M \) and regular with respect to \( \varepsilon \), then there exists a 4-coloured graph \( \Gamma' \), representing \( M \), and satisfying the following conditions:

\( 1) \quad \varrho_e(\Gamma') = \varrho_e(\Gamma); \)

\( 2) \quad \forall v \in V(\Gamma'), \quad \deg v \geq 3 \quad \text{and} \quad \forall i = 1, \ldots, r, \quad V_i(\Gamma') \cap (V^{(\beta)}(\Gamma') \cup V^{(\alpha)}(\Gamma') \cup V^{(\beta)}(\Gamma')) = \emptyset \) or \( V_i(\Gamma') \cap (V^{(\alpha)}(\Gamma') \cup V^{(\beta)}(\Gamma') \cup V^{(\beta)}(\Gamma')) = \emptyset. \)

**Proof.** Let \( i \in \{1, \ldots, r\} \) be such that \( V_i(\Gamma) \cap V^{(\alpha)}(\Gamma) \neq \emptyset \) and \( V_i(\Gamma) \cap V^{(\beta)}(\Gamma) \neq \emptyset \); let \( w \) be a \( \alpha \)-coloured vertex of \( \partial_i K(\Gamma) \).

Let us consider the 4-coloured graph \( b\Gamma \) obtained by performing a bisection of type \( (\alpha, \beta) \) around \( w \) (see [9]) i.e. we perform a stellar subdivision on each edge having as endpoints \( w \) and a \( \beta \)-coloured vertex and colour \( w \) by \( \beta \) and the new vertices by \( \alpha \), keeping the colours of \( K(\Gamma) \) for the remaining vertices (see [9]).

Note that \( \text{card} (V_i(b\Gamma) \cap V^{(\alpha)}(b\Gamma)) = \text{card} (V_i(\Gamma) \cap V^{(\alpha)}(\Gamma)) - 1. \)

We claim that \( \varrho_e(b\Gamma) = \varrho_e(\Gamma). \)

In fact, let \( \Lambda_w \) be the \( \overline{g} \)-residue of \( \Gamma \) representing the disjoined link of \( w \) in \( K(\Gamma) \).

We have:
$\forall j \neq \beta, \quad  \partial_a \partial (b \Gamma) = \partial_a \partial (\Gamma) + \partial_a \partial (\Lambda_w)$

$\forall i \neq \alpha, \quad \partial_{\beta a} \partial (b \Gamma) = \partial_{\beta a} \partial (\Gamma) - \partial_{\beta a} \partial (\Lambda_w) + q^{(i)}(\Lambda_w)$

where \(q^{(i)}(\Lambda_w)\) is the number of \(i\)-coloured edges of \(\Lambda_w\).

\[p(b \Gamma) = p(\Gamma) + p(\Lambda_w) \quad q(b \Gamma) = q(\Gamma) + q^{(\alpha)}(\Lambda_w) + q^{(\beta)}(\Lambda_w) + p(\Lambda_w)\]

Therefore:

\[\chi_x(b \Gamma) = 3 \partial_{\alpha a}(b \Gamma) + \partial_{\alpha a}(b \Gamma) + \partial_{\beta a}(b \Gamma) + \partial_{\beta a}(b \Gamma) - q(b \Gamma) + p(b \Gamma) =
= \partial_{\alpha a}(\Gamma) + \partial_{\alpha a}(\Lambda_w) + \partial_{\beta a}(\Gamma) - \partial_{\beta a}(\Lambda_w) + q^{(\alpha)}(\Lambda_w) + \partial_{\beta a}(\Gamma) - \partial_{\beta a}(\Lambda_w) - q^{(\alpha)}(\Lambda_w) - q^{(\beta)}(\Lambda_w) + p(\Lambda_w) = \chi_x(\Gamma).
\]

Moreover, note that, for each \(i \in \Delta_3\), if \(j\) is the colour non-consecutive to \(i\) in \(\varepsilon\), \(\partial_{\alpha a}(\Gamma)\) equals the number of \(j\)-coloured vertices in the components of \(\partial K(\Gamma)\) missing colour \(i\).

