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THREE WEAK SOLUTIONS TO A DEGENERATE QUASILINEAR ELLIPTIC SYSTEM

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Sufficient conditions are established to guarantee the existence of at least three weak solutions to a degenerate quasilinear elliptic system with three parameters and Dirichlet boundary conditions. An application of the main theorem to a scalar elliptic problem is also presented. The proofs in the paper mainly make use of a variational argument and an abstract critical point theorem due to Ricceri.

1. Introduction

In this paper, we study the existence of multiple weak solutions to the degenerate quasilinear elliptic system

$$\begin{cases} -\nabla\left(m_i(x)|\nabla u_i|^{p_i-2}\nabla u_i\right) \\ \quad = \varepsilon F_{u_i}(x,u_1,\ldots,u_n) - \lambda G_{u_i}(x,u_1,\ldots,u_n) - \nu H_{u_i}(x,u_1,\ldots,u_n), \quad x \in \Omega, \\ u_i|_{\partial\Omega} = 0, \quad i = 1,\ldots,n, \end{cases} \tag{1.1}$$

where Ω is a bounded and connected subset of $\mathbb{R}^N$ $(N \geq 2)$, $p_i > 1$ are constants and m_i are nonnegative weight functions on Ω, $i = 1,\ldots,n$, ε, λ, and ν are non-

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negative parameters, F, G, $H \in C^1(\overline{\Omega} \times \mathbb{R}^n, \mathbb{R})$, and $\nabla u = (u_{t_1}, \ldots, u_{t_N})$ denotes the gradient of u with respect to $(t_1, \ldots, t_N) \in \mathbb{R}^N$.

The degeneracy of system (1.1) is considered in the case that the weight functions m_i, $i = 1, \ldots, n$, are allowed to be unbounded and/or not separated from zero. Degenerate phenomena frequently occur in many areas such as in phenomena related to the equilibrium of anisotropic continuous media [6] and in the study of population dynamics [3, 4]. Scalar degenerated quasilinear elliptic equations with p-Laplacians have been extensively studied in the 1990s and the reader may refer to the the monograph [7] for some related results. In recent years, there has been renewed and increasing interest in degenerate elliptic problems. Below, we only briefly mention several works on degenerate quasilinear elliptic systems. In 2008, Zographopoulos [19] studied the properties of the positive principal eigenvalues of the degenerate quasilinear elliptic system

$$\begin{cases} -\nabla(\nu_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha}|v|^{\beta}v, & x \in \Omega, \\ -\nabla(\nu_2(x)|\nabla v|^{p-2}\nabla v) = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta}u, & x \in \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

and proved that this eigenvalue is simple, unique up to positive eigenfunctions, and isolated. In 2014, An, Lu, and Suo [1] applied the properties of the principle eigenvalues of system (1.2) to obtain the existence and multiplicity of weak solutions to the degenerate quasilinear elliptic system

$$\begin{cases} -\nabla(\nu_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha}|v|^{\beta}v + F_u(x,u,v), x \in \Omega, \\ -\nabla(\nu_2(x)|\nabla v|^{p-2}\nabla v) = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta}u + F_v(x,u,v), x \in \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases}$$

Recently, using a variational approach and some recent theory on weighted Lebesgue and Sobolev spaces with variable exponents, Kong [5] studied the existence of at least two nontrivial weak solutions of the elliptic system with degenerate $p_i(x)$-Laplacian operators

$$\begin{cases} -\nabla(w_i(x)|\nabla u_i|^{p_i(x)-2}\nabla u_i) = \lambda f_i(x, u_1, \ldots, u_n) \quad \text{in } \Omega,\ i = 1, \ldots, n, \\ u_i = 0 \quad \text{on } \partial\Omega,\ i = 1, \ldots, n. \end{cases}$$

Motivated by the above mentioned work, in this paper, we investigate the existence of weak solutions to the degenerate quasilinear system (1.1). In particular, we obtain the existence of at least three weak solutions to system (1.1) by applying a variational approach and a recent abstract critical point theorem proved by Ricceri in [12], which can be seen in Lemma 2.1 in Section 2. Lemma 2.1 below has been successfully employed in [11, 18] to obtain the existence of three weak solutions for other problems, and the reader may refer to the papers [12–15] and the monograph [10] for more related results. We also consider the application of our results to a scalar degenerate elliptic problem. To prove our results, many new ideas have been developed throughout the paper. Our results extend and complement some existing results in the literature on degenerate quasilinear systems, such as those in [1, 12, 19].

The rest of this paper is organized as follows. Section 2 contains some preliminary results, Section 3 contains the main theorem and its proof, and the application of the main theorem to a scalar problem is given in Section 4.

2. Preliminary results

In this section, we present some preliminaries. To this end, let E be a non-empty set and $I, \Psi, \Phi : \mathrm{E} \to \mathbb{R}$ be three given functionals. For $\mu > 0$ and $r \in (\inf_{\mathrm{E}} \Phi, \sup_{\mathrm{E}} \Phi)$, let

$$\beta(\mu I + \Psi, \Phi, r) = \sup_{u \in \Phi^{-1}((r,\infty))} \frac{\mu I(u) + \Psi(u) - \inf_{\Phi^{-1}((-\infty,r])}(\mu I + \Psi)}{r - \Phi(u)}.$$

If $\Psi + \Phi$ is bounded from below, for each $r \in (\inf_{\mathrm{E}} \Phi, \sup_{\mathrm{E}} \Phi)$ such that

$$\inf_{\Phi^{-1}((-\infty,r])} I(u) < \inf_{\Phi^{-1}(r)} I(u),$$

we set

$$\mu^{\star}(I, \Psi, \Phi, r) = \inf\left\{ \frac{\Psi(u) - \gamma + r}{\eta_r - I(u)} : u \in \mathrm{E},\, \Phi(u) < r, I(u) < \eta_r \right\},$$

where

$$\gamma = \inf_{\mathrm{E}}(\Psi(u) + \Phi(u)) \quad \text{and} \quad \eta_r = \inf_{u \in \Phi^{-1}(r)} I(u).$$

We now present an abstract critical point theorem due to Ricceri. Here, recall that the derivative $I' : X \to X^*$ of a C^1-functional I is said to admit a continuous inverse on X^* provided that there exists a continuous operator $h : X^* \to X$ such that $h(I'(u)) = u$ for every $u \in X$.

