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SPECTRAL ANALYSIS FOR A DISCONTINUOUS SECOND ORDER ELLIPTIC OPERATOR

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The spectrum of a second order elliptic operator S, with ellipticity constant α discontinuous in a point, is studied in L^p spaces. It turns out that, for (α, p) in a set $\mathcal{A}$, classical results for the spectrum of smooth elliptic operators (see e.g. [3]) remain true for S; in particular, it is proved that S is the infinitesimal generator of an holomorphic semigroup . If $(\alpha, p) \notin \mathcal{A}$, then the spectrum of S is the whole complex plane.

Let $S = S_\alpha$ be the second order uniformly elliptic operator, in two dimensions, defined as:

$$S := \alpha\Delta + (1-2\alpha)\sum_{h,k=1}^{2} \frac{x^h x^k}{(x^1)^2 + (x^2)^2} \frac{\partial}{\partial x^h} \frac{\partial}{\partial x^k}; \tag{1}$$

$\alpha \in (0, \frac{1}{2})$ is the (lower) ellipticity constant, $1 - \alpha$ is the (upper) ellipticity constant (if $\alpha = \frac{1}{2}$, then: $S = \frac{1}{2}\Delta$). The operator above, discontinuous at the origin, has been mainly used to construct counterexamples (see e.g. [5], [7], [11]). The existence and uniqueness theorem for the Dirichlet problem in Sobolev spaces has been proved in [8].

In the present work the spectrum of S is studied in L^p spaces (in a disk, with Dirichlet boundary conditions). It turn out that there exists $\mathcal{A} \subset \mathbf{R}^2$ with

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the property that, if $(\alpha, p) \in \mathcal{A}$, then S behaves as the Laplace operator: (i) it has a pure point spectrum, with eigenvalues and eigenvectors explicitly constructed using Bessel functions; (ii) the resolvent is a compact operator and its norm goes to zero as $O(|\lambda|^{-1})$ when $\lambda \to +\infty$ in a sector around the positive real axis, with opening larger than π. On the other hand, if $(\alpha, p) \notin \mathcal{A}$, then the spectrum of the operator coincides with the complex plane.

The classical techniques, used to get (i) and (ii) above for the Laplacian and for elliptic equations, cannot be used in the present case, due to the discontinuity of the coefficients; we had to construct ad hoc a priori bounds. The asymptotic behavior of the resolvent will enables us, in a forthcoming paper, to study the Cauchy problem for the parabolic operator, discontinuous on an axis: $\dot{u} = Su$, $u|_{t=0} = u_0$.

In section 1, notations, preliminary facts on Bessel functions and separation of variables techniques for S are considered. In section 2 the closed operator $\mathcal{S}$, acting in L^p, defined by S with homogeneous Dirichlet boundary conditions in a disk, is considered; the spectrum of $\mathcal{S}$ and $\mathcal{S}^*$ is studied. In section 3 several a priori bounds are proved and the asymptotic behavior of $(\mathcal{S} - \lambda)^{-1}$ is studied, when $\lambda \to \infty$ in a sector $|\arg\lambda| \le \frac{\pi}{2} + \epsilon, \epsilon > 0$.

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1. Notations and preliminary results.

Let us introduce several constants and functions used later: let $\alpha \in (0, \frac{1}{2})$, $\alpha' := 1 - \alpha$, $p \in (1, +\infty)$, $p' := \frac{p}{p-1}$. Let $\{l_\nu, \nu \in \mathbf{Z}\}$ be the sequence, defined as:

$$(2) \quad l_0 := 0, \quad l_\nu = l_\nu(\alpha) := 1 - \frac{1}{2\alpha'} + \sqrt{(1 - \frac{1}{2\alpha'})^2 + \frac{\alpha}{\alpha'}\nu^2} \quad \nu \in \mathbf{Z} \setminus \{0\}.$$

Notice that: $l_{\pm 1} = 1$, $l_\nu \in (1, |\nu|)$ if $\nu \in \mathbf{Z} \setminus \{-1, 0, 1\}$; for fixed ν, l_ν is an increasing function of $\alpha \in (0, \frac{1}{2})$.

Let us also introduce the sequence:

$$(3) \qquad h_\nu = h_\nu(\alpha) := l_\nu(\alpha) - (1 - \frac{1}{2\alpha'}).$$

Let us notice that: $h_0 = -1 + \frac{1}{2\alpha'} \in (-\frac{1}{2}, 0)$, and $h_\nu (\nu \in \mathbf{Z} \setminus \{0\})$ is positive.

From now on, the spaces $\mathbf{R}^2$ and $\mathbf{C}$ will be identified, by writing: $z = (x, y) = \rho e^{i\theta}$, where: $x = \Re z = \rho \cos\theta$, $y = \Im z = \rho \sin\theta$.

Let $z \in \mathbf{C}$; in what follows $\sqrt{z}$ will always be the principal value of the square root.

Bessel functions of first kind $J_{\pm h_\nu}(z)$, linearly independent solutions to the Bessel equation:

(4) $$z^2 w_{zz} + z w_z + (z^2 - h_\nu^2) w \; = \; 0$$

will be considered (h_ν not an integer). If h_ν is an integer, then linearly independent solutions to (4) are $J_{h_\nu}(z)$ and $Y_{h_\nu}(z) \; := \; (\pi)^{-1} \; [\frac{\partial}{\partial\mu} J_\mu(z) - (-1)^{h_\nu} \frac{\partial}{\partial\mu} J_{-\mu}(z)]_{\mu = h_\nu}$.

Modified Bessel functions of first kind $I_{\pm h_\nu}(z) \; = \; e^{\mp h_\nu \frac{\pi}{2} i} J_{\pm h_\nu}(iz)$, of index $\pm h_\nu$, $\nu \in \mathbf{Z}$ will also be used.

Let us list some of the properties of the functions above, for later use (for them, see e.g. Watson's book [13]).

(i)

(5) $$J_{\pm h_\nu}(z) = (\frac{z}{2})^{\pm h_\nu} \frac{1}{\Gamma(1 \pm h_\nu)} \, {}_0F_1(1 \pm h_\nu, -(\frac{z}{2})^2)$$

(6) $$I_{\pm h_\nu}(z) = (\frac{z}{2})^{\pm h_\nu} \frac{1}{\Gamma(1 \pm h_\nu)} \, {}_0F_1(1 \pm h_\nu, (\frac{z}{2})^2);$$

here ${}_0F_1$ is the entire generalized hypergeometric function, defined as :

$${}_0F_1(a, z) := 1 + \sum_{m=1}^{\infty} \frac{z^m}{a(a+1)\cdots(a+m-1)m!}.$$

The functions $Y_{h_\nu}(z)$ (h_ν integer) can also be written as:

(7) $$Y_{h_\nu}(z) = 2[\gamma + \log(\frac{z}{2})] J_{h_\nu}(z) - \sum_{m=0}^{h_\nu - 1} \frac{(h_\nu - m - 1)!}{m!} (\frac{z}{2})^{2m - h_\nu}$$

(here γ is Euler's constant).

(ii) Let $0 < \omega_0 \leq \frac{\pi}{4}$; as a consequence of the asymptotic expansion of I_{h_ν}, we have:

(8) $$\lim_{z \to \infty, \; |arg z| \leq \omega_0} I_{h_\nu}(z) \, e^{-z} \sqrt{2\pi z} \; = \; 1$$

(see e.g. [13]).

(iii) For any integer ν, the functions: $z^{-h_\nu} J_{h_\nu}(z)$ are entire even functions with real simple zeros only; let:

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \cdots \qquad n \in \mathbf{N} \tag{9}$$

be the increasing sequence of the positive zeros of $z^{-h_\nu} J_{h_\nu}(z)$; then:

$$J_{h_\nu}(z) = \frac{(\frac{z}{2})^{h_\nu}}{\Gamma(1+h_\nu)} \prod_{n=1}^{\infty} \big(1 - \frac{z^2}{j_{\nu,n}^2}\big), \tag{10}$$

$$I_{h_\nu}(z) = \frac{(\frac{z}{2})^{h_\nu}}{\Gamma(1+h_\nu)} \prod_{n=1}^{\infty} \big(1 + \frac{z^2}{j_{\nu,n}^2}\big). \tag{11}$$

Lemma 1. *Let $\nu \in \mathbf{Z}$, $|\omega| \le \frac{\pi}{4}$; $\sigma \ge 0$, $\rho > 0$, $0 \le r \le \rho$. The bound:*

$$\left| \left(\frac{r}{\rho}\right)^{1-\frac{1}{2\alpha'}} \frac{I_{h_\nu}(r\sigma e^{i\omega})}{I_{h_\nu}(\rho\sigma e^{i\omega})} \right| \le \big(\frac{r}{\rho}\big)^{l_\nu} \tag{12}$$

holds.

Proof. By (11),

$$\left| \left(\frac{r}{\rho}\right)^{1-\frac{1}{2\alpha'}} \frac{I_{h_\nu}(r\sigma e^{i\omega})}{I_{h_\nu}(\rho\sigma e^{i\omega})} \right| = \big(\frac{r}{\rho}\big)^{l_\nu} \cdot \left| \frac{\prod_{n=1}^{\infty} \big(1 + \frac{(r\sigma e^{i\omega})^2}{j_{\nu,n}^2}\big)}{\prod_{n=1}^{\infty} \big(1 + \frac{(\rho\sigma e^{i\le})^2}{j_{\nu,n}^2}\big)} \right|.$$

As:

$$\left| \frac{\big(1 + \frac{(r\sigma e^{i\omega})^2}{j_{\nu,n}^2}\big)}{\big(1 + \frac{(\rho\sigma e^{i\omega})^2}{j_{\nu,n}^2}\big)} \right| \le \frac{|(r\sigma e^{i\omega})^2 + j_{\nu,n}^2|}{|(\rho\sigma e^{i\omega})^2 + j_{\nu,n}^2|} \le 1, \tag{13}$$

the thesis follows. □

Lemma 2. *Let $\nu \in \mathbf{Z}$; there exists $K(\alpha, \nu)$, depending on α, ν only, such that if $\rho > 0, 0 \le r \le \rho$, $\sigma \ge 0$, $|\omega| \le \frac{\pi}{4}$, the bound:*

$$\left| \left(\frac{r}{\rho}\right)^{1-\frac{1}{2\alpha'}} \frac{I_{h_\nu}(r\sigma e^{i\omega})}{I_{h_\nu}(\rho\sigma e^{i\omega})} \right| \le K(\alpha, \nu)\, e^{\frac{\sigma}{2\sqrt{2}}(r-\rho)} \tag{14}$$

holds.

Proof. Let $\mathcal{J}$ be the left hand side of (14) and let κ be any constant (not necessarily the same) depending on α, ν only.