Therefore:

\[\partial_{\alpha a}(b \Gamma) = \partial_{\alpha a}(\Gamma) + 1 \quad \partial_{\alpha a}(b \Gamma) = \partial_{\alpha a}(\Gamma)\]

and \(\chi_x(b \Gamma) = \chi_x(\Gamma)\).

Finally, we have that \(\partial_{\varepsilon}(b \Gamma) = \partial_{\varepsilon}(\Gamma)\).

By performing a finite number of bisection of type \((\alpha, \beta)\) on the components of \(\partial K(\Gamma)\) missing \(\alpha\) and \(\beta\) and, similarly a finite number of bisection of type \((\alpha', \beta')\) on the components missing \(\alpha'\) and \(\beta'\), we obtain the graph \(\Gamma'\).

Suppose now that \(\Gamma\) is a 4-coloured graph satisfying condition (2) of Lemma 4, with respect to a cyclic permutation \(\varepsilon\) of \(\Delta_3\) and suppose that \(\partial \Gamma(\Gamma)\) has \(r\) connected components. Let us choose one of them, say \(\partial K(\Gamma)\). Then there exists \(j \in \Delta_3\) such that for each \(k \in \Delta_3 - \{j\}\), \(V_i(\Gamma) \cap V^{(k)}(\Gamma) = \emptyset\).

Let us denote by \(\Gamma^{(j)}\) the 4-coloured graph obtained from \(\Gamma\) by the following rule:

- \(\forall v, w \in V_i(\Gamma) \cap V^{(j)}(\Gamma)\), join the vertices \(v\) and \(w\) by a \(j\)-coloured edge iff \(v\) and \(w\) belong to the same \([j, j + 1]\)-residue of \(\Gamma\).

It is easy to see that, if \(\Gamma\) represents a 3-manifold \(M\) with \(r\) boundary components, \(\Gamma^{(j)}\) represents the singular 3-manifold obtained from \(M\) by capping off the \(i\)-th boundary component by a cone over it.

Moreover, we have
Lemma 5. \( \varrho_e(\Gamma_i^{(j)}) = \varrho_e(\Gamma) \)

Proof. We have

\[
p(\Gamma_i^{(j)}) = p(\Gamma) \\
q(\Gamma_i^{(j)}) = q(\Gamma) + \frac{p_i^{(j)}(\Gamma)}{2}
\]

\[
\dot{g}_{kk+1}(\Gamma_i^{(j)}) = \dot{g}_{kk+1}(\Gamma) \quad \forall k \in \Delta_3 - \{j - 1, j + 1\}
\]

\[
\dot{g}_{jj+1}(\Gamma_i^{(j)}) = \dot{g}_{jj+1}(\Gamma) + \frac{p_i^{(j)}(\Gamma)}{2} \quad \dot{g}_{j-1j}(\Gamma_i^{(j)}) = \dot{g}_{j-1j}(\Gamma) + a_{g_i^{(j)}}(\Gamma)
\]

where \( p^{(j)}(\Gamma) = \text{card} \left(V_i(\Gamma) \cap V^{(j)}(\Gamma)\right) \) and \( a_{g_i^{(j)}}(\Gamma) \) is the number of closed walks defined as for \( a_{g_i}(\Gamma) \), whose boundary vertices belong only to \( V_i(\Gamma) \).

Then

\[
\chi_e(\Gamma_i^{(j)}) = \sum_{k \in \mathbb{Z}_4} \dot{g}_{kk+1}(\Gamma_i^{(j)}) - q(\Gamma_i^{(j)}) + p(\Gamma_i^{(j)})
\]

\[
= \sum_{k \in \mathbb{Z}_4} \dot{g}_{kk+1}(\Gamma) + \frac{p_i^{(j)}(\Gamma)}{2} + a_{g_i^{(j)}}(\Gamma) - q(\Gamma) - \frac{p_i^{(j)}(\Gamma)}{2} + p(\Gamma)
\]

\[
= \chi_e(\Gamma) + a_{g_i^{(j)}}(\Gamma).
\]

Moreover \( \lambda_e(\Gamma_i^{(j)}) = \lambda_e(\Gamma) - a_{g_i^{(j)}}(\Gamma) \). Therefore \( \varrho_e(\Gamma_i^{(j)}) = \varrho_e(\Gamma) \). \( \Box \)

Proof. (Theorem 3) Let \( M = T_g \times \mathbb{D}^1 \) or \( M = U_g \times \mathbb{D}^1 \). Suppose \( \tilde{\varrho}(M) < g \).