Lemma 2.1. ([12, Theorem 3]) *Let* $(\mathrm{E}, \|.\|)$ *be a reflexive Banach space;* $I : \mathrm{E} \to \mathbb{R}$ *be a sequentially weakly lower semicontinuous, coercive, bounded on each bounded subset of* E, C^1*-functional whose derivative admits a continuous inverse on the topological dual* E^**;* Φ, $\Psi : \mathrm{E} \to \mathbb{R}$ *be two* C^1*-functionals with compact derivatives. Assume also that the functional* $\Psi + \lambda\Phi$ *is bounded below for all* $\lambda > 0$ *and*

$$\liminf_{\|u\| \to \infty} \frac{\Psi(u)}{I(u)} = -\infty.$$

Then, for each $r > \sup_S \Phi$, where S is the set of all global minima of I, for each $\mu > \max\{0, \mu^\star(I, \Psi, \Phi, r)\}$, and each compact interval $[\bar{a}, \bar{b}] \subset (0, \beta(\mu I + \Psi, \Phi, r))$, there exists a number $\rho > 0$ with the following property: for every $\lambda \in [\bar{a}, \bar{b}]$ and every C^1-functional $\Gamma : \mathrm{E} \to \mathbb{R}$ with a compact derivative, there exists $\delta > 0$ such that for each $\nu \in [0, \delta]$, the equation

$$\mu I'(u) + \Psi'(u) + \lambda \Phi'(u) + \nu \Gamma'(u) = 0$$

has at least three solutions in E whose norms are less than ρ.

Remark 2.2. In Lemma 2.1, we implicitly use the fact that $\beta(\mu I + \Psi, \Phi, r) > 0$, which is guaranteed by [12, Theorem 2].

In the sequel, we let the function space $(N)_p$ be the set consisting of all the functions $m : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ such that $m \in \mathrm{L}^1(\Omega)$, $m^{-\frac{1}{p-1}} \in \mathrm{L}^1(\Omega)$ and $m^{-s} \in \mathrm{L}^1(\Omega)$ for some $p > 1$ and $s > \max\{\frac{N}{p}, \frac{1}{p-1}\}$ satisfying $ps < N(s+1)$. Throughout this paper, we assume that the weight functions $m_1, \ldots, m_n$ appearing in system (1.1) satisfy the following hypothesis:

(N) For $i = 1, \ldots, n$, there exist constants s_{p_i} with $s_{p_i} > \max\left\{\dfrac{N}{p_i}, \dfrac{1}{p_i - 1}\right\}$ and functions $r_i \in (N)_{p_i}$ such that

$$\frac{r_i(x)}{l_i} \le m_i(x) \le l_i r_i(x) \tag{2.1}$$

a.e. in Ω for some constants $l_i > 1$ and $i = 1, \ldots, n$.

Let $m \in (N)_p$ be a nonnegative weight function in Ω. The weighted Sobolev space $\mathrm{D}_0^{1,p}(\Omega, m)$ is defined as the closure of $\mathrm{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathrm{D}_0^{1,p}(\Omega,m)} = \left(\int_\Omega m(x)|\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Then, $\mathrm{D}_0^{1,p}(\Omega, m)$ is a reflexive Banach space. For a discussion of this space, we refer the reader to [7] and the references therein. Let

$$p_s^* = \frac{Nps}{N(s+1) - ps}. \tag{2.2}$$

Lemma 2.3. ([7, 19]) *Assume that Ω is a bounded domain in $\mathbb{R}^N$ and $m \in (N)_p$. Then, the following embeddings hold:*

(i) $\mathrm{D}_0^{1,p}(\Omega, m) \hookrightarrow \mathrm{L}^{p_s^*}(\Omega)$ *continuously for $1 < p_s^* < N$;*

(ii) $D_0^{1,p}(\Omega,m) \hookrightarrow L^r(\Omega)$ *compactly for* $r \in [1, p_s^*)$.

For simplicity, in the remainder of this paper, we denote by p_i^* the quantities $p^*_{s_{p_i}}$, $i = 1,\dots,n$, where s_{p_i} is induced by condition (N). Moreover, we use the symbol $\|\cdot\|_m$ for the norm $\|\cdot\|_{D_0^{1,p}(\Omega,m)}$ for any function $m \in (N)_p$; for any $r > 1$, we use the usual notation $\|\cdot\|_{L^r}$ to denote the norm in the space $L^r(\Omega)$, i.e., $\|u\|_{L^r} = \int_\Omega |u|^r dx$.

The space setting for system (1.1) is taken as the product space

$$X = D_0^{1,p_1}(\Omega,m_1) \times \ldots \times D_0^{1,p_n}(\Omega,m_n)$$

equipped with the norm

$$\|u\|_X = \|u_1\|_{m_1} + \ldots + \|u_n\|_{m_n}, \quad u = (u_1,\ldots,u_n) \in X.$$

Observe that X is a reflexive Banach space and inequalities (2.1) in condition (N) imply that the functional spaces

$$D_0^{1,p_1}(\Omega,m_1) \times \ldots \times D_0^{1,p_n}(\Omega,m_n)$$

and

$$D_0^{1,p_1}(\Omega,r_1) \times \ldots \times D_0^{1,p_n}(\Omega,r_n)$$

are equivalent.

Next, let the functionals $S, J_F, J_G, J_H, J : X \to \mathbb{R}$ be defined by

$$S(u) = S(u_1,\ldots,u_n) = \sum_{i=1}^n \frac{1}{p_i} \int_\Omega m_i(x)|\nabla u_i|^{p_i} \mathrm{d}x, \tag{2.3}$$

$$J_F(u) = J_F(u_1,\ldots,u_n) = \int_\Omega F(x,u_1(x),\ldots,u_n(x))\mathrm{d}x, \tag{2.4}$$

$$J_G(u) = J_G(u_1,\ldots,u_n) = \int_\Omega G(x,u_1(x),\ldots,u_n(x))\mathrm{d}x, \tag{2.5}$$

$$J_H(u) = J_H(u_1,\ldots,u_n) = \int_\Omega H(x,u_1(x),\ldots,u_n(x))\mathrm{d}x, \tag{2.6}$$

and

$$J(u) = J(u_1,\ldots,u_n) = \sum_{i=1}^n \frac{1}{p_i} \int_\Omega |u_i|^{p_i} \mathrm{d}x. \tag{2.7}$$

We denote by $\mathcal{F}$ the class of all continuous functions $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\sup_{(x,t)\in\Omega\times\mathbb{R}^n} \frac{|F(x,t)|}{|t| + |t|^q} < \infty,$$

where $1 < q < \min\{p_1^*,\ldots,p_n^*\}$, $t = (t_1,\ldots,t_n)$, and $|t| = \sqrt{t_1^2 + \ldots + t_n^2}$. In this paper, we assume that the functions F, G, H further satisfy the condition

(C) $F,G,H \in \mathcal{F}$.

Now, it is a standard procedure to prove the following properties of the above functionals. See, for example, the proof of [8, Lemma 2.1].