From (8), there exists σ_0 depending on α, ν only, with the property that, if $|z| \geq \frac{\sigma_0}{2}$, and $|\arg z| \leq \frac{\pi}{4}$, then:

$$k_1 |z|^{-\frac{1}{2}} e^{\Re z} \leq \left| I_{h_\nu}(z) \right| \leq k_2 |z|^{-\frac{1}{2}} e^{\Re z} \tag{15}$$

$(k_1 = (2\sqrt{2\pi})^{-1},\ k_2 = 2(\sqrt{2\pi})^{-1}$ will do) .

Let us write $w := \rho\sigma e^{i\omega}$, $\epsilon := \frac{r}{\rho} \in [0, 1]$; then:

$$\mathcal{J} = |\epsilon^{1-\frac{1}{2\alpha'}} I_{h_\nu}(\epsilon w) \cdot (I_{h_\nu}(w))^{-1}|.$$

The condition $|\omega| \leq \frac{\pi}{4}$, implies that $w \in W := \{|w| \leq \sqrt{2}\Re w\}$ and $|w| e^{-\frac{\Re w}{2}} \leq \sqrt{2} e^{-1}$ in W.

Several subcases will be considered.

(A) Let: $|w| \geq \sigma_0$, $\epsilon \geq \frac{1}{2}$; then: $|\epsilon w| \geq \frac{\sigma_0}{2}$, and (15) gives us:

$$\begin{aligned} \mathcal{J} &\leq \kappa \cdot k_2 |\epsilon w|^{-\frac{1}{2}} e^{\epsilon \Re w} \, [k_1 |w|^{-\frac{1}{2}} e^{\Re w}]^{-1} \\ &= \kappa \cdot \frac{k_2}{k_1} \, \epsilon^{-\frac{1}{2}} e^{(\epsilon - 1)\Re w} \\ &\leq \kappa \cdot \frac{k_2}{k_1} \, 2^{\frac{1}{2}} e^{\frac{\sigma}{2\sqrt{2}}(r-\rho)}. \end{aligned}$$

(B) Let: $|w| \geq \sigma_0$, $0 \leq \epsilon \leq \frac{1}{2}$, $|\epsilon w| \leq \sigma_0$; then (6) gives us:

$$|(\epsilon w)^{1-\frac{1}{2\alpha'}} \, I_{h_\nu}(\epsilon w)| \leq \kappa |\epsilon w|^{l_\nu} \, {}_0F_1(1 + h_\nu, (\frac{\sigma_0}{2})^2) \leq \kappa;$$

Last inequality, (15) and the fact that $w \in W$, give us:

$$\begin{aligned} \mathcal{J} &\leq \kappa \, [k_1 |w|^{-\frac{1}{2}+1-\frac{1}{2\alpha'}} e^{\Re w}]^{-1} \leq \kappa |w|^{\frac{1}{2}} e^{-\Re w} \\ &\leq \kappa (\frac{\sqrt{2}}{e})^{\frac{1}{2}} e^{-(1-\epsilon)\frac{\Re w}{2}} \\ &\leq \kappa \, e^{\frac{\sigma}{2\sqrt{2}}(r-\rho)}. \end{aligned}$$

(C) Let: $|w| \geq \sigma_0$, $0 \leq \epsilon \leq \frac{1}{2}$, $|\epsilon w| > \sigma_0$. As in (A) :

$$\mathcal{J} \leq \kappa \cdot \frac{k_2}{k_1}\, \epsilon^{-\frac{1}{2}} e^{(\epsilon-1)\Re w}.$$

then:

$$\mathcal{J} \leq \kappa \left(\frac{1}{\epsilon |w|}\right)^{\frac{1}{2}} |w|^{\frac{1}{2}} e^{\frac{\Re w}{2}}\, e^{\frac{\epsilon-1}{2}\Re w} \leq \kappa e^{\frac{\sigma}{2\sqrt{2}}(r-\rho)}.$$

(D) Let: $|w| < \sigma_0$. As in the previous lemma:

$$\mathcal{J} \leq \left(\frac{r}{\rho}\right)^{l_\nu} \leq 1 \leq e^{\frac{\sigma_0}{2\sqrt{2}}}\, e^{\frac{\sigma}{2\sqrt{2}}(r-\rho)}.$$

From (A), (B), (C), (D), the thesis follows. □

The operator S, defined in (1), is a second order, uniformly elliptic operator, regular in $\mathbf{R}^2 \setminus \{(0,0)\}$ and discontinuous at $(0,0)$. Its formal adjoint is:

$$S^* u = \alpha \Delta u + (1-2\alpha) \sum_{h,k=1}^{2} \frac{\partial}{\partial x^h} \frac{\partial}{\partial x^k} \Big(\frac{x^h x^k}{(x^1)^2 + (x^2)^2} u\Big);$$

in polar coordinates:

$$Su(\rho e^{i\theta}) = \big(\alpha' \frac{\partial^2}{\partial \rho^2} u + \frac{1-\alpha'}{\rho} \frac{\partial}{\partial \rho} u + \frac{\alpha}{\rho^2} u_{\theta\theta}\big)(\rho e^{i\theta}), \tag{16}$$

$$S^* u(\rho e^{i\theta}) = \Big[\alpha\Big(u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta}\Big) + \frac{1-2\alpha}{\rho} (\rho u)_{\rho\rho}\Big](\rho e^{i\theta}). \tag{17}$$

In polar form, solutions to both S and S^* can be constructed by using separation of variables.

Remark 1.1. *Let u be of the form: $u(\rho e^{i\theta}) = u_\nu(\rho) e^{i\nu\theta}$; then:*

$$Su(\rho e^{i\theta}) = (s_\nu u_\nu(\rho)) e^{i\nu\theta} \tag{18}$$

$$S^* u(\rho e^{i\theta}) = (s_\nu^* u_\nu(\rho)) e^{i\nu\theta} \tag{19}$$

where:

$$s_\nu u_\nu(\rho) := \Big(\alpha' u_\nu'' + \frac{1-\alpha'}{\rho} u_\nu' - \frac{\nu^2 \alpha}{\rho^2} u_\nu\Big)(\rho)$$

$$s_\nu^* u_\nu(\rho) := \Big(\alpha u_\nu'' + \frac{\alpha}{\rho} u_\nu' - \frac{\nu^2 \alpha}{\rho^2} u_\nu + \frac{1-2\alpha}{\rho} \big(\rho u_\nu\big)''\Big)(\rho).$$

By using Lommel transformations, the following facts can be proved.

Remark 1.2. *Let* $\lambda \in \mathbf{C}$, $\alpha \in (0, \frac{1}{2})$, $\nu \in \mathbf{Z}$. *The equation:*

$$s_\nu u(\rho) - \lambda u(\rho) = 0 \quad in \quad (0, +\infty) \tag{20}$$

has the two independent solutions:

$$\rho^{(1-\frac{1}{2\alpha'})} J_{h_\nu}\Big(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\Big), \quad \rho^{(1-\frac{1}{2\alpha'})} J_{-h_\nu}\Big(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\Big).$$

(If h_ν is not an integer, $J_{-h_\nu} = J_{h_\nu}$ must be substituted by the Bessel function of second kind, Y_{h_ν}).

The two independent solutions (if h_ν is not an integer) can also been written as:

$$\rho^{(1-\frac{1}{2\alpha'})} I_{h_\nu}\Big(\sqrt{\frac{\lambda}{\alpha'}}\,\rho\Big), \quad \rho^{(1-\frac{1}{2\alpha'})} I_{-h_\nu}\Big(\sqrt{\frac{\lambda}{\alpha'}}\,\rho\Big)$$

The equation:

$$s^*_\nu u(\rho) - \lambda u(\rho) = 0 \quad in \quad (0, +\infty) \tag{21}$$

has the two independent solutions:

$$\rho^{(\frac{1}{2\alpha'}-1)} J_{h_\nu}\Big(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\Big), \quad \rho^{(\frac{1}{2\alpha'}-1)} J_{-h_\nu}\Big(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\Big)$$

(If h_ν is integer, the Bessel function $J_{-h_\nu} = J_{h_\nu}$ in the second solution must be substituted by the Bessel function of second kind, Y_{h_ν}).

The two independent solutions (if h_ν is not an integer) can also been written as:

$$\rho^{(\frac{1}{2\alpha'}-1)} I_{h_\nu}\Big(\sqrt{\frac{\lambda}{\alpha'}}\,\rho\Big), \quad \rho^{(\frac{1}{2\alpha'}-1)} I_{-h_\nu}\Big(\sqrt{\frac{\lambda}{\alpha'}}\,\rho\Big).$$

2. Spectral properties of S and S^*.

It will always assumed in what follows (except in theorem 4 below) that the lower ellipticity constant α of S and the exponent p belong to:

$$\mathcal{A} = \left\{(\alpha, p) \in R^2 : 0 < \alpha < \frac{1}{2}, \qquad 2\alpha' < p < \frac{2}{2 - l_2(\alpha)}\right\}.$$

The operator S will be studied in the open disk B_R, centered in (0, 0) with radius R. $W^{2,p}$ will be the space of (complex valued) functions in L^p with first and second derivatives in L^p; $W^{2,p}_{\gamma_0}(B_R) := \{u \in W^{2,p}(B_R) : u = 0 \text{ on } \partial B_R\}$.

Let us recall an existence-uniqueness theorem for the Dirichlet problem for S (see [8]).

Fact 1. Let $(\alpha, p) \in \mathcal{A}$; for every (complex valued) $f \in L^p(B_R)$ the Dirichlet problem:

$$u \in W^{2,p}_{\gamma_0}(B_R), \qquad Su = f \text{ a.e. in } B_R \tag{23}$$

has a unique solution, satisfying the a priori bound:

$$||u||_{W^{2,p}(B_R)} \leq k(\alpha, p, R)\, ||Su||_{L^p(B(R)};\tag{24}$$

the constant $k(\alpha, p, R)$ depends on α, p, R only.

Let $(\alpha, p) \in \mathcal{A}$; let us define the operator $\mathcal{S} = \mathcal{S}_{\alpha,p,R}$ in $L^p(B_R)$ in the following way: let $D(\mathcal{S}) = W^{2,p}_{\gamma_0}(B_R)$; then, if $u \in D(\mathcal{S})$, let us set: $\mathcal{S}u := Su$.

Theorem 1. *Let $(\alpha, p) \in \mathcal{A}$, $R > 0$. The following properties hold.*

(i) $\mathcal{S}$ is closed, densely defined, with range $L^p(B_R)$.
(ii) $\mathcal{S}^{-1}$ is defined in $L^p(B_R)$ and is compact.
(iii) The spectrum $\Sigma(\mathcal{S})$ consists of isolated eigenvalues of finite multiplicity.

