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SOBOLEV INEQUALITIES VIA MURAMATU'S INTEGRAL FORMULA

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For the Sobolev space $W_p^m(\mathbb{R}^n)$ with positive integer m and $1 < p < \infty$, sometimes replaced by $1 \leq p < \infty$, we consider the case $m - n/p < 0$ and the case $m - n/p = 0$, and give new proofs of the Sobolev embedding theorems by Muramatu's integral formula. When $m - n/p < 0$, the embedding into $L_q(\mathbb{R}^n)$ with q satisfying $m - n/p = -n/q$ is derived without the Hardy-Littlewood-Sobolev inequality by incorporating the method to prove it. When $m - n/p = 0$, we prove the embedding into the BMO space or the VMO space as well as Trudinger's inequality.

1. Introduction

This paper supplements