Lemma 2.4. *The functionals S, J_F, J_G, J_H, and J are well defined. Moreover, S, J_F, J_G, and J_H are continuous and J is compact.*

Moreover, it is easy to see that S, J_F, J_G, J_H, and J are continuously Gâteaux differentiable. More precisely, for every $u=(u_1,\dots,u_n)$, $v=(v_1,\dots,v_n)\in X$, we have

$$S'(u)(v)=\sum_{i=1}^{n}\int_{\Omega} m_i(x)|\nabla u_i(x)|^{p_i-2}\nabla u_i\nabla v_i \mathrm{d}x,$$

$$J'_F(u)(v)=\int_{\Omega}\sum_{i=1}^{n}F_{u_i}(x,u(x))v_i(x)\mathrm{d}x, \tag{2.8}$$

$$J'_G(u)(v)=\int_{\Omega}\sum_{i=1}^{n}G_{u_i}(x,u(x))v_i(x)\mathrm{d}x,$$

$$J'_H(u)(v)=\int_{\Omega}\sum_{i=1}^{n}H_{u_i}(x,u(x))v_i(x)\mathrm{d}x,$$

and

$$J'(u)(v)=\sum_{i=1}^{n}\int_{\Omega}|u_i|^{p_i-2}u_i v_i \mathrm{d}x.$$

Definition 2.5. We say that $u=(u_1,\dots,u_n)\in X$ is a weak solution of system (1.1) if and only if $(S'(u)-\varepsilon J'_F(u)+\lambda J'_G(u)+\nu J'_H(u))v=0$ for all $v=(v_1,\dots,v_n)\in X$.

Remark 2.6. In view of Definition 2.5, $u=(u_1,\dots,u_n)\in X$ is a weak solution of system (1.1) if and only if u is a critical point of the functional P defined by

$$P(u)=S(u)-\varepsilon J_F(u)+\lambda J_G(u)+\nu J_H(u). \tag{2.9}$$

In the following, for convenience, we let

$$\overline{\alpha}=\max\{\alpha_i : i=1,\dots,n\},\quad \underline{\alpha}=\min\{\alpha_i : i=1,\dots,n\},$$

and

$$\overline{p}=\max\{p_i : i=1,\dots,n\},\quad \underline{p}=\min\{p_i : i=1,\dots,n\}.$$

We now present two more results that are necessary in the proof of our main theorem.

Lemma 2.7. *Assume that either $\underline{p} \geq 2$ or $1 < \overline{p} < 2$. Let $\mathcal{J} := S' : X \longrightarrow X^*$ be the operator defined by*

$$\mathcal{J}(u)(v) = \sum_{i=1}^{n} \int_{\Omega} m_i(x) |\nabla u_i(x)|^{p_i - 2} \nabla u_i . \nabla v_i \mathrm{d}x$$

for every $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in X$. Then $\mathcal{J}$ admits a continuous inverse on X^*.

Below, we let C_i denote a generic positive constant.

Proof. Let $u = (u_1, \ldots, u_n) \in X \backslash \{0\}$. Note that

$$\langle \mathcal{J}(u), u \rangle = \sum_{i=1}^{n} \int_{\Omega} m_i(x) |\nabla u_i(x)|^{p_i} \mathrm{d}x \geq \|u_i\|_{m_i}^{p_i} \quad \text{for all } i = 1, \ldots, n,$$

and

$$\|u\|_X = \sum_{i=1}^{n} \|u_i\|_{m_i} \leq n \max\{\|u_i\|_{m_i} \ : \ i = 1, \ldots, n\}.$$

Then, we have

$$\langle \mathcal{J}(u), u \rangle \geq \frac{1}{n^{p_{i(u)}}} \|u\|_X^{p_{i(u)}}, \tag{2.10}$$

where $i(u) \in \{1, \ldots, n\}$ satisfies $\|u_{i(u)}\|_{m_{i(u)}} = \max\{\|u_i\|_{m_i} \ : \ i = 1, \ldots, n\}$. Thus,

$$\frac{\langle \mathcal{J}(u), u \rangle}{\|u\|_X} \geq \frac{1}{n^{\overline{p}}} \|u\|_X^{\underline{p}-1} \to \infty \quad \text{as } \|u\|_X \to \infty.$$

Thus, $\mathcal{J}$ is a coercive operator. Moreover, $\mathcal{J}(u)$ is a linear and continuous functional on X. Now, for any $u = (u_1, \ldots, u_n) \in X$ and $v = (v_1, \ldots, v_n) \in X$, we have

$$\begin{aligned} &\langle \mathcal{J}(u) - \mathcal{J}(v), u - v \rangle \\ &= \sum_{i=1}^{n} \int_{\Omega} m_i(x) \big(|\nabla u_i(x)|^{p_i - 2} \nabla u_i - |\nabla v_i(x)|^{p_i - 2} \nabla v_i \big) (\nabla u_i - \nabla v_i) \mathrm{d}x. \end{aligned}$$

Then by (2.2) in [16], we see that

$$\langle \mathcal{J}(u) - \mathcal{J}(v), u - v \rangle \geq \begin{cases} C_1 \sum_{i=1}^{n} \int_{\Omega} m_i(x) |\nabla u_i(x) - \nabla v_i(x)|^{p_i} \mathrm{d}x, & \underline{p} \geq 2, \\ C_2 \sum_{i=1}^{n} \int_{\Omega} \frac{m_i(x) |\nabla u_i(x) - \nabla v_i(x)|^2}{(|\nabla u_i(x)| + |\nabla v_i(x)|)^{2 - p_i}} \mathrm{d}x, & 1 < \overline{p} < 2. \end{cases} \tag{2.11}$$

If $\underline{p} \geq 2$, as in obtaining (2.10), it follows from (2.11) that

$$\langle \mathcal{J}(u) - \mathcal{J}(v), u - v \rangle \geq \frac{C_1}{n^{p_{i(u,v)}}} \|u - v\|_X^{p_{i(u,v)}},$$

where $i(u,v) \in \{1,\ldots,n\}$ satisfies $\|u_{i(u,v)} - v_{i(u,v)}\|_{m_{i(u,v)}} = \max\{\|u_i - v_i\|_{m_i} : i = 1,\ldots,n\}$. Thus, $\mathcal{J}$ is uniformly monotone. By [17, Theorem 26.A (d)], $\mathcal{J}^{-1}$ exists and is continuous on X^*.

If $1 < \overline{p} < 2$, by Hölder's inequality, we obtain that

$$\begin{aligned}
&\int_\Omega m_i(x)|\nabla u_i(x) - \nabla v_i(x)|^{p_i}\mathrm{d}x \\
&\leq \left(\int_\Omega \frac{m_i(x)|\nabla u_i(x) - \nabla v_i(x)|^2}{(|\nabla u_i(x)| + |\nabla v_i(x)|)^{2-p_i}}\mathrm{d}x\right)^{\frac{p_i}{2}} \\
&\qquad \times \left(\int_\Omega m_i(x)(|\nabla u_i(x)| + |\nabla v_i(x)|)^{p_i}\mathrm{d}x\right)^{\frac{2-p_i}{2}} \\
&\leq C_3\left(\int_\Omega \frac{m_i(x)|\nabla u_i(x) - \nabla v_i(x)|^2}{(|\nabla u_i(x)| + |\nabla v_i(x)|)^{2-p_i}}\mathrm{d}x\right)^{\frac{p_i}{2}} \\
&\qquad \times \left(\int_\Omega m_i(x)(|\nabla u_i(x)|^{p_i} + |\nabla v_i(x)|^{p_i})\mathrm{d}x\right)^{\frac{2-p_i}{2}} \\
&\leq C_4\left(\int_\Omega \frac{m_i(x)|\nabla u_i(x) - \nabla v_i(x)|^2}{(|\nabla u_i(x)| + |\nabla v_i(x)|)^{2-p_i}}\mathrm{d}x\right)^{\frac{p_i}{2}} (\|u\|_X + \|v\|_X)^{\frac{(2-p_i)p_i}{2}}. \qquad (2.12)
\end{aligned}$$