Proof. Proposition (i) is immediate consequence of Fact 1.

Again by Fact 1, the map:

$$L^p(B_R) \ni f \mapsto u \in W^{2,p}_{\gamma_0}(B_R)$$

is continuous; thus, by Rellich-Kondrachev theorem, the map:

$$L^p(B_R) \ni f \mapsto u \in L^p(B_R)$$

is a compact operator in $L^p(B_R)$; (ii) and (iii) are consequence of classical theorems in the theory of compacts operators (see e. g. [6], III, thm. 6.29). □

Remark 2.1. *Let $(\alpha, p) \in \mathcal{A}$, $R > 0$, $\lambda \in P(\mathcal{S}) := \mathbf{C} \setminus \Sigma(\mathcal{S})$; then, there exists $k(\alpha, p, R, \lambda)$, such that, for every $f \in L^p(B_R)$, the problem:*

$$u \in W^{2,p}_{\gamma 0}(B_R), \qquad Su - \lambda u = f \text{ a.e. in } B_R \tag{25}$$

has a unique solution, satisfying:

$$||u||_{W^{2,p}_{\gamma 0}(B_R)} \leq k(\alpha, p, R, \lambda)||f||_{L^p(B_R)}. \tag{26}$$

Let C be a compact subset of $P(\mathcal{S})$. Then, there exists $k(\alpha, p, R, C)$, such that, for every $f \in L^p(B_R)$, $\lambda \in C$ the solution of (25) satisfies:

$$\begin{aligned} &||D^2u||_{L^p(B_R)} + \sqrt{|\lambda|}\, ||Du||_{L^p(B_R)} + |\lambda|\, ||u||_{L^p(B_R)} \leq \\ &\leq k(\alpha, p, R, C)\, ||f||_{L^p(B_R)}. \end{aligned} \tag{27}$$

Proof. Let $\lambda \in P(\mathcal{S})$ and $u = u_\lambda := (\mathcal{S} - \lambda)^{-1} f$. Clearly u is the unique solution of (25).

As $||u_\lambda||_{L^p(B_R)}$ is a continuous function of λ in $P(\mathcal{S})$, then there exists $k(\alpha, p, R, C)$, depending on α, p, R, C only, such that, $|\lambda|\ ||u||_{L^p(B_R)} \leq k(\alpha, p, R, C)\ ||f||_{L^p(B_R)}$.

By (24) and interpolation theorems, then (26) and (27) follow.

□

By using remarks 1.1 and 1.2 a more precise analysis of the spectrum of $\mathcal{S}$ can be done. If $(\alpha, p) \in \mathcal{A}$, spectrum and eigenvectors closely look like those of Laplace operator.

Theorem 2. *Let $(\alpha, p) \in \mathcal{A}$. The spectrum $\Sigma(\mathcal{S})$ of $\mathcal{S}$ lies on the real negative axis. More precisely:*

(28) $$\Sigma(\mathcal{S}) = \{-\alpha'(j_{\nu,m})^2 R^{-2} :\ \nu \in \mathbf{N} \cup \{0\}, m \in \mathbf{N}\}.$$

Let:

(29) $$\omega_{\nu,m}(\rho) = \rho^{(1-\frac{1}{2\alpha'})} J_{h_\nu}(j_{\nu,m}\frac{\rho}{R}).$$

If J_{h_ν} and $J_{h_{\nu'}} (\nu \neq \nu')$ have no common zeros outside of the origin: (i) the eigenvalues $-\alpha' j_{0,m}^2 R^{-2}$ $(m \in \mathbf{N})$, have multiplicity one, with eigenfunction $\omega_{0,m}(\rho)$ in $C^\infty(\overline{B_R})$; (ii) the eigenvalues $-\alpha' j_{1,m}^2 R^{-2}$ $(m \in \mathbf{N}\ )$, have two linearly independent eigenfunctions $\omega_{1,m}(\rho)e^{i\theta}$, $\omega_{1,m}(\rho)e^{-i\theta}$, in $C^\infty(\overline{B_R})$; (iii) the eigenvalues $-\alpha' j_{\nu,m}^2 R^{-2}(\nu \in \mathbf{N}, \nu \geq 2, m \in \mathbf{N})$, have two linearly independent eigenfunctions $\omega_{\nu,m}(\rho)e^{i\nu\theta}$, $\omega_{\nu,m}(\rho)e^{-i\nu\theta}$, in $C^\infty(\overline{B_R} \setminus \{(0,0)\})$, that in a neighborhood of $(0,0)$ are of the form: a smooth even function of ρ times $\rho^{l_\nu} e^{i\nu\theta}, \rho^{l_\nu} e^{-i\nu\theta}$ (respectively); if $1 < l_\nu(\alpha) < 2$, the second factor is in $W^{2,\tilde{p}}_{\gamma_0}(B_R)$, for every $\tilde{p} \in [2, \frac{2}{2-l_\nu(\alpha)})$; if $l_\nu(\alpha) \geq 2$ then $\rho^{l_\nu} e^{\pm i\nu\theta}$ is in $C^2(\overline{B_R})$.

Proof. Let λ be an eigenvalue of $\mathcal{S}$, defined as a complex number such that there exists a nonzero $u \in D(\mathcal{S}) = W^{2,p}_{\gamma_0}(B_R)$ satisfying:

(30) $$Su - \lambda u = 0 \qquad \text{in } B_R, \qquad u_{\partial B_R} = 0.$$

As S is smooth in $\mathbf{R}^2 \setminus (0,0)$, then $u \in C^\infty(\overline{B}_R) \setminus (0,0)$; by Fact 1, for some $\tilde{p} > 2$, $u \in W^{2,\tilde{p}}_{\gamma_0}(B_R)$; then u has Hölder continuous first derivatives in $\overline{B}_R$.

Let us expand u in Fourier series in θ :

(31) $$u(\rho e^{i\theta}) \sim \sum_{\nu=-\infty}^{\infty} u_\nu(\rho) e^{i\nu\theta}$$

where:

$$u_\nu(\rho) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta})e^{-i\nu\theta}d\theta;$$

the functions u_ν satisfy the conditions: (i) $u_\nu \in C^\infty(0, R]$ with Hölder continuous first derivatives in $[0, R]$; (ii):

$$s_\nu u_\nu = \lambda u_\nu \quad \text{in } (0, R], \quad u_\nu(R) = 0;$$

(iii) if $\nu \neq 0$, then $u_\nu(0) = 0$; if $\nu = 0$, then $u_0'(0) = 0$. By remark 1.2, u_ν must be of the form:

$$u_\nu(\rho) = c_\nu^{(1)}\,\rho^{(1-\frac{1}{2\alpha'})} J_{h_\nu}\Big(\sqrt{-\frac{\lambda}{\alpha'}}\rho\Big) \;+\; c_\nu^{(2)}\,\rho^{(1-\frac{1}{2\alpha'})} J_{-h_\nu}\Big(\sqrt{-\frac{\lambda}{\alpha'}}\rho\Big). \tag{32}$$

(if h_ν is a positive integer, then J_{-h_ν} must be substituted with Y_{h_ν}).

Let ν be nonzero, then, by (5), the first term in (32) is of the form ρ^{l_ν} times a smooth even function of ρ; the second term does not have continuous first derivative and in almost all the cases is unbounded; then $c_\nu^{(2)}$ must be zero. It is not difficult to see that $u_\nu(\rho)e^{i\nu\theta} \in W^{2,\tilde{p}}_{\gamma_0}(B_R)$.

Let $\nu = 0$; by (5), the first term in (32) is a smooth even function of ρ; the second term is $\rho^{2-\frac{1}{\alpha'}}$ times a smooth even function of ρ, not vanishing at zero; as $2 - \frac{1}{\alpha'} \in (0, 1)$, the second term cannot satisfy the condition $u_0'(0) = 0$, so $c_0^{(2)}$ must be zero.

The condition $u_\nu(R) = 0$, implies that:

$$\sqrt{-\frac{\lambda}{\alpha'}}R = j_{\tilde{\nu},\tilde{m}} \qquad \text{for some } \tilde{\nu} \in \mathbf{N} \cup \{0\},\ \tilde{m} \in \mathbf{N}.$$

Then, the eigenvalues of S are of as in (28) and the eigenvectors of S related to the eigenvalue $-\alpha'(j_{\tilde{\nu},\tilde{m}})^2 R^{-2}$ are of the form:

$$\sum_{\{\nu\in\mathbf{Z}:\, j_{|\nu|,m}=j_{\tilde{\nu},\tilde{m}}\}} c_\nu^{(1)}\,\rho^{(1-12\alpha')} J_{h_\nu}\Big(j_{|\nu|,m}\frac{\rho}{R}\Big)e^{i\nu\theta}. \tag{33}$$

The thesis follows. □

Let $\mathcal{S}^*$ be the adjoint operator to $\mathcal{S}$, defined and valued in $L^{p'}(B_R)$; then,

$$D(\mathcal{S}^*) \;:=\; \{v \in L^{p'}(B_R):\ \exists\ g \in L^{p'}(B_R) \ \text{ such that } \forall u \in D(\mathcal{S})$$

$$\int_{B_R} v\,\overline{\mathcal{S}u}\,dxdy \;=\; \int_{B_R} g\,\overline{u}\,dxdy\ \}.$$

The properties of $\mathcal{S}$ stated in theorem 1 imply similar properties for $\mathcal{S}^*$.

Remark 2.2. *Let* $(\alpha, p) \in \mathcal{A}$, $R > 0$.

(i) $\mathcal{S}^*$ *is closed, densely defined, with range* $L^{p'}(B_R)$.

(ii) $(\mathcal{S}^*)^{-1}$ *is defined in* $L^p(B_R)$ *and is compact.*

(iii) *As* $\Sigma(\mathcal{S})$ *is real, then:* $\mathcal{S}^*$ *and* $\mathcal{S}$ *have the same eigenvalues with same multiplicity.*

An equivalent definition of $D(\mathcal{S}^*)$ follows.