Let \( \Gamma \) be a 4-coloured graph representing \( M \) such that \( \Gamma \) is regular with respect to a cyclic permutation \( \epsilon \) of \( \Delta_3 \) and \( \varrho_e(\Gamma) < g \).

By Lemma 4, we can suppose, without loss of generality, that \( \Gamma \) satisfy condition (2) of the Lemma. Moreover we can also suppose, up to a change of colours, that \( V_2(\Gamma) \subseteq V^{(3)}(\Gamma) \) (i.e. the vertices corresponding to one of the boundary components miss colour 3).

If also \( V_1(\Gamma) \subseteq V^{(3)}(\Gamma) \), then the graph is regular with respect to the colour 3 and \( \varrho(M) \leq \varrho_e(\Gamma) < g \), which is clearly impossible.

If, on the contrary, \( V_1(\Gamma) \subseteq V^{(2)}(\Gamma) \), let us consider the graph \( \Gamma^{(2)}_1 \). Then \( \tilde{M} = |K(\Gamma^{(2)}_1)| \) is obtained from \( M \) by capping off one boundary component by a cone, i.e. it is a cone over the surface \( T_g \) or \( U_g \).

Since \( \Gamma^{(2)}_1 \) is regular with respect to the colour 3, by Lemma 5, we have \( \varrho(\tilde{M}) \leq \varrho_e(\Gamma^{(2)}_1) < g \); on the other hand it is well-known ([110]) that \( \varrho(\tilde{M}) \geq \varrho(\partial M) = g \), since \( \partial \tilde{M} = T_g \) or \( \partial \tilde{M} = U_g \). \( \Box \)

If \( g = 1 \) the previous result is a corollary of the following theorem, which gives a characterization of punctured 3-spheres.
Theorem 6. Let $M$ be a 3-manifold with boundary and let $r$ be the number of its boundary components, then
\[ \tilde{b}(M) = 0 \iff M \text{ is a sphere with } r \text{ holes (punctured 3-sphere)}. \]

Proof. If $M$ is a punctured 3-sphere, its generalized regular genus is clearly zero since its regular genus is zero (see [4]). Conversely let $M$ be a 3-manifold such that $\tilde{b}(M) = 0$, $\varepsilon$ a cyclic permutation of $\Delta_3$ and $\Gamma$ a 4-coloured graph representing $M$ such that $\Gamma$ is regular with respect to $\varepsilon$ and $\varphi_\varepsilon(\Gamma) = 0$.

Again by Lemma 4, we can suppose, without loss of generality, that $\Gamma$ satisfy condition (2) of the Lemma. Therefore we can consider the 4-coloured graph (without boundary) $\tilde{\Gamma}$ obtained from $\Gamma$ by joining, $\forall j \in \Delta_3$ and $\forall v, w \in V^{(j)}(\Gamma)$, the vertices $v$ and $w$ by a $j$-coloured edge iff $v$ and $w$ belong to the same $\{j, j + 1\}$-residue of $\Gamma$, i.e. $\tilde{\Gamma}$ is obtained by performing $r$ times the operation involved in Lemma 5.

Therefore $\tilde{\Gamma}$ represents the singular 3-manifold $\hat{M}$ obtained from $M$ by capping each component of $\partial M$ by a cone.

By Lemma 5 we have that $\varphi_\varepsilon(\tilde{\Gamma}) = \varphi_\varepsilon(\Gamma) = 0$ and by [4] (Corollary 3.), $\hat{M} \cong \mathbb{S}^n$; therefore for each $i = 1, \ldots, r$, $\partial_i M$ is a sphere and $M$ is a punctured 3-sphere. \hfill \square

Remark 2. The proof of Lemma 4 tells us that, as far as 3-manifolds are concerned, we can always suppose that the generalized regular genus is obtained by a 4-coloured graph satisfying condition (2). Let us denote by $\mathcal{G}_4$ the class of such graphs.