Now, let

$$p^\dagger = \begin{cases} \underline{p}, & \text{if } \|u\|_X + \|v\|_X \geq 1, \\ \overline{p}, & \text{if } \|u\|_X + \|v\|_X < 1. \end{cases}$$

Then, from (2.11) and (2.12), it follows that

$$\begin{aligned}
\langle \mathcal{J}(u) - \mathcal{J}(v), u - v\rangle &\geq C_5 \sum_{i=1}^n \frac{\left(\int_\Omega m_i(x)|\nabla u_i(x) - \nabla v_i(x)|^2\mathrm{d}x\right)^{\frac{2}{p_i}}}{(\|u\|_X + \|v\|_X)^{2-p_i}} \\
&\geq \frac{C_5}{(\|u\|_X + \|v\|_X)^{2-p^\dagger}} \sum_{i=1}^n \left(\int_\Omega m_i(x)|\nabla u_i(x) - \nabla v_i(x)|^2\mathrm{d}x\right)^{\frac{2}{p_i}} \\
&\geq \frac{C_6}{(\|u\|_X + \|v\|_X)^{2-p^\dagger}} \left(\sum_{i=1}^n \left(\int_\Omega m_i(x)|\nabla u_i(x) - \nabla v_i(x)|^2\mathrm{d}x\right)^{\frac{1}{p_i}}\right)^2 \\
&= \frac{C_6\|u - v\|_X^2}{(\|u\|_X + \|v\|_X)^{2-p^\dagger}}. \qquad (2.13)
\end{aligned}$$

Thus, $\mathcal{J}$ is strictly monotone. By [17, Theorem 26.A (d)], $\mathcal{J}^{-1}$ exists and is bounded. Given $w_1, w_2 \in X^*$, from (2.13), we have

$$\|\mathcal{J}^{-1}(w_1) - \mathcal{J}^{-1}(w_2)\|_X \leq \frac{1}{C_6}(\|\mathcal{J}^{-1}(w_1)\|_X + \|\mathcal{J}^{-1}(w_2)\|)^{2-p^\dagger}\|w_1 - w_2\|_{X^*}.$$

Thus, $\mathcal{J}^{-1}$ is locally Lipschitz continuous and hence continuous. This completes the proof of the lemma. $\square$

Lemma 2.8. *Let the functionals S and J be defined by* (2.3) *and* (2.7)*, respectively. Assume that condition (N) is satisfied. Denote by L the set*

$$\left\{ u = (u_1, \dots, u_n) \in X \ : \ J(u) = 1 \right\}.$$

Then, the equation $S'(u) = \lambda J'(u)$ admits a positive principal eigenvalue λ_1 satisfying

$$\lambda_1 = \inf_{(u_1,\dots,u_n)\in L} S(u). \tag{2.14}$$

Moreover, the associated normalized eigenfunction $\phi_1 = (\phi_{11}, \dots, \phi_{1n})$ belongs to X and each component is nonnegative.

This lemma can be proved by the same argument as contained in the proof of [19, Theorem 2.4]. For the completeness, we give its proof below.

Proof. In view of Lemma 2.4, we see that the operators S and J are continuously Fréchet differentiable such that

(i) S is coercive on $X \cap \{J(u) \leq c\}$, where c is a constant,

(ii) J is compact and $J'(u_1, \dots, u_n) = 0$ only at $(u_1, \dots, u_n) = (0, \dots, 0)$.

Then, from [2, Theorem 6.3.2], the equation $S'(u) = \lambda J'(u)$ admits a positive principal eigenvalue λ_1 satisfying (2.14). Moreover, if $(u_1, \dots, u_n)$ is a minimizer of (2.14), then $(|u_1|, \dots, |u_n|)$ should be also a minimizer. Hence, corresponding to λ_1 there exists an eigenfunction $(u_1, \dots, u_n)$ such that $u_i \geq 0$, $i = 1, \dots, n$, a.e. in Ω. $\square$

3. Main theorem

Let us first fix some notations that we will adopt in the sequel. For each $r > 0$ and each pair of functions F, $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ belonging $\mathcal{F}$ such that $G - F$ is bounded from below, let

$$\widetilde{\mu}(F, G, r) = \inf_{u \in X} \left\{ \frac{r - \widetilde{\gamma} - J_F(u)}{\widetilde{\eta}_r - \sum_{i=1}^n \frac{1}{p_i} \|u_i\|_{m_i}^{p_i}} \ : J_G(u) < r, \ \sum_{i=1}^n \frac{1}{p_i} \|u_i\|_{m_i}^{p_i} < \widetilde{\eta}_r \right\},$$

where

$$\widetilde{\gamma} = \int_\Omega \inf_{\xi \in \mathbb{R}^n} \big(G(x, \xi) - F(x, \xi) \big)$$

and

$$\widetilde{\eta}_r = \inf_{u\in J_G^{-1}(r)} \sum_{i=1}^{n} \frac{1}{p_i}\|u_i\|_{m_i}^{p_i}.$$

Moreover, for each $\varepsilon \in \left(0, \frac{1}{\max\{0,\widetilde{\mu}(F,G,r)\}}\right)$, we let

$$\widetilde{\beta}(\varepsilon,F,G,r) = \sup_{u\in J_G^{-1}((r,\infty))} \frac{1}{(r-J_G(u))}\left(\sum_{i=1}^{n}\frac{1}{p_i}\|u_i\|_{m_i}^{p_i} - \varepsilon J_F(u)\right.$$
$$\left. - \inf_{u\in J_G^{-1}((-\infty,r])}\left(\sum_{i=1}^{n}\frac{1}{p_i}\|u_i\|_{m_i}^{p_i} - \varepsilon J_F(u)\right)\right).$$

Now, we are in a position to state and prove the main result in this paper.