Lemma 3. $D(\mathcal{S}^*)$ *is the subset of* $\{v \in W^{2,p'}_{loc}(\overline{B_R} \setminus \{(0,0)\}) \cap L^{p'}(B_R)\}$, *satisfying: (i)* $\mathcal{S}^* v \in L^{p'}(B_R)$, $v|_{\partial B_R} = 0$; *(ii)* $v \in C^0(\overline{B_R} \setminus \{(0,0)\})$ *and, for every* $r \in (0, R)$, $v_r(re^{i(\cdot)}) \in L^{p'}(0, 2\pi)$; *(iii)* $\forall u \in C^\infty(\overline{B_R})$, $u|_{\partial B_R} = 0$:

$$0 = \lim_{r\to 0^+} \int_0^{2\pi} \{(1-2\alpha)\overline{u}(re^{i\theta})\, v(re^{i\theta}) + \\ - \alpha' r\, [\overline{u_r}(re^{i\theta})\, v(re^{i\theta}) - v_r(re^{i\theta})\, \overline{u}(re^{i\theta})]\} d\theta. \tag{34}$$

Proof. Let $v \in D(\mathcal{S}^*)$; thus there exists $g \in L^{p'}(B_R)$, with the property that, for every complex valued $u \in W^{2,p}_{\gamma_0}(B_R)$, vanishing in a neighborhood of the origin,

$$\int_{B_R} v\, \overline{u}\, dxdy = \int_{B_R} g\, \overline{u}\, dxdy.$$

Classical regularity theorems for elliptic equations with smooth coefficients, imply that $v \in W^{2,p'}_{loc}(\overline{B_R} \setminus \{(0,0)\}\})$, $v|_{\partial B_R} = 0$; and that $g = S^* v$ a. e.; then, (i) follows. (ii) follows from (i) and immersion theorems in Sobolev spaces.

Let us recall (see e. g. [8]) that $\forall r > 0, \forall u \in W^{2,p}(B_R \setminus B_r)$, $\forall v \in W^{2,p'}(B_R \setminus B_r)$, satisfying $u|_{\partial B_R} = 0$, $v|_{\partial B_R} = 0$, the equality

$$\int_{B_R\setminus B_r} [\, v\, \overline{Su} - \overline{u}\, S^* v]\, dxdy = \int_0^{2\pi} \{(1-2\alpha)\overline{u}(re^{i\theta})\, v(re^{i\theta}) + \\ - \alpha' r\, [\overline{u_r}(re^{i\theta})\, v(re^{i\theta}) - v_r(re^{i\theta})\, \overline{u}(re^{i\theta})]\} d\theta. \tag{35}$$

holds.

Let us use (35) in the present case: as $r \to 0^+$, the left hand side of last equation tends to zero; (iii) follows.

On the other hand, if $v \in W^{2,p'}_{loc}(\overline{B_R} \setminus \{(0,0)\}\})$, and satisfies (i) and (34), then by (35), $v \in D(\mathcal{S}^*)$. □

Remark 2.3. *Let* $v \in W^{2,p'}_{loc}(\overline{B_R} \setminus \{(0,0)\})$. *Sufficient condition for (34) to hold is: there exist* δ, K *positive, such that the bounds* $|v(\rho e^{i\theta})| \le K\rho^\delta$, $|v_\rho(\rho e^{i\theta})| \le K\rho^{\delta-1}$, *hold.*

Remark 2.4. *Let* $v \in W^{2,p'}_{loc}(\overline{B_R} \setminus \{(0,0)\})$. *Sufficient condition for (34) to hold is: there exist a constant* $c \in \mathbf{C}$ *and a function* w *satisfying the conditions of previous remark, such that:* $v(\rho e^{i\theta}) = c\rho^{\frac{1}{\alpha'}-2} + w(\rho e^{i\theta})$.

By using previous statements, the eigenvectors of $\mathcal{S}^*$ can be studied, as in theorem 2.

Theorem 3. *Let* $(\alpha, p) \in \mathcal{A}$. *Let us consider the spectrum of* $\mathcal{S}^*$. *If* J_{h_ν} *and* $J_{h_{\nu'}} (\nu \neq \nu')$ *have no common zeros outside of the origin: (i) The eigenvalues:* $-\alpha' j^2_{0,m} R^{-2} (m \in \mathbf{N})$, *have one eigenfunction of the form:* $\rho^{\frac{1}{\alpha'}-2} \cdot \omega_{0,m}(\rho)$; *(ii) the eigenvalues:* $-\alpha' j^2_{\nu,m} R^{-2} (\nu \in \mathbf{N},\ m \in \mathbf{N})$, *have two linearly independent eigenfunctions, of the form:* $\rho^{\frac{1}{\alpha'}-2}\omega_{\nu,m}(\rho)e^{i\nu\theta}$, $\rho^{\frac{1}{\alpha'}-2} \cdot \omega_{\nu,m}(\rho)e^{-i\nu\theta}$.

If $(\alpha, p) \notin \mathcal{A}$, the spectrum of $\mathcal{S}$ is completely different from the previous case.

Theorem 4. *Let* $(\alpha, p) \in (0, \frac{1}{2}) \times (1, +\infty) \setminus \mathcal{A}$; *then* $\mathcal{S}$ *is a closable operator, but:*

$$\Sigma(\mathcal{S}_{\alpha,p,R}) = \Sigma(\mathcal{S}^*_{\alpha,p,R}) = \mathbf{C}.$$

Proof. Let us prove that $\mathcal{S}$ is a closable operator. Let $u_n \in D(\mathcal{S})$, and $u_n \to 0$, $\mathcal{S}u_n \to g$ in $L^p(B_R)$. Let $0 < r < R, v \in W^{2,p'}_{\gamma_0}(B_R), v \equiv 0.$ in a neighbourhood of B_r. Let us write (35) with $u = u_n$ and pass to the limit as $n \to \infty$; then: $\int_{B_R \setminus B_r} v\, \overline{g}\, dxdy = 0$; thus $g = 0$ a. e. in $B_R \setminus B_r$. Then, $g = 0$ a. e. in B_R, i.e. $\mathcal{S}$ is closable.

Let us study now the spectrum of $\mathcal{S}$.

Several subcases will be considered.

(A) Let: $0 < \alpha < \frac{1}{2}$, $1 < p < 2\,(1-\alpha)$. Let us show that every $\mathcal{A} \in \mathbf{C}$, is an eigenvalue to $\mathcal{S}$.

Let $\sqrt{-\frac{\lambda}{\alpha'}}R = j_{0,m}$, (for some $m \in \mathbf{N}$); then, as in theorem 2, λ is an eigenvalue for $\mathcal{S}$, with eigenfunction $u_\lambda(\rho) = \omega_{0,m}(\rho)$ in $C^\infty(\overline{B_R})$.

Let $\sqrt{-\frac{\lambda}{\alpha'}}R = \hat{j}_{0,m}$ (for some $m \in \mathbf{N}$), where $\{\hat{\beth}_{0,m}\}$ $(m \in \mathbf{N}$), are the positive zeros of $J_{-h_0}(z)$ (clearly J_{h_0} and J_{-h_0} have no common zero outside of the origin); then, λ is an eigenvalue for $\mathcal{S}$, with eigenfunction: $u_\lambda(\rho) = \rho^{(1-\frac{1}{2\alpha'})} J_{-h_0}\big(\sqrt{-\frac{\lambda}{\alpha'}}\ \rho\big)$ of the form: $\rho^{2-\frac{1}{\alpha'}}$ times a smooth even function of ρ; thus: $u_\lambda \in W^{2,p}_{\gamma_0}(B_R)$ if $p < 2\,(1-\alpha)$.

Let $\lambda \in \mathbf{C}$, satisfying the condition:

$$\sqrt{-\frac{\lambda}{\alpha'}}R \neq j,$$

where j is an arbitrary positive zero to $J_{h_0}(z)$ or to $J_{-h_0}(z)$. Then, by remarks (1.1), (1.2), λ is an eigenvalue to $\mathcal{S}$ with the eigenfunction:

$$u_\lambda(\rho) := \left(\frac{\rho}{R}\right)^{(1-\frac{1}{2\alpha'})} \frac{J_{h_0}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\right)}{J_{h_0}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,R\right)} - \left(\frac{\rho}{R}\right)^{(1-\frac{1}{2\alpha'})} \frac{J_{-h_0}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\right)}{J_{-h_0}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,R\right)} \quad \lambda \neq 0,$$

$$u_\lambda(\rho) := 1 - \left(\frac{\rho}{R}\right)^{2-\frac{1}{\alpha'}} \quad \lambda = 0;$$

the function u_λ is of the form: a smooth even function of ρ minus $\rho^{2-\frac{1}{\alpha'}}$ times a smooth even function of ρ; thus $u_\lambda \in W^{2,p}_{\gamma_0}(B_R)$ if $p < 2\,(1-\alpha)$.

(B) Let: $0 < \alpha < \frac{1}{2}$, $p = 2\,(1-\alpha)$.

Let $\sqrt{-\frac{\lambda}{\alpha'}}R = j_{\nu,m}$, (for some $\nu \in \mathbf{N} \cup \{0\}$, $m \in \mathbf{N}$); then, as in theorem 2, λ is an eigenvalue for $\mathcal{S}$.

Let $\lambda \in \mathbf{C}$ be not an eigenvalue for $\mathcal{S}$. For every $\epsilon > 0$, let us introduce the functions:

$$b_{\lambda,\epsilon}(\rho) := \epsilon^{\frac{1}{2\alpha'}-1} \rho^{\epsilon+1-\frac{1}{2\alpha'}} J_{-h_0}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\right) \quad \lambda \neq 0,$$

$$b_{\lambda,\epsilon}(\rho) := \epsilon^{\frac{1}{2\alpha'}-1} \rho^{\epsilon+2-\frac{1}{\alpha'}} \quad \lambda = 0.$$

Notice that, by (5), $b_{\lambda,\epsilon}$ is of the form: $\epsilon^{\frac{1}{2\alpha'}-1} \rho^{\epsilon+1-\frac{1}{2\alpha'}} f(\rho^2)$, where f is a smooth function and $f(0) = 1$.

It is not difficult to show that:

$$\lim_{\epsilon \to 0} \int_{B_R} |(S-\lambda) b_{\lambda,\epsilon}|^{2\alpha'} dx dy = \text{finite and positive} \tag{36}$$

$$\lim_{\epsilon \to 0} \int_{B_R} |b_{\lambda,\epsilon}|^{2\alpha'} dx dy = +\infty. \tag{37}$$

Let $\sqrt{-\frac{\lambda}{\alpha'}}R = \hat{j}_{0,m}$ (for some $m \in \mathbf{N}$), where $\{\hat{j}_{0,m}\}$ ($m \in \mathbf{N}$), are the positive zeros of $J_{-h_0}(z)$; then $b_{\lambda,\epsilon} \in W^{2,2\alpha'}_{\gamma_0}(B_R)$ and (36), (37) imply that a bound of the form:

$$||(S-\lambda)u||_{L^{2\alpha'}(B_R)} \geq K||u||_{L^{2\alpha'}(B_R)}$$

(for every $u \in D(\mathcal{S})$), cannot hold. Thus λ is in the spectrum of $\mathcal{S}$.