For each $\Gamma \in \mathcal{G}_4$ we can define a "boundary graph" $\partial \Gamma$ in the following way:
- $V(\partial \Gamma) = \partial V(\Gamma)$;
- $\forall i = 1, \ldots, r, j \in \Delta_3$ and $\forall v, w \in V_i \cap V^{(j)}$ join $v$ and $w$ by a $c$-coloured edge ($c \in \Delta_3$) iff $v$ and $w$ belong to the same $\{c, j\}$-residue of $\Gamma$.

Note that $\partial \Gamma$ is not a 3-coloured graph, but has $r$ connected components $\partial_1 \Gamma, \ldots, \partial_r \Gamma$ each of them being a 3-coloured graph with colour set $\Delta_3 - \{j\}$ for some $j \in \Delta_3$. Of course, for each $i = 1, \ldots, r$, $\partial_i \Gamma$ represents $\partial_i M$.

Remark 3. Note that, as proved by the graphs we constructed in this section for $T_g \times \mathbb{D}^1$ and $U_h \times \mathbb{D}^1$, the generalized regular genus, still unlike the regular one (see [10]), is generally strictly less the sum of the genera of the boundary components.
3. Regular embeddings of 4-coloured graphs and generalized Heegaard splittings.

In this section we shall show that there exists a correspondence between regular embeddings of 4-coloured graphs in $\overline{G}_4$, representing a 3-manifold, and generalized Heegaard splittings of the same manifold. We briefly recall the basic concepts about generalized Heegaard splittings.

We shall denote by $S_g$ either the orientable closed surface of genus $g$ or the closed non orientable surface of genus $2g$.

A hollow handlebody of genus $g$ is a 3-manifold with boundary $X_g$, obtained from $S_g \times [0, 1]$ by attaching 2- and 3-handles along $S_g \times \{1\}$. We call $S_g \times \{0\}$ the free boundary of $X_g$.

Note that the orientability of $X_g$ depends on that of $S_g$ and conversely.

A generalized Heegaard splitting of genus $g$ of a 3-manifold with boundary $M$ is a pair $(X_g, Y_g)$ of hollow handlebodies of genus $g$, such that $X_g \cup Y_g = M$ and $X_g \cap Y_g$ is the free boundary of both $X_g$ and $Y_g$.

The generalized Heegaard genus of a 3-manifold $M$ is the non negative integer

$$\tilde{H}(M) = \min \{ g \mid \text{there exists a generalized Heegaard splitting of genus } g \text{ of } M \}.$$

Let $\Gamma$ be a 4-coloured graph of $\overline{G}_4$, regular with respect to a cyclic permutation $\varepsilon$ of $\Delta_3$ and such that the “boundary” colours are consecutive in $\varepsilon$. Then, up to a change of colours, we can suppose that

$$(*) \ V^{(\varepsilon_0)} = V^{(\varepsilon_1)} = \emptyset$$

We can state the following

**Proposition 7.** Let $M$ be a connected 3-manifold, $\Gamma \in \overline{G}_4$ a 4-coloured graph representing $M$, regular with respect to a cyclic permutation $\varepsilon$ of $\Delta_3$ and satisfying condition $(*)$, then there exists a generalized Heegaard splitting for $M$ of genus $\tilde{g}_*(\Gamma)$.

**Proof.** To avoid long notations let us suppose $\varepsilon = id$.

Given the graph $\Gamma$, representing $M$ and regular with respect to $\varepsilon$, we know, from the proof of Theorem 6, that there exists a 4-coloured graph without boundary $\tilde{\Gamma}$ such that $\tilde{\varphi}_\varepsilon(\tilde{\Gamma}) = \varphi_\varepsilon(\Gamma)$ and $\tilde{\Gamma}$ represents the singular 3-manifold $\tilde{M}$ obtained from $M$ by capping off each boundary component by a cone.

$\tilde{\Gamma}$ is obtained from $\Gamma$ by adding a 3-coloured edge (resp. 2-coloured edge) between two vertices $v, w \in V^{(3)}$ (resp. $v, w \in V^{(2)}$) iff $v$ and $w$ belong to the same connected component of $\Gamma_{[0,3]}$ (resp. $\Gamma_{[1,2]}$).