Theorem 3.1. *Assume that the conditions (N) and (C) hold and either $\underline{p} \geq 2$ or $1 < \overline{p} < 2$. Suppose further $F,\ G : \Omega\times\mathbb{R}^n \to \mathbb{R}$ satisfy that*

$$\lim_{|\xi|\to\infty} \frac{\inf_{x\in\Omega} F(x,\xi)}{|\xi|^{\overline{p}}} = \infty, \tag{3.1}$$

$$\limsup_{|\xi|\to\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{\sigma}} < \infty, \tag{3.2}$$

and

$$\lim_{|\xi|\to\infty} \frac{\inf_{x\in\Omega} G(x,\xi)}{|\xi|^{\sigma}} = \infty, \tag{3.3}$$

where $\xi = (\xi_1,\dots,\xi_n)$, $|\xi| = \sqrt{\xi_1^2+\dots+\xi_n^2}$, and $1 < \sigma < \min\{p_1^*,\dots,p_n^*\}$. Then, for each $r > 0$, for each $\varepsilon \in \left(0, \frac{1}{\max\{0,\widetilde{\mu}(F,G,r)\}}\right)$, and for each compact interval $[\bar{a},\bar{b}] \subset \left(0,\widetilde{\beta}(\varepsilon,F,G,r)\right)$, there exists a number $\rho > 0$ with the property: for every $\lambda \in [\bar{a},\bar{b}]$ and every function $H \in \mathcal{F}$, there exists $\delta > 0$ such that, for each $\nu \in [0,\delta]$, system (1.1) *has at least three weak solutions whose norms in X are less than ρ.*

Proof. Note that, according to the discussion in Section 2, $(X,\|\cdot\|_X)$ is a reflexive Banach space. Let the functionals S, J_F, J_G, J_H, and J be defined by (2.3)–(2.7), respectively. Then, in view of Lemma 2.7, it is easy to check that S is a sequentially weakly lower semicontinuous C^1-functional whose derivative admits a continuous inverse on X^*, and J_F, J_G, and J_H are C^1-functionals with compact derivatives. Moreover, from (2.3), we have

$$\frac{1}{\overline{p}}\sum_{i=1}^{n}\|u_i\|_{m_i}^{p_i} \leq S(u) \leq \frac{1}{\underline{p}}\sum_{i=1}^{n}\|u_i\|_{m_i}^{p_i} \quad \text{for all } u=(u_1,\dots,u_n)\in X. \tag{3.4}$$

which clearly implies that S is coercive and bounded on each bounded subset of X.

Next, we prove that

$$\limsup_{\|u\|_X\to\infty} \frac{J_F(u)}{\sum_{i=1}^n \|u_i\|_{m_i}^{p_i}} = \infty. \tag{3.5}$$

By Lemma 2.8, λ_1, defined by (2.14), is the positive principal eigenvalue of the equation $S'(u) = \lambda J'(u)$ and the associated normalized eigenfunction $\phi_1 = (\phi_{11}, \ldots, \phi_{1n})$ belongs to X and each component is nonnegative. Thus, we have

$$\sum_{i=1}^n \|\phi_{1i}\|_{m_i}^{p_i} = \lambda_1 \sum_{i=1}^n \|\phi_{1i}\|_{L^{p_i}}^{p_i}. \tag{3.6}$$

To prove (3.5), it suffices to show that

$$\lim_{k\to\infty} \frac{J_F(k\phi_1)}{\sum_{i=1}^n \|k\phi_{1i}\|_{m_i}^{p_i}} = \infty. \tag{3.7}$$

To this end, fix two positive numbers L_1 and L_2 such that $L_1 < \frac{L_2}{\overline{p}}$. From (3.1), there exists $\eta > 0$ such that

$$\begin{aligned} F(x,\xi) &\geq \frac{\lambda_1 L_2 \sum_{i=1}^n \|\phi_{1i}\|_{L^{p_i}}^{p_i}}{\sum_{i=1}^n \|\phi_{1i}\|_{L^{\overline{p}}}^{\overline{p}}} |\xi|^{\overline{p}} = \frac{\lambda_1 L_2 \sum_{i=1}^n \|\phi_{1i}\|_{L^{p_i}}^{p_i}}{\sum_{i=1}^n \|\phi_{1i}\|_{L^{\overline{p}}}^{\overline{p}}} \left(\sum_{i=1}^n |\xi_i|^2\right)^{\overline{p}/2} \\ &\geq \frac{\lambda_1 L_2 \sum_{i=1}^n \|\phi_{1i}\|_{L^{p_i}}^{p_i}}{\sum_{i=1}^n \|\phi_{1i}\|_{L^{\overline{p}}}^{\overline{p}}} \sum_{i=1}^n |\xi_i|^{\overline{p}} \quad \text{uniformly for all } (x,|\xi|) \in \Omega\times[\eta,\infty). \end{aligned} \tag{3.8}$$

For each $k \in \mathbb{N}$, set

$$A_k = \left\{x \in \Omega : |\phi_1(x)| \geq \frac{\eta}{k}\right\} = \left\{x \in \Omega : \phi_{11}^2(x) + \ldots + \phi_{1n}^2(x) \geq \frac{\eta^2}{k^2}\right\}.$$

Note that, for every $k \in \mathbb{N}$, we have $A_k \subseteq A_{k+1}$. Then, the numerical sequence

$$\left\{\sum_{i=1}^n \int_{A_k} |\phi_{1i}(x)|^{\overline{p}} \mathrm{d}x\right\}_{k\in\mathbb{N}}$$

is nondecreasing, i.e.,

$$\sum_{i=1}^n \int_{A_k} |\phi_{1i}(x)|^{\overline{p}} \mathrm{d}x \leq \sum_{i=1}^n \int_{A_{k+1}} |\phi_{1i}(x)|^{\overline{p}} \mathrm{d}x \quad \text{for every } k \in \mathbb{N}.$$

Moreover, we have

$$\sum_{i=1}^{n}\int_{A_k}|\phi_{1i}(x)|^{\overline{p}}\mathrm{d}x \to \sum_{i=1}^{n}\int_{\Omega}|\phi_{1i}(x)|^{\overline{p}}\mathrm{d}x = \sum_{i=1}^{n}\|\phi_{1i}\|_{L^{\overline{p}}}^{\overline{p}} \quad \text{as } k\to\infty.$$

Then, we can fix $\tilde{k}\in\mathbb{N}$ so that

$$\sum_{i=1}^{n}\int_{A_k}|\phi_{1i}(x)|^{\overline{p}}\mathrm{d}x > \frac{\overline{p}L_1}{L_2}\sum_{i=1}^{n}\|\phi_{1i}\|_{L^{\overline{p}}}^{\overline{p}} \quad \text{for any } k\geq\tilde{k}. \tag{3.9}$$

Since $F\in\mathcal{F}$, there exists a constant $c>0$ such that

$$|F(x,\xi)|\leq c(|\xi|+|\xi|^q) \quad \text{for all } \xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n.$$