Let $\lambda \in \mathbf{C}$, satisfying the condition:

$$\sqrt{-\frac{\lambda}{\alpha'}}R \neq j,$$

where j is an arbitrary positive zero of $J_{h_0}(z)$ or of $J_{-h_0}(z)$. Then: $b_{\lambda,\epsilon}(R) \neq 0$.

Let us define :

$$v_{\lambda,\epsilon}(\rho) := \epsilon^{\frac{1}{2\alpha'}-1}\left\{\left(\frac{\rho}{R}\right)^{(\frac{1}{2\alpha'}-1)}\frac{J_{h_0}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\right)}{J_{h_0}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,R\right)} - \frac{b_{\lambda,\epsilon}(\rho)}{b_{\lambda,\epsilon}(R)}\right\} \quad \lambda \neq 0,$$

$$v_{0,\epsilon}(\rho) := \epsilon^{\frac{1}{2\alpha'}-1}\left\{1 - \frac{b_{0,\epsilon}(\rho)}{b_{0,\epsilon}(R)}\right\} \quad \lambda = 0;$$

the functions $v_{\lambda,\epsilon} \in W^{2,2\alpha'}_{\gamma_0}(B_R)$ and by (36), (37):

$$\lim_{\epsilon \to 0}\int_{B_R} |(S-\lambda)v_{\lambda,\epsilon}|^{2\alpha'}dxdy \;=\; \text{finite and positive}$$

$$\lim_{\epsilon \to 0}\int_{B_R} |v_{\lambda,\epsilon}|^{2\alpha'}dxdy = +\infty$$

Therefore, a bound of the form:

$$||(S-\lambda)u||_{L^{2\alpha'}(B_R)} \geq K||u||_{L^{2\alpha'}(B_R)}$$

(for every $u \in D(\mathcal{S})$), cannot hold. Thus λ is in the spectrum of $\mathcal{S}$.

(C) Let $0 < \alpha < \frac{1}{2}$, $\frac{2}{2-l_2(\alpha)} < p$. Let us show that every $\lambda \in \mathbf{C}$, is an eigenvalue of $\mathcal{S}^*$.

Let $\sqrt{-\frac{\lambda}{\alpha'}}R = j_{2,m}$, (for some $m \in \mathbf{N}$); then (theorem 3), λ is an eigenvalue for $\mathcal{S}^*$, with eigenfunctions: $v_\lambda(\rho e^{i\theta}) := \rho^{(-1+\frac{1}{2\alpha'})}J_{h_2}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\right)e^{\pm 2i\theta}$.

Let $\sqrt{-\frac{\lambda}{\alpha'}}R = \hat{j}_{2,m}$ (for some $m \in \mathbf{N}$), where $\{\hat{\beth}_{2,m}\}$ ($m \in \mathbf{N}$), are the zeros of $J_{-h_2}(z)$ (clearly J_{h_2} and J_{-h_2} have no common zeros); then, λ is an eigenvalue for $\mathcal{S}^*$, with eigenfunctions: $\tilde{v_\lambda}(\rho e^{i\theta}) := \rho^{(-1+\frac{1}{2\alpha'})}J_{-h_2}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\right)$ $e^{\pm 2i\theta}$; let us notice explicitly that $\tilde{v_\lambda} \in L^{p'}(B_R)$ if and only if $p' < \frac{2}{l_2}$ i. e. if and only if $p > \frac{2}{2-l_2}$.

Let $\lambda \in \mathbf{C}$, satisfying the condition:

$$\sqrt{-\frac{\lambda}{\alpha'}}R \neq j,$$

where j is an arbitrary positive zero of $J_{h_2}(z)$ or of $J_{-h_2}(z)$. Then, λ is an eigenvalue to $\mathcal{S}^*$ with eigenfunctions:

$$\left(\frac{\rho}{R}\right)^{(1-\frac{1}{2\alpha'})}\left\{\frac{J_{h_2}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\right)}{J_{h_2}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,R\right)} - \frac{J_{-h_2}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\right)}{J_{-h_2}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,R\right)}\right\}e^{\pm 2i\theta} \quad \lambda \neq 0,$$

$$\left\{\left(\frac{\rho}{R}\right)^{(l_2-2+\frac{1}{\alpha'})} - \left(\frac{\rho}{R}\right)^{(-l_2)}\right\}e^{\pm 2i\theta} \quad \lambda = 0.$$

(D) Let $0 < \alpha < \frac{1}{2}$, $p = \frac{2}{2-l_2(\alpha)}$.
Let $\psi \in C^\infty(\overline{B_R})$, $\psi|_{\partial B_R} = 0$, $\psi = 1$ in a neighbourhood $\mathcal{U}$ of $(0, 0)$.
Let:

$$v_0(\rho e^{i\theta}) := \psi(\rho e^{i\theta}) \cdot \rho^{(1-\frac{1}{2\alpha'})} J_{h_2}\left(\sqrt{-\frac{\lambda}{\alpha'}}\,\rho\right)e^{2i\theta} \quad \lambda \neq 0,$$

$$v_0(\rho e^{i\theta}) := \psi(\rho e^{i\theta}) \cdot \rho^{l_2} e^{2i\theta} \quad \lambda = 0.$$

Notice that: (i) in $\mathcal{U}$, by (5), v_0 is of the form: $\rho^{l_2} f(\rho^2)e^{2i\theta}$, where f is a smooth function and $f(0) = 1$; (ii) $(S - \lambda)v_0 \in C^\infty(\overline{B_R})$ and $v_0 \in W^{2,\tilde{p}}_{\gamma_0}(B_R)$ (for every $1 < \tilde{p} < \frac{2}{2-l_2(\alpha)}$), but $v_0 \notin W^{2,\frac{2}{2-l_2(\alpha)}}_{\gamma_0}(B_R)$.

Therefore, the problem

$$(S - \lambda)u = (S - \lambda)v_0 \ \text{ in } B_R,\ u \in W^{2,\tilde{p}}_{\gamma_0}(B_R)$$

has the unique solution $u = v_0$ if $1 < \tilde{p} < \frac{2}{2-l_2(\alpha)}$, and no solution if $\tilde{p} = \frac{2}{2-l_2(\alpha)}$. Thus λ is in the spectrum of $\mathcal{S}^*$. $\square$

3. A priori bounds and asymptotic behaviour of $\mathcal{S}$.

Theorem 5. *Let* $(\alpha, p) \in \lambda$, $R > 0$, $s \in [1, \infty)$, $\lambda \in \mathbf{C}$, $\Re\lambda \geq 0$, $\phi \in L^s[0, 2\pi]$, $\phi(\theta) \sim \sum_{\nu=-\infty}^{\infty} \phi_\nu e^{i\nu\theta}$.

The problem:

$$w \in W^{2,p}_{loc}(B_R), \qquad Sw - \lambda w = 0 \quad \text{a.e. in } B_R, \tag{38}$$

$$\lim_{\rho \to R} \int_0^{2\pi} |w(\rho e^{i\theta}) - \phi(\theta)|^s d\theta = 0, \tag{39}$$

has a unique solution:

$$w(\rho e^{i\theta}) = \sum_{\nu=-\infty}^{+\infty} \gamma_{\nu,\rho,R} \phi_\nu \, e^{i\nu\theta}, \tag{40}$$

where:

$$\gamma_{\nu,\rho,R} := \left(\frac{\rho}{R}\right)^{1-\frac{1}{2\alpha'}} \cdot \frac{I_{h_\nu}\left(\sqrt{\frac{\lambda}{\alpha'}}\rho\right)}{I_{h_\nu}\left(\sqrt{\frac{\lambda}{\alpha'}}R\right)}. \tag{41}$$

The function w satisfies the bounds:

$$\left(\int_0^{2\pi} |w(\rho e^{i\theta})|^s \, d\theta\right)^{\frac{1}{s}} \leq \left(\int_0^{2\pi} |\phi(\theta)|^s \, d\theta\right)^{\frac{1}{s}}, \quad 0 \leq \rho \leq R; \tag{42}$$

$$||w||_{W^{2,p}_{loc}(B_r)} \leq k(\alpha, p, \lambda, r, s)||\phi||_{L^s(\partial B_R)}, \quad 0 < r < R; \tag{43}$$

here $k(\alpha, p, \lambda, r, s)$ depends on α, p, λ, r, s only. If, moreover , $\phi_\nu = 0$ for $|\nu| < \nu_0$, then there exists a constant $k(\alpha, \nu_0, s)$ depending on α, ν_0, s only, such that the bounds:

$$\left(\int_0^{2\pi} |w(\rho e^{i\theta})|^s \, d\theta\right)^{\frac{1}{s}} \leq k(\alpha, \nu_0, s) \left(\frac{\rho}{R}\right)^{l_{\nu_0}} \left(\int_0^{2\pi} |\phi(\theta)|^s \, d\theta\right)^{\frac{1}{s}} \tag{44}$$

($0 \leq \rho \leq R$) hold.

Proof. The condition $\Re\lambda \geq 0$ gives us $\left| \arg\sqrt{\frac{\lambda}{\alpha'}} \right| \leq \frac{\pi}{4}$. By lemma 1:

$$\left|\left(\frac{\rho}{R}\right)^{1-\frac{1}{2\alpha'}} \frac{I_{h_\nu}(\rho\sqrt{\frac{\lambda}{\alpha'}})}{I_{h_\nu}(R\sqrt{\frac{l}{\alpha'}})}\right| \leq \left(\frac{\rho}{R}\right)^{l_\nu}. \tag{45}$$

Let us assume for a moment that ϕ is a trigonometric polynomial; one can look for a solution w to $Sw - \lambda w = 0$ a. e. in B_R, trigonometric polynomial in θ; by remark 1.2 and (39), it is not difficult to see that w is of the form (40); then, using standard techniques, one can see that the problem: (38), (39) has such a function as unique solution in $W^{2,\tilde{p}}(B_R) \cap C^\infty(\overline{B_R} \setminus (0,0))$, for some $\tilde{p} > 2$;

By using a complex maximum principle (see [2]), we have that $S|w| \geq 0$ in $B_R \setminus \{|w| = 0\}$.

In [9] it has been shown that the problem: $v \in W^{2,\tilde{p}}_{loc}(B_R) \cap C^0(\overline{B_R})$, $Sv = 0$ in B_R, $v|_{\partial B_R} = |\phi|$ has a unique positive solution, satisfying the inequalities:

$$\int_0^{2\pi} \left(v(\rho e^{i\theta})\right)^s d\theta \leq \int_0^{2\pi} (|\phi(\theta)|)^s d\theta \quad 0 \leq \rho \leq R. \tag{46}$$

By the maximum principle $|w| \leq v$ in B_R. Then (42) follows from (46).

The bounds (43) follow with standard arguments using Fact 1, (42) and classical bounds for smooth elliptic equations.