Let $K'$ (resp. $K''$) the 1-dimensional subcomplex of $K(\tilde{\Gamma})$ generated by its 0- and 2-coloured (resp. 1- and 3-coloured) vertices and $H$ the largest
subcomplex of $SdK(\tilde{\Gamma})$ (where $Sd$ means first barycentric subdivision) disjoint from $SdK' \cup SdK''$; then $H$ splits $SdK(\tilde{\Gamma})$ into two subcomplexes $N'$ and $N''$ such that $N' \cap N'' = \partial N' \cap \partial N'' = H$. Set $A' = |N'|$, $A'' = |N''|$ and $S = |H|$. $A'$ and $A''$ are handlebodies, $S$ is a closed connected surface of genus $0_0(\tilde{\Gamma})$, where $\tilde{\Gamma}$ regularly embeds.

Let $C$ be a collar of $S$ in $A'$; let $C_0$, $C_1$ be the surfaces corresponding to $S \times \{0\}$ and $S \times \{1\}$ respectively. For each 1-simplex $e$ of $K(\tilde{\Gamma})$ whose endpoints are 0- and 2-coloured, let $H_e^{02}$ be a regular neighbourhood in $A'$ of the 2-cell dual of $e$ (see Figure 2).

![Figure 2.](image)

Set $X = C \cup (\bigcup_e H_e^{02})$. $X$ is a hollow handlebody, since the $H_e^{02}$s are 2-handles attached along $C_1 \cong S \times \{1\}$.

Moreover $A' - X$ is the union of regular neighbourhoods of the 0- and 2-coloured vertices of $K(\tilde{\Gamma})$.

Let $\tilde{X}$ be the hollow handlebody obtained by adding to $X$ the neighbourhoods corresponding to non singular vertices.

Similarly we can define a hollow handlebody $\tilde{Y}$ by starting from a collar of $S$ in $A''$ and attaching on it:

- the 2-handles $H_1^{13}$ dual to the 1-simplices of $K(\tilde{\Gamma})$ having endpoints coloured by 1 and 3;
- the 3-handles corresponding to the neighbourhoods of the non singular 1- and 3-coloured vertices.

We have that $\tilde{X} \cup \tilde{Y} = M$ and $\tilde{X} \cap \tilde{Y} = S$. 
Therefore $(\tilde{X}, \tilde{Y})$ is a generalized Heegaard splitting for $M$ of genus $g(\delta) = \varphi_\epsilon(\tilde{\Gamma}) = \varphi_\epsilon(\Gamma)$. $\Box$

As a consequence of Proposition 7 and Lemma 4, we have the following:

**Corollary 8.** For each 3-manifold $M$, $\overline{H}(M) \leq \overline{\mathcal{G}}(M)$.

**Proof.** Let $\Gamma$ be a 4-coloured graph representing $M$ and $\epsilon$ a cyclic permutation of $\Delta_3$ such that $\Gamma$ is regular with respect to $\epsilon$ and $\overline{\mathcal{G}}(M) = \varphi_\epsilon(\Gamma)$.

By Lemma 4 we know that we can always suppose that $\Gamma$ misses at most two colours.

If these colours are non consecutive in $\epsilon$, then, by means of suitable bisections, we can obtain a new graph, still representing $M$, with the same genus as $\Gamma$ and missing only one colour, i.e. a graph regular with respect to a colour, that we can always suppose to be 3.

In this case by Lemma 1 of [2] there exists a proper ([2]) Heegaard splitting of $M$ of genus $\overline{\mathcal{G}}(M) = \mathcal{G}(M)$.

On the other hand, if the “boundary” colours are consecutive in $\epsilon$, we can apply Proposition 7 to get a Heegaard splitting of $M$ of the required genus. $\Box$

Note that the splitting is always proper in the case of $M$ having connected boundary. In this case $\overline{\mathcal{G}}(M) = \mathcal{G}(M) = \mathcal{H}(M) = \overline{H}(M)$ (see [3]).

**REFERENCES**


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