Thus,

$$\sup_{(x,\xi)\in\Omega\times[0,\eta]^n}|F(x,\xi)|\leq c(\eta+\eta^q)<\infty,$$

where $[0,\eta]^n=[0,\eta]\times\ldots\times[0,\eta]$. Then, for each $k\in\mathbb{N}$ satisfying

$$k>\max\left\{\tilde{k},\ \left(\frac{\operatorname{meas}(\Omega)\sup_{(x,\xi)\in\Omega\times[0,\eta]^n}|F(x,\xi)|}{L_1\sum_{i=1}^{n}\|\phi_{1i}\|_{m_i}^{p_i}}\right)^{\frac{1}{\overline{p}}}\right\},$$

from (3.6), (3.8), and (3.9), it follows that

$$\begin{aligned}
\frac{J_F(k\phi_1)}{\sum_{i=1}^{n}\|k\phi_{1i}\|_{m_i}^{p_i}} &= \frac{\int_{A_k}F(x,k\phi_1(x))\mathrm{d}x}{\sum_{i=1}^{n}k^{p_i}\|\phi_{1i}\|_{m_i}^{p_i}} + \frac{\int_{\Omega\setminus A_k}F(x,k\phi_1(x))\mathrm{d}x}{\sum_{i=1}^{n}k^{p_i}\|\phi_{1i}\|_{m_i}^{p_i}}\\
&\geq \frac{\int_{A_k}F(x,k\phi_1(x))\mathrm{d}x}{k^{\overline{p}}\sum_{i=1}^{n}\|\phi_{1i}\|_{m_i}^{p_i}} + \frac{\int_{\Omega\setminus A_k}F(x,k\phi_1(x))\mathrm{d}x}{k^{\overline{p}}\sum_{i=1}^{n}\|\phi_{1i}\|_{m_i}^{p_i}}\\
&\geq \frac{\frac{\lambda_1L_2\sum_{i=1}^{n}\|\phi_{1i}\|_{L^{p_i}}^{p_i}}{\sum_{i=1}^{n}\|\phi_{1i}\|_{L^{\overline{p}}}^{\overline{p}}}\left(\sum_{i=1}^{n}\int_{A_k}|\phi_{1i}(x)|^{\overline{p}}\mathrm{d}x\right)}{\lambda_1\sum_{i=1}^{n}\|\phi_{1i}\|_{L^{p_i}}^{p_i}} + \frac{\int_{\Omega\setminus A_k}F(x,k\phi_1(x))\mathrm{d}x}{k^{\overline{p}}\sum_{i=1}^{n}\|\phi_{1i}\|_{m_i}^{p_i}}\\
&\geq \frac{\overline{p}L_1\sum_{i=1}^{n}\|\phi_{1i}\|_{L^{\overline{p}}}^{\overline{p}}}{\sum_{i=1}^{n}\|\phi_{1i}\|_{L^{\overline{p}}}^{\overline{p}}} - \frac{\operatorname{meas}(\Omega)\sup_{(x,\xi)\in\Omega\times[0,\eta]^n}|F(x,\xi)|}{k^{\overline{p}}\sum_{i=1}^{n}\|\phi_{1i}\|_{m_i}^{p_i}}\\
&> \overline{p}L_1-L_1=(\overline{p}-1)L_1,
\end{aligned}$$

which shows (3.7) holds. Thus, by (3.4) and (3.7), we have

$$\liminf_{\|u\|_X\to\infty}\frac{-J_F(u)}{S(u)}=-\infty.$$

Now, from (3.2), there exists $\kappa>0$ such that

$$F(x,\xi)\leq\kappa(|\xi|^\sigma+1), \quad \text{for every } (x,\xi)\in\Omega\times\mathbb{R}^n, \tag{3.10}$$

and from (3.3), for each $\iota > 0$, there exists a constant $c_\iota > 0$ such that

$$G(x,\eta) \geq \iota|\xi|^\sigma - c_\iota \quad \text{for every } (x,\xi) \in \Omega \times \mathbb{R}^n. \tag{3.11}$$

By (3.10) and (3.11), we see that, for each $\lambda > 0$, the functional $\lambda J_G - J_F$ is bounded from below in $\mathbb{R}^n$. In fact, for any fixed $\lambda > 0$, we can choose $\iota > 0$ large enough so that $\lambda\iota > \kappa$. Hence, by (3.10) and (3.11), it follows that

$$\begin{aligned}(\lambda G - F)(x,\xi) &\geq \lambda\iota|\xi|^\sigma - \lambda c_\iota - \kappa(|\xi|^\sigma + 1)\\ &= (\lambda\iota - \kappa)|\xi|^\sigma - (\lambda c_\iota + \kappa)\\ &\geq -(\lambda c_\iota + \kappa) \quad \text{for any } (x,\xi) \in \Omega \times \mathbb{R}^n.\end{aligned}$$

Thus,

$$\int_\Omega (\lambda G(x,u(x)) - F(x,u(x)))\mathrm{d}x \geq -(c_\iota + \lambda\kappa)\mathrm{meas}(\Omega).$$

Then, $\lambda J_G - J_F$ is bounded from below in X. Now, the conclusion of the theorem follows directly from Lemma 2.1, where it is taken that $E = X$, $I(u) = S(u)$, $\Psi(u) = -J_F(u)$, $\Phi(u) = J_G(u)$, and $\Gamma(u) = J_H(u)$. This completes the proof of the theorem. □

4. A scalar problem

As an application of Theorem 3.1, in this section, we consider the scalar problem

$$\begin{cases}-\nabla\left(m(x)|\nabla u|^{p-2}\nabla u\right) = \varepsilon f(u) - \lambda g(u) - \nu h(u), & x \in \Omega,\\ u|_{\partial\Omega} = 0,\end{cases} \tag{4.1}$$

where Ω is a bounded and connected subset of $\mathbb{R}^N$ $(N \geq 2)$, $p > 1$, $m \in (N)_p$ is a nonnegative weight function, ε, λ, and ν are nonnegative parameters, f, g, $h \in \mathrm{C}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and $\nabla u = (u_{t_1}, \ldots, u_{t_N})$ denotes the gradient of u with respect to $(t_1, \ldots, t_N) \in \mathbb{R}^N$.

We introduce the functions

$$F(t) = \int_0^t f(\xi)\mathrm{d}\xi \quad \text{for all } t \in \mathbb{R},$$

$$G(t) = \int_0^t g(\xi)\mathrm{d}\xi \quad \text{for all } t \in \mathbb{R},$$

and

$$H(t) = \int_0^t h(\xi)\mathrm{d}\xi \quad \text{for all } t \in \mathbb{R}.$$

Now consider the functionals T, J_f, J_g, $J_h : X \to \mathbb{R}$ defined by

$$T(u) = \frac{1}{p}\int_\Omega m(x)|\nabla u(x)|^p \mathrm{d}x,$$

$$J_f(u) = \int_\Omega F(u(x))\mathrm{d}x,$$

$$J_g(u) = \int_\Omega G(u(x))\mathrm{d}x,$$

and

$$J_h(u) = \int_\Omega H(u(x))\mathrm{d}x.$$

Then, T, J_f, J_g, and J_h are continuously Gâteaux differentiable. More precisely, for every u, $v \in \mathrm{D}_0^{1,p}(\Omega, m)$, we have

$$T'(u)(v) = \frac{1}{p}\int_\Omega m(x)|\nabla u(x)|^{p-2}\nabla u \nabla v \mathrm{d}x,$$

$$J_f'(u)(v) = \int_\Omega f(u(x))v(x)\mathrm{d}x,$$

$$J_g'(u)(v) = \int_\Omega g(u(x))v(x)\mathrm{d}x,$$

and

$$J_h'(u)(v) = \int_\Omega h(u(x))v(x)\mathrm{d}x.$$

We denote by $\mathcal{F}_1$ the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \frac{f(t)}{1+|t|^q} < \infty$$

for some $q \in (0, p_s^* - 1)$, where p_s^* is defined by (2.2).