Let us prove (44). Let $\phi_\nu = 0$ for $|\nu| < \nu_0$ and $0 < \rho \leq \frac{R}{2}$; thus, by (45):

$$\int_0^{2\pi} |w(\rho e^{i\theta})|^s d\theta \leq 2\pi \cdot \left(\sum_{|\nu| \geq \nu_0} \left| \left(\frac{\rho}{R}\right)^{1-\frac{1}{2\alpha'}} \frac{I_{h_\nu}\left(\sqrt{\frac{\lambda}{\alpha'}}\rho\right)}{I_{h_\nu}\left(\sqrt{\frac{\lambda}{\alpha'}}R\right)} \right| |\phi_\nu| \right)^s \leq$$

$$2\pi \cdot \left(\sum_{|\nu| \geq \nu_0} \left(\frac{\rho}{R}\right)^{l_\nu} |\phi_\nu| \right)^s \leq 2\pi \cdot \left(\frac{\rho}{R}\right)^{sl_{\nu_0}} \left(\sup{}_\nu\, |\phi_\nu|\right)^s \left(\sum_{|\nu \geq \nu_0} \left(\frac{1}{2}\right)^{l_\nu - l_{\nu_0}} \right)^s.$$

Since: $l_\nu \geq |\nu|\sqrt{\frac{\alpha}{\alpha'}}$, the last series has a finite sum depending on α, ν_0. Moreover, as: $|\phi_\nu| \leq$ const $\cdot ||\phi||_{L^s[0,2\pi]}$, then (44) holds when $0 \leq \rho \leq \frac{R}{2}$. If $\frac{R}{2} \leq \rho \leq R$, by (42):

$$\left(\int_0^{2\pi} |w(\rho e^{i\theta})|^s d\theta\right)^{\frac{1}{s}} \leq 2^{l_{\nu_0}} \left(\frac{\rho}{R}\right)^{l_{\nu_0}} ||\phi||_{L^s[0,2\pi]}$$

and (44) follows.

Assume now that ϕ is an arbitrary function in $L^s[0, 2\pi]$; the thesis follows using a sequence of trigonometric polynomials that have limit ϕ, (42), (43), (44) and classical approximation theorems. □

Definition 1. *Let $N \in \mathbf{N}$, $R > 0$; let us define:*

$$\mathcal{F}_N(B_R) := \{u \in L^1(B_R) : u(\rho e^{i\theta}) \sim \sum_{|\nu|\leq N} u_\nu(\rho) e^{i\nu\theta} \quad \rho \text{ a. e. in } (0, R)\}.$$

Lemma 4. *Let $(\alpha, p) \in \mathcal{A}$, $R > 0$, $\lambda \in \mathbf{C}\Re\lambda \geq 0$, $N \in \mathbf{N}$, $f \in \mathcal{F}_N(B_R) \cap L^p(B_R)$, of the form: $f(re^{i\theta}) = \sum_{|\nu|\leq N} f_\nu(r)e^{i\nu\theta}$. Then, the solution u of the problem:*

$$u \in W^{2,p}_{\gamma_0}(B_R), \quad Su - \lambda u = f$$

can be written as:

$$u(re^{i\theta}) = \sum_{|\nu|\leq N} \frac{1}{\alpha'} \int_R^r \gamma_{\nu,r,\rho} d\rho \int_0^\rho \big(\frac{t}{\rho}\big)^{\frac{1}{\alpha'}-1} \gamma_{\nu,t,\rho}\, f_\nu(t) dt\, e^{i\nu\theta}. \tag{47}$$

Proof. The proof is similar to the proof of lemma 4 in [8]. □

Lemma 5. *Let: $(\alpha, p) \in \mathcal{A}$, $R > 0$, $N \in \mathbf{N}$, $\lambda \in \mathbf{C}$, $\Re\lambda \geq 0$, let $u \in W^{2,p}_{\gamma_0}(B_R) \cap \mathcal{F}_N$; there exists $k(\alpha, p, N, R)$, depending on α, p, N, R only, for which the bound:*

$$\begin{aligned} &||D^2u||_{L^p(B_R)} + \sqrt{|\lambda|}\, ||Du||_{L^p(B_R)} + |\lambda|\, ||u||_{L^p(B_R)} \leq \\ &\leq k(\alpha, p, R, N)\, ||Su - \lambda u||_{L^p(B_R)}. \end{aligned} \tag{48}$$

holds.

Proof. By Fact 1 and interpolation theorems, the left hand side of (48) can be bound by $k(\alpha, p, R)[|\lambda|\, ||u||_{L^p(B_R)} + ||Su - \lambda u||_{L^p(B_R)}]$.

Therefore it is enough to prove:

$$|\lambda|\, ||u||_{L^p(B_R)} \quad \leq \quad K(\alpha, p, N)\, ||Su - \lambda u||_{L^p(B_R)}. \tag{49}$$

Let:

$$u(re^{i\theta}) = \sum_{|\nu|\leq N} u_\nu(r) e^{i\nu\theta},$$

$f := Su - \lambda u$ and:

$$f(re^{i\theta}) := \sum_{|\nu|\leq N} f_\nu(r) e^{i\nu\theta}.$$

Let:

$$v_\nu(r) := \int_0^r \big(\frac{t}{r}\big)^{\frac{1}{\alpha'}-1} \gamma_{\nu,t,r}\, f_\nu(t) dt;$$

then $u_\nu(r)$ can be written as:

$$u_\nu(r) := \frac{1}{\alpha'} \int_R^r \gamma_{\nu,r,\rho}\, v_\nu(\rho) d\rho.$$

Notice that for some $K(p, N)$:

$$\|u\|_{L^p(B_R)} \le K(p, N) \sum_{|\nu|\le N} \Big(\int_0^R |u_\nu(r)|^p r dr \Big)^{\frac{1}{p}},$$

$$\sum_{|\nu|\le N} \Big(\int_0^R |f_\nu(r)|^p r dr \Big)^{\frac{1}{p}} \le K(p, N) \, \|f\|_{L^p(B_R)};$$

thus, the bound (49) will be proved if, for every $\nu \in \mathbf{Z}$, there exists $K(\alpha, p, \nu)$ such that the bounds:

$$|\lambda|^{\frac{1}{2}} \Big(\int_0^R |u_\nu(r)|^p r dr \Big)^{\frac{1}{p}} \le K(\alpha, p, \nu) \Big(\int_0^R |v_\nu(r)|^p r dr \Big)^{\frac{1}{p}} \tag{50}$$

and:

$$|\lambda|^{\frac{1}{2}} \Big(\int_0^R |v_\nu(r)|^p r dr \Big)^{\frac{1}{p}} \le K(\alpha, p, \nu) \Big(\int_0^R |f_\nu(r)|^p r dr \Big)^{\frac{1}{p}} \tag{51}$$

hold.

In the remaining part of the proof, κ will be any constant (not necessarily the same), depending on α, p, ν only.

Let $\sqrt{\frac{\lambda}{\alpha'}} := \sigma e^{i\omega}$; $|\omega| \le \frac{\pi}{4}$; then, by (14):

$$|\gamma_{\nu,r,\rho}| \le K(\alpha, p, \nu)\, e^{\frac{\sigma}{2\sqrt{2}}(r-\rho)}; \tag{52}$$

thus:

$$\mathcal{J} := \Big(|\lambda|^{\frac{1}{2}} \Big(\int_0^R |u_\nu(r)|^p r dr \Big)^{\frac{1}{p}} \Big)^p$$

$$\le \sigma^p \int_0^R \Big| K(\alpha, p, \nu) \int_r^R e^{\frac{\sigma}{2\sqrt{2}}(r-\rho)} v_\nu(\rho) d\rho \Big|^p r dr;$$

let us make the change of variables: $r' = \sigma \frac{r}{R}$, $\rho' = \sigma \frac{\rho}{R}$; then:

$$\mathcal{J} \le \kappa \frac{R^2}{\sigma^2} \int_0^\sigma \Big[\int_{r'}^\sigma e^{\frac{R(r'-\rho')}{2\sqrt{2}}} |v_\nu(\frac{R\rho'}{\sigma})| R d\rho' \Big]^p r' dr';$$

let us use Hölder inequality in the inside integral; then:

$$\mathcal{J} \leq \kappa \frac{R^{p+2}}{\sigma^2} \int_0^\sigma \Big[\Big(\int_{r'}^\sigma e^{\frac{R(r'-\rho')}{2\sqrt{2}}} d\rho' \Big)^{\frac{p}{p'}} \int_{r'}^\sigma e^{\frac{R(r'-\rho')}{2\sqrt{2}}} |v_\nu(\frac{R\rho'}{\sigma})|^p d\rho' \Big] r' dr'; \tag{53}$$

as:

$$\int_{r'}^\sigma e^{\frac{R(r'-\rho')}{2\sqrt{2}}} d\rho' = (\frac{R}{2\sqrt{2}})^{-1}(1 - e^{\frac{R(r'-\sigma)}{2\sqrt{2}}}) \leq \frac{2\sqrt{2}}{R}, \tag{54}$$

(53) becomes:

$$\mathcal{J} \leq \kappa \frac{R^{p+2-\frac{p}{p'}}}{\sigma^2} \int_0^\sigma r' \, dr' \int_{r'}^\sigma e^{\frac{R(r'-\rho')}{2\sqrt{2}}} |v_\nu(\frac{R\rho'}{\sigma})|^p d\rho';$$

let us exchange the last two integrals; then, as $p + 2 - \frac{p}{p'} = 3$:

$$\mathcal{J} \leq \kappa \frac{R^3}{\sigma^2} \int_0^\sigma [\int_0^{\rho'} e^{\frac{R(r'-\rho')}{2\sqrt{2}}} r' dr'] |v_\nu(\frac{R\rho'}{\sigma})|^p d\rho'; \tag{55}$$

as:

$$\int_0^{\rho'} e^{\frac{R(r'-\rho')}{2\sqrt{2}}} r' dr' \leq \rho' \frac{2\sqrt{2}}{R},$$

by making the change of variable $\rho' = (R)^{-1}\sigma\rho$, (55) becomes:

$$\mathcal{J} \leq \kappa \frac{R^2}{\sigma^2} \int_0^\sigma |v_\nu(\frac{R\rho'}{\sigma})|^p \rho' d\rho' = \kappa \int_0^R |v_\nu(\rho)|^p \rho d\rho$$

and (50) follows.

Let us prove (51). By (52):

$$\mathcal{J}_1 := \Big(|\lambda|^{\frac{1}{2}} \big(\int_0^R |v_\nu(\rho)|^p \rho d\rho \big)^{\frac{1}{p}} \Big)^p$$

$$\leq \kappa \sigma^p \int_0^R \Big| \int_0^\rho e^{\frac{\sigma(t-\rho)}{2\sqrt{2}}} \big(\frac{t}{\rho}\big)^{\frac{1}{\alpha'}-1} \big| f_\nu(t) \big| dt \Big|^p \rho d\rho;$$

let us make the change of variables: $\rho' = \sigma \frac{\rho}{R}, \; t' = \sigma \frac{t}{R}$; then:

$$\mathcal{J}_1 \leq \kappa \frac{R^{p+2}}{\sigma^2} \int_0^\sigma \Big[\int_0^{\rho'} (\frac{t'}{\rho'})^{\frac{1}{\alpha'}-1-\frac{1}{p}} e^{\frac{R(t'-\rho')}{2\sqrt{2}}} |f_\nu(\frac{Rt'}{\sigma})| (t')^{\frac{1}{p}} dt' \Big]^p d\rho';$$

let us use Hölder inequality in the inside integral; then:

$$\mathcal{J}_1 \leq \kappa \frac{R^{p+2}}{\sigma^2} \int_0^\sigma \left[\int_0^{\rho'} (\tfrac{t'}{\rho'})^{p'(\frac{1}{\alpha'}-1-\frac{1}{p})} e^{\frac{R(t'-\rho')}{2\sqrt{2}}} \, dt' \right]^{\frac{p}{p'}} \cdot \left[\int_0^{\rho'} e^{\frac{R(t'-\rho')}{2\sqrt{2}}} |f_\nu(\tfrac{Rt'}{\sigma})|^p t' dt' \right] d\rho'; \tag{56}$$

let us notice that :

$$\left[\int_0^{\rho'} (\tfrac{t'}{\rho'})^{p'(\frac{1}{\alpha'}-1-\frac{1}{p})} e^{\frac{R(t'-\rho')}{2\sqrt{2}}} \, dt' \right]^{\frac{p}{p'}} \leq \kappa R^{-\frac{p}{p'}}; \text{ then :} \quad \mathcal{J}_1 \leq \kappa \frac{R^{p+2-\frac{p}{p'}}}{\sigma^2} \int_0^\sigma \left[\int_0^{\rho'} e^{\frac{R(t'-\rho')}{2\sqrt{2}}} |f_\nu(\tfrac{Rt'}{\sigma})|^p t' dt' \right] d\rho' \tag{57}$$

let us exchange the integrals:

$$\mathcal{J}_1 \leq \kappa \frac{R^3}{\sigma^2} \int_0^\sigma |f_\nu(\frac{Rt'}{\sigma})|^p \left[\int_{t'}^\sigma e^{\frac{R(t'-\rho')}{2\sqrt{2}}} d\rho' \right] t' dt';$$

let us use (54) and let us come back to the original variable t; thus:

$$\mathcal{J}_1 \leq \kappa \int_0^R |f_\nu(t)|^p t dt;$$

(51) follows. □

Lemma 6. *Let* $(\alpha, p) \in \mathcal{A}$, $R > 0$, $\lambda \in \mathbf{C}$, $\Re\mathcal{A} \geq 0$, $u \in W^{2,p}_{\gamma_0}(B_R) \setminus \mathcal{F}_2(B_R)$. *Then, there exists* $K(\alpha, p)$ *depending on* α, p *only, such that the bound:*

$$|| \, | \cdot |^{-2} u ||_{L^p(B_R)} \leq K(\alpha, p) || Su - \lambda u ||_{L^p(B_R)} \tag{58}$$

holds. Moreover, for every $\epsilon \in (0, 1)$*, there exists* $C(\alpha, p, R, \epsilon)$ *depending on* α, p, R, ϵ *only, such that the bound:*

$$|| \, | \cdot |^{-1} Du ||_{L^p(B_R)} \leq \epsilon || D^2 u ||_{L^p(B_R)} + C(\alpha, p, R, \epsilon) || Su - \lambda u ||_{L^p(B_R)} \tag{59}$$

holds.

Proof. It is sufficient to prove the bound (58) for every $N \in \mathbf{N}$, $u \in (\mathcal{F}_N \setminus \mathcal{F}_2) \cap C^\infty(\overline{B_R})$, $u|_{\partial B_R} = 0$.

Let $f := Su - \lambda u$; then: $f(re^{i\theta}) = \sum_{3 \leq |\nu| \leq N} f_\nu(r) e^{i\nu\theta}$ and by lemma 4:

$$u(re^{i\theta}) = \int_R^r d\rho \int_0^\rho dt \sum_{3 \leq |\nu| \leq N} \frac{1}{\alpha'} \gamma_{\nu,r,\rho} \big(\frac{t}{\rho}\big)^{\frac{1}{\alpha'}-1} \gamma_{\nu,t,\rho} \, f_\nu(t) \, e^{i\nu\theta}$$

Let $(\mathcal{G}_{\rho,R}\phi)(\theta)$ be the solution of problem (38); then:

$$\sum_{3\le|\nu|\le N} \gamma_{\nu,r,\rho}\gamma_{\nu,t,\rho}\ f_\nu(t)\, e^{i\nu\theta} = [\mathcal{G}_{r,\rho}\ \cdot\ \mathcal{G}_{t,\rho} f(te^{i(\cdot)})](\theta)$$

and u can also be written as:

$$u(re^{i\theta}) = \frac{1}{\alpha'}\int_R^r d\rho \int_0^\rho [\mathcal{G}_{r,\rho}\big(\frac{t}{\rho}\big)^{\frac{1}{\alpha'}-1}\mathcal{G}_{t,\rho} f(te^{i(\cdot)})](\theta)dt$$

Thus, by (44) and (42):

$$|| \, r^{-2}u(re^{i\cdot})||_{L^p(0,2\pi)} \quad \le$$

$$\le \frac{k(\alpha,p)}{\alpha' r^2} \quad \int_r^R \big(\frac{r}{\rho}\big)^{l_3} d\rho \int_0^\rho \big(\frac{t}{\rho}\big)^{\frac{1}{\alpha'}-1} ||\mathcal{G}_{t,\rho}\ f(te^{i(\cdot)})||_{L^p(0,2\pi)}dt$$

$$\le \frac{k(\alpha,p)}{\alpha' r^2} \quad \int_r^R \big(\frac{r}{\rho}\big)^{l_3} d\rho \int_0^\rho \big(\frac{t}{\rho}\big)^{\frac{1}{\alpha'}-1} ||f(te^{i(\cdot)})||_{L^p(0,2\pi)}dt.$$

This formula, as in [8] (lemma 6), gives (58).

Let $\epsilon > 0$ and let κ_ϵ be any constant depending on α, p, R, ϵ only.

Let us prove the the bound:

$$(60) \qquad ||\,|\cdot|^{-1}\frac{\partial u}{\partial \rho}||_{L^p(B_R)} \le \epsilon||D^2u||_{L^p(B_R)} \ + \ \kappa_\epsilon||Su - \lambda u||_{L^p(B_R)}.$$

Let $k := \frac{2}{p} - 1 + \frac{1}{\epsilon}$; as $u(0,0) = u_\rho(0,0) = 0$, the identity:

$$\rho^{-1}\frac{\partial u}{\partial \rho}(\rho e^{i\theta}) \ = \ k\rho^{-2}u(\rho e^{i\theta}) + \rho^{-1}\int_0^\rho \big(\frac{t}{\rho}\big)^k\big[\frac{\partial^2 u}{\partial \rho^2}(te^{i\theta}) - k(k-1)t^{-2}u(te^{i\theta})\big]dt$$

holds; by Hardy inequality in ρ and L^p norm in θ, one gets the bound:

$$||\,|\cdot|^{-1}\frac{\partial u}{\partial \rho}||_{L^p(B_R)} \le \epsilon||\frac{\partial^2 u}{\partial \rho^2}||_{L^p(B_R)} + \kappa_\epsilon|||\cdot|^{-2}u||_{L^p(B_R)}.$$

This inequality and (58) give (60).

Let us prove the the bound:

$$(61) \qquad ||\,|\cdot|^{-2}\frac{\partial u}{\partial \theta}||_{L^p(B_R)} \le \epsilon||D^2u||_{L^p(B_R)} \ + \ \kappa_\epsilon||Su - \lambda u||_{L^p(B_R)}.$$

Let us fix $\rho \in (0, R)$; the interpolation bound:

$$||\frac{\partial u}{\partial \theta}(\rho e^{i\cdot})||_{L^p(0,2\pi)} \leq \epsilon ||\frac{\partial^2 u}{\partial \theta^2}(\rho e^{i\cdot})||_{L^p(0,2\pi)} + \kappa_\epsilon ||u(\rho e^{i\cdot})||_{L^p(0,2\pi)}$$

holds; the $(\int_0^R |\rho^{-2} (\,\cdot\,) |^p \rho d\rho)^{\frac{1}{p}}$ norm of both members gives:

$$||\, |\cdot|^{-2} \frac{\partial u}{\partial \theta} \,||_{L^p(B_R)} \leq \epsilon ||\, |\cdot|^{-2} \frac{\partial^2 u}{\partial \theta^2} \,||_{L^p(B_R)} + \kappa ||\, |\cdot|^{-2}\, u \,||_{L^p(B_R)}. \tag{62}$$

As $|\cdot|^{-2} \frac{\partial^2 u}{\partial \theta^2} = \Delta u - \frac{\partial^2 u}{\partial \rho^2} - |\cdot|^{-1} \frac{\partial u}{\partial \rho}$, the bounds (62), (60), (58) give (61).

From (60), (61), then (59) follows. □

Let us recall the following result (see e. g. [3]).

Fact 2. Let $p > 1$, $R > 0$, $\Re\lambda \geq 0$, $w \in W^{2,p}_{\gamma_0}(B_R)$. Then, there exists $K(p, R)$ depending on p, R only, such that the bound:

$$\begin{aligned} ||D^2 w||_{L^p(B_R)} \; + \; \sqrt{|\lambda|}\, ||Dw||_{L^p(B_R)} \; + \; |\lambda|\, ||w||_{L^p(B_R)} \leq \\ \leq \quad K(p, R) \quad ||\Delta w - \lambda w||_{L^p(B_R)} \end{aligned} \tag{63}$$

holds.

Lemma 7. *Let $(\alpha, p) \in \mathcal{A}$, $R > 0, \lambda \in \mathbf{C}$, $\Re\lambda \geq 0$, $u \in W^{2,p}_{\gamma_0}(B_R) \setminus \mathcal{F}_2(B_R)$. Then, there exists $K(\alpha, p, R)$ depending on α, p, R only, such that the bound:*

$$\begin{aligned} ||D^2 u||_{L^p(B_R)} \; + \; \sqrt{|\lambda|}\, ||Du||_{L^p(B_R)} \; + \; |\lambda|\, ||u||_{L^p(B_R)} \leq \\ \leq \quad K(\alpha, p, R) \quad ||Su - \lambda u||_{L^p(B_R)} \end{aligned} \tag{64}$$

holds.