We introduce the following notations. For each $r > 0$ and each pair of functions f, $g \in \mathcal{F}_1$ such that $G - F$ is bounded from below, let

$$\widetilde{\mu}(f,g,r) = p \inf_{u\in X}\left\{ \frac{r - \widetilde{\gamma} - J_f(u)}{\widetilde{\eta}_r - \|u\|_m^p} : J_g(u) < r,\, \|u\|_m^p < \widetilde{\eta}_r \right\},$$

where

$$\widetilde{\gamma} = \inf_{\xi\in\mathbb{R}} (G(\xi) - F(\xi)),$$

and

$$\widetilde{\eta}_r = \inf_{u\in J_g^{-1}(r)} \|u\|_m^p.$$

Moreover, for each $\varepsilon \in \left(0, \frac{1}{\max\{0,\widetilde{\mu}(f,g,r)\}}\right)$, define

$$\widetilde{\beta}(\varepsilon, f, g, r) = \sup_{u \in J_g^{-1}((r,\infty))} \frac{\|u\|_m^p - \varepsilon p J_f(u) - \inf_{u\in J_g^{-1}((-\infty,r])}(\|u\|_m^p - \varepsilon p J_f(u))}{p(r - J_g(u))}.$$

The following result is a direct consequence of Theorem 3.1.

Theorem 4.1. *Assume that* $f, g \in \mathcal{F}_1$ *and*

$$\lim_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|^{p-2}\xi} = \infty, \quad \limsup_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|^{\sigma-2}\xi} < \infty, \quad \text{and} \quad \lim_{|\xi|\to\infty} \frac{g(\xi)}{|\xi|^{\sigma-2}\xi} = \infty, \tag{4.2}$$

where $1 < \sigma < p_s^*$. *Then, for each* $r > 0$*, for each* $\varepsilon \in \left(0, \frac{1}{\max\{0,\widetilde{\mu}(f,g,r)\}}\right)$*, and for each compact interval* $[\bar{a}, \bar{b}] \subset \left(0, \widetilde{\beta}(\varepsilon, f, g, r)\right)$*, there exists a number* $\rho > 0$ *with the property: for every* $\lambda \in [\bar{a}, \bar{b}]$ *and every function* $h \in \mathcal{F}_1$*, there exists* $\delta > 0$ *such that, for each* $\nu \in [0, \delta]$*, system* (4.1) *has at least three weak solutions whose norms in* $\mathrm{D}_0^{1,p}(\Omega, m)$ *are less than* ρ.

We now give one example to apply Theorem 4.1.

Example 4.2. Let $N = 3$, $p = 4$, and $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \le 9\} \subset \mathbb{R}^3$. Consider the problem

$$\begin{cases} -\nabla\left((2 + \sin(x_1^2 + x_2^2 + x_3^2))|\nabla u|^{p-2}\nabla u\right) = \varepsilon f(u) - \lambda g(u) - \nu h(u) & \text{in}\,\Omega, \\ u = 0 & \text{on}\,\partial\Omega, \end{cases} \tag{4.3}$$

where

$$f(t) = \left(1 + t^4\right)\operatorname{sgn}(t) \quad \text{and} \quad g(t) = \left(1 + t^{\frac{28}{5}}\right)\operatorname{sgn}(t).$$

We observe that $m(x) = 2 + \sin x \in (N)_p$ with $s = 1$, $\operatorname{meas}(\Omega) = 36\pi$, and $p_s^* = 6$. Moreover, it is easy to see that f, $g \in \mathcal{F}_1$ and (4.2) holds with $\sigma = 26/5$. Then, the conclusion of Theorem 4.1 holds for problem (4.3).

We comment that, for some special functions f and g in (4.1), it is possible to obtain some estimates for the constants $\widetilde{\mu}(f, g, r)$ and $\widetilde{\beta}(\varepsilon, f, g, r)$. For instance, let $p < \kappa < \zeta < p^*$, $f(t) = |t|^{\kappa-2}t$, and $g(t) = |t|^{\zeta-2}t$, where $p^* = \frac{Np}{2N-p}$. Then, we have

$$F(t) = \frac{1}{\kappa}|t|^{\kappa}, \quad G(t) = \frac{1}{\zeta}|t|^{\zeta},$$

$$J_f(u) = \frac{1}{\kappa}\int_\Omega |u(x)|^{\kappa} dx, \quad \text{and} \quad J_g(u) = \frac{1}{\zeta}\int_\Omega |u(x)|^{\zeta} dx.$$

As a consequence, $\widetilde{\gamma} = \inf_{\xi\in\mathbb{R}}(G(\xi) - F(\xi)) = \frac{1}{\zeta} - \frac{1}{\kappa}$, $J_f(0) = J_g(0) = 0$, and

$$\|u\|_{L^\zeta}^\zeta = \int_\Omega |u(x)|^\zeta dx = \zeta r \quad \text{for any } u \in J_g^{-1}(r) \text{ with } r > 0. \tag{4.4}$$

By Lemma 2.3, there exists $C > 0$ such that

$$C\|u\|_{L^\zeta} \le \|u\|_m \quad \text{for all } u \in \mathrm{D}_0^{1,p}(\Omega, m). \tag{4.5}$$

For any $r > 0$, define a constant K_1 by

$$K_{1,r} = \frac{p\left(r - \frac{1}{\zeta} + \frac{1}{\kappa}\right)}{C^p(\zeta r)^{\frac{p}{\zeta}}}. \tag{4.6}$$