Proof. In the proof $\kappa, \kappa', \kappa''$ will be constants depending on α, p, R only. Let $I_1, \cdots, I_m$ be open sectors in $\mathbf{R}^2$ with vertex in $(0, 0)$, of the form: $I_j = \{(\rho\cos\theta, \rho\sin\theta) : \rho > 0, |\theta - \theta_j| < \delta_j,\;$ with $2\delta_j < \pi \sqrt{\alpha/\alpha'}\,\} j = 1, \cdots, m$, satisfying: $\cup_{j=1}^m I_j = \mathbf{R}^2 \setminus (0, 0)$.

Let $I_1^*, \cdots, I_m^*$ be open sectors in $\mathbf{R}^2$ with vertex in $(0, 0)$ of the form: $I_j^* = \{(\rho\cos\theta, \rho\sin\theta) : \rho > 0, |\theta - \theta_j| < \delta_j^*,\;$ with $2\delta_j < 2\delta_j^* < \pi \sqrt{\alpha/\alpha'}\,\} j = 1, \cdots, m$.

Let $\Phi_1, \cdots, \Phi_m$ be a partition of unity in $\mathbf{R}^2 \setminus (0, 0)$, satisfying:

(i) $\Phi_j \in C^\infty(\mathbf{R}^2 \setminus (0, 0))$, Φ_j homogeneous of degree zero;
(ii) $1/2 \leq \Phi_j \leq 1$ in I_j, $\Phi_j \equiv 0$ outside I_j^*, $j = 1, \cdots, m$;

(iii) $\sum_{j=1}^{m} \Phi_j \equiv 1$ in $\mathbf{R}^2 \setminus (0,0)$.

Let $u \in C^\infty(\overline{B_R}) \setminus \mathcal{F}_2(B_R)$, $u|_{\partial B_R} = 0$. Then (see e.g. [12]) u is of the form $u(\rho e^{i\theta}) = \rho^3 v(\rho e^{i\theta})$, where $v \in C^\infty(\overline{B_R})$.

Let $g := \frac{Su-\lambda u}{\alpha'} - (\frac{1}{\alpha'} - 2)\frac{1}{\rho}\frac{\partial u}{\partial \rho}$; then:

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\alpha}{\alpha'}\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{\lambda}{\alpha'} u = g.$$

Let us define:

$$u^{(j)} := u\Phi_j,$$

$$g^{(j)} := g\Phi_j + \frac{\alpha}{\alpha'\rho^2}\big[2\frac{\partial \Phi_j}{\partial \theta}\frac{\partial u}{\partial \theta} + u\frac{\partial^2 \Phi_j}{\partial \theta^2}\big]$$

$j = 1, \cdots, m$; then, $u^{(j)} \in C^\infty(\overline{B_R} \setminus \{(0,0)\})$; moreover u can be extended to a function $\in C^2(\overline{B_R})$; furthermore:

$$\frac{\partial^2 u^{(j)}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u^{(j)}}{\partial \rho} + \frac{\alpha}{\alpha'}\frac{1}{\rho^2}\frac{\partial^2 u^{(j)}}{\partial \theta^2} - \frac{\lambda}{\alpha'} u^{(j)} = g^{(j)}$$

$j = 1, \cdots, m$.

Let us define: $v^{(j)}(\rho e^{i\theta'}) := u^{(j)}(\rho e^{i(\theta_j + \sqrt{\frac{\alpha}{\alpha'}}\theta')})$ $(\rho > 0, |\theta'| \leq 2\pi)$, $j = 1, \cdots, m$. Then:

$v^{(j)} \in C^2(\overline{B_R})$, $v^{(j)} := 0$ in $x \leq 0$ and

$$[\Delta v^{(j)} - \frac{\lambda}{\alpha'} v^{(j)}](\rho e^{i\theta'}) = g^{(j)}(\rho e^{i(\theta_j + \sqrt{\frac{\alpha}{\alpha'}}\theta')}).$$

The following relations hold.

$$||\, u^{(j)}||_{L^p(B_R)} \;\leq\; \kappa||\, v^{(j)}||_{L^p(B_R)}$$

$$||\, Du^{(j)}||_{L^p(B_R)} \;\leq\; \kappa'||\, Dv^{(j)}||_{L^p(B_R)}$$

$$||\, D^2 u^{(j)}||_{L^p(B_R)} \;\leq\; \kappa"||\, D^2 v^{(j)}||_{L^p(B_R)} + ||\, |\cdot|^{-1}\, \frac{\partial u^{(j)}}{\partial \rho}\, ||_{L^p(B_R)}$$

$j = 1, \cdots, m$.

Now let us prove (64). The properties of $u^{(j)}$, $v^{(j)}$ give us:

$$\mathcal{J} \;:=\; ||D^2 u||_{L^p(B_R)} \;+\; \sqrt{|\lambda|}\; ||Du||_{L^p(B_R)} \;+\; |\lambda|\; ||u||_{L^p(B_R)} \leq$$

$$\leq \sum_{j=1}^{m} \big[||D^2 u^{(j)}||_{L^p(B_R)} + \sqrt{|\lambda|}\, ||Du^{(j)}||_{L^p(B_R)} + |\lambda|\, ||u^{(j)}||_{L^p(B_R)} \big]$$

$$\leq \kappa \sum_{j=1}^{m} \big[||D^2 v^{(j)}||_{L^p(B_R)} + \sqrt{\frac{|\lambda|}{\alpha'}}\, ||Dv^{(j)}||_{L^p(B_R)} + \frac{|\lambda|}{\alpha'}\, ||v^{(j)}||_{L^p(B_R)} +$$

$$+ ||\, |\cdot|^{-1} \frac{\partial u^{(j)}}{\partial \rho}\, ||_{L^p(B_R)} \big];$$

then, by Fact 2,

$$\mathcal{J} \leq \kappa \sum_{j=1}^{m} \big[||\Delta v^{(j)} - \frac{\lambda}{\alpha'} v^{(j)}||_{L^p(B_R)} + ||\, |\cdot|^{-1} \frac{\partial u^{(j)}}{\partial \rho}\, ||_{L^p(B_R)} \big] \leq$$

$$\leq \kappa \sum_{j=1}^{m} \big[||g^{(j)}||_{L^p(B_R)} + ||\, |\cdot|^{-1} \frac{\partial u^{(j)}}{\partial \rho}\, ||_{L^p(B_R)} \big]$$

$$\leq \kappa \big[\, ||g||_{L^p(B_R)} + ||\, |\cdot|^{-2} \frac{\partial u}{\partial \theta}\, ||_{L^p(B_R)} +$$

$$+ ||\, |\cdot|^{-2}\, u\, ||_{L^p(B_R)} \big] + ||\, |\cdot|^{-1} \frac{\partial u}{\partial \rho}\, ||_{L^p(B_R)} \big].$$

Thus:

$$\mathcal{J} \leq \kappa \big[\, ||\, Su - \lambda u\, ||_{L^p(B_R)} + ||\, |\cdot|^{-1} \frac{\partial u}{\partial \rho}\, ||_{L^p(B_R)} +$$

$$+ ||\, |\cdot|^{-2} \frac{\partial u}{\partial \theta}\, ||_{L^p(B_R)} + ||\, |\cdot|^{-2}\, u\, ||_{L^p(B_R)} \big].$$

The last three terms on the right hand side can be bound using lemma 6. Then:

$$\mathcal{J} \leq \kappa\, ||\, Su - \lambda u\, ||_{L^p(B_R)} + \epsilon ||D^2 u||_{L^p(B_R)} + C(\alpha, p, R, \epsilon) ||Su - \lambda u||_{L^p(B_R)}.$$

By choosing ϵ sufficiently small, the thesis follows. □

Theorem 6. *Let* $(\alpha, p) \in \mathcal{A}$*,* $R > 0$*,* $\lambda \in \mathbf{C} \Re\lambda \geq 0$*,* $u \in W^{2,p}_{\gamma_0}(B_R)$*. There exists* $K(\alpha, p)$ *depending on* α*,* p *only, such that the bound:*

$$||D^2 u||_{L^p(B_R)} + \sqrt{|\lambda|}\, ||Du||_{L^p(B_R)} + |\lambda|\, ||u||_{L^p(B_R)} \leq \leq K(\alpha, p)\, ||Su - \lambda u||_{L^p(B_R)} \tag{65}$$

holds.

Proof. A function $u \in W^{2,p}_{\gamma_0}(B_R)$ can be decomposed as $u = u_2 + v$, where $u_2 \in W^{2,p}_{\gamma_0}(B_R) \cap \mathcal{F}_2(B_R)$ and $v \in W^{2,p}_{\gamma_0}(B_R) \setminus \mathcal{F}_2(B_R)$.

Then by lemmas 5 and 7,

$$||D^2 u_2||_{L^p(B_R)} + \sqrt{|\lambda|}\, ||Du_2||_{L^p(B_R)} + |\lambda|\, ||u_2||_{L^p(B_R)} \leq$$

$$\leq K(\alpha, p, R)||Su_2 - \lambda u_2||_{L^p(B_R)},$$

$$||D^2 v||_{L^p(B_R)} + \sqrt{|\lambda|}\, ||Dv||_{L^p(B_R)} + |\lambda|\, ||v||_{L^p(B_R)} \leq$$

$$\leq K(\alpha, p, R)||Sv - \lambda v||_{L^p(B_R)}.$$

As:

$$||Su_2 - \lambda u_2||_{L^p(B_R)} \leq K(\alpha, p, R)||Su - \lambda u||_{L^p(B_R)},$$

and (65) follows, with a constant on the right hand side depending on α, p, R .

A simple scaling technique shows that in (65), the constant on the right hand side, actually, does not depend on R. □

Immediate consequence of previous theorem is the asymptotic behaviour of the resolvent of $\mathcal{S}$. Let us denote by $|||\cdot|||$ the operator norm for bounded operators acting in $L^p(B_R)$.

Theorem 7. *Let* $(\alpha, p) \in \mathcal{A}$, $R > 0$*; then, there exists* $\eta > 0$ *and* M_η*, such that, if* $|arg\lambda| < \frac{\pi}{2} + \eta$*, then*

$$|||(\mathcal{S} - \lambda)^{-1}||| \leq \frac{M_\eta}{|\lambda|}.$$

Proof. The thesis follows from previous theorem using a classical fact (see e.g. [3] 1.19, [6] IV 1.1. □

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