Fix a small positive ε so that the set $\Omega_1 = \{x \in \Omega \ : \ \text{dist}(x, \partial\Omega) \ge 2\varepsilon\} \neq \emptyset$. Choose $u_\varepsilon \in \mathrm{D}_0^{1,p}(\Omega, m)$ such that $0 \le u_\varepsilon(x) \le 2\left(\frac{r\zeta}{|\Omega_1|}\right)^{1/\zeta}$ and

$$u_\varepsilon(x) = \begin{cases} 2\left(\frac{r\zeta}{|\Omega_1|}\right)^{1/\zeta}, & x \in \Omega_1, \\ 0, & x \in \Omega_2, \end{cases}$$

where $\Omega_2 = \{x \in \Omega \ : \ \text{dist}(x, \partial\Omega) \le \varepsilon\}$ and $|S|$ denotes the Lebesgue measure of a set S. Then,

$$J_g(u_\varepsilon) \ge \frac{1}{\zeta}\int_{\Omega_1} |u_\varepsilon(x)|^\zeta dx \ge 2^\zeta r > r.$$

and

$$\|u_\varepsilon\|_m^p = \int_{\Omega_3} m(x)|\nabla u_\varepsilon|^p dx,$$

where $\Omega_3 = \{x \in \Omega \ : \ \varepsilon \le \text{dist}(x, \partial\Omega) \le 2\varepsilon\}$. Thus, $u_\varepsilon \in J_g^{-1}((r, \infty))$. For any $u \in J_g^{-1}((-\infty, r])$, we have $J_g(u) = \frac{1}{\zeta}\int_\Omega |u(x)|^\zeta dx \le r$. Hence, $\int_\Omega |u(x)|^\zeta dx \le \zeta r$. Recall that $\kappa < \zeta$. Then, from Hölder's inequality, it follows that

$$\begin{aligned} J_f(u) &= \frac{1}{\kappa}\int_\Omega |u(x)|^\kappa dx \\ &\le \frac{1}{\kappa}|\Omega|^{\frac{\zeta-\kappa}{\zeta}}\left(\int_\Omega |u(x)|^\zeta dx\right)^{\frac{\kappa}{\zeta}} \le \frac{1}{\kappa}(\zeta r)^{\frac{\kappa}{\zeta}}|\Omega|^{\frac{\zeta-\kappa}{\zeta}}, \end{aligned}$$

which in turn implies that

$$\|u\|_m^p - \varepsilon p J_f(u) \ge \gamma := -\frac{1}{\kappa}(\zeta r)^{\frac{\kappa}{\zeta}}|\Omega|^{\frac{\zeta-\kappa}{\zeta}} \quad \text{for all } u \in J_g^{-1}((-\infty, r]).$$

Thus, we have

$$\inf_{u\in J_g^{-1}((-\infty,r])} (\|u\|_m^p - \varepsilon p J_f(u)) \geq \gamma. \tag{4.7}$$

Note that $\kappa > p$ and

$$\|ku_\varepsilon\|_m^p - \varepsilon p J_f(ku_\varepsilon)) = k^p \|u_\varepsilon\|_m^p - k^\kappa \varepsilon p J_f(u_\varepsilon) \quad \text{for any } k > 0.$$

Then, we can choose a large $k_1 = k_1(\varepsilon, r, p, \zeta, \kappa) > 1$ such that $J_g(k_1 u_\varepsilon) = k_1^\zeta J_g(u_\varepsilon) > r$ and

$$\|k_1 u_\varepsilon\|_m^p - \varepsilon p J_f(k_1 u_\varepsilon) < \gamma - p J_g(u_\varepsilon). \tag{4.8}$$

Define a constant $K_{2,r} = K_{2,r}(\varepsilon, r, p, \zeta, \kappa) > 0$ by

$$K_{2,r} = \frac{1}{k_1^\zeta}. \tag{4.9}$$

The following result provides an upper bound for $\widetilde{\mu}(f,g,r)$ and a lower bound for $\widetilde{\beta}(\varepsilon, f, g, r)$.

Corollary 4.3. *Assume that* $f(t) = |t|^{\kappa-2}t$ *and* $g(t) = |t|^{\zeta-2}t$ *in* (4.1)*, where* $p < \kappa < \zeta < p^* := \frac{Np}{2N-p}$*. Then,*

$$\widetilde{\mu}(f,g,r) \leq K_{1,r} \quad \textit{and} \quad \widetilde{\beta}(\varepsilon, f, g, r) \geq K_{2,r}, \tag{4.10}$$

where $K_{1,r}$ *and* $K_{2,r}$ *are defined by* (4.6) *and* (4.9)*, respectively.*

Proof. From (4.4), (4.5), and the definition of $\widetilde{\eta}_r$, we see that, for any $n \in \mathbb{N}$, there exists $u_n \in J_g^{-1}(r)$ such that

$$\frac{1}{n} + \widetilde{\eta}_r \geq \|u_n\|_m^p \geq C^p \|u_n\|_{L^\zeta}^p = C^p (\zeta r)^{\frac{p}{\zeta}},$$

i.e,

$$\frac{1}{n} + \widetilde{\eta}_r \geq C^p (\zeta r)^{\frac{p}{\zeta}} \quad \text{for any } n \in \mathbb{N}.$$

Hence, $\widetilde{\eta}_r \geq C^p (\zeta r)^{\frac{p}{\zeta}}$. Then, in view of the definition of $\widetilde{\mu}(f,g,r)$, we have

$$\widetilde{\mu}(f,g,r) \leq \frac{p\left(r - \frac{1}{\zeta} + \frac{1}{\kappa}\right)}{\widetilde{\eta}_r} \leq \frac{p\left(r - \frac{1}{\zeta} + \frac{1}{\kappa}\right)}{C^p (\zeta r)^{\frac{p}{\zeta}}} = K_{1,r}.$$

Thus, the first inequality in (4.10) holds.

Now, from (4.7), (4.8), and the definition of $\widetilde{\beta}(\varepsilon, f, g, r)$, it follows that

$$\begin{aligned}\widetilde{\beta}(\varepsilon, f, g, r) &\geq \frac{\gamma - (\|k_1 u_\varepsilon\|_m^p - \varepsilon p J_f(k_1 u_\varepsilon))}{p(J_g(k_1 u_\varepsilon) - r)} \\ &\geq \frac{p J_g(u_\varepsilon)}{p J_g(k_1 u_\varepsilon)} = \frac{1}{k_1^\zeta} = K_{2,r}.\end{aligned}$$

Hence, the second inequality in (4.10) also holds. This completes the proof of the corollary. □

The following corollary is a consequence of Theorem 4.1 and Corollary 4.3.

Corollary 4.4. *Assume that $p > 1$ and $m \in (N)_p$ with $s = 1$ in $(N)_p$. Let $p < \kappa < \zeta < p^*$ with $p^* = \frac{Np}{2N-p}$. Then, for each $r > 0$, for each $\varepsilon \in \left(0, \frac{1}{K_{1,r}}\right)$, and for every compact interval $[\bar{a}, \bar{b}] \subset (0, K_{2,r})$ there exists $\rho > 0$ with the property: for every $\lambda \in [\bar{a}, \bar{b}]$ and every continuous function $h \in \mathcal{F}_1$, there exists $\delta > 0$ such that for every $\nu \in [0, \delta]$, the problem*

$$\begin{cases} -\nabla\left(m(x)|\nabla u|^{p-2}\nabla u\right) = \varepsilon|u|^{\kappa-2}u - \lambda|u|^{\zeta-2}u - \nu h(u) & \text{in}\,\Omega, \\ u = 0 & \text{on}\,\partial\Omega, \end{cases}$$

has at least three weak solutions whose norms in $\mathrm{D}_0^{1,p}(\Omega, m)$ are less than ρ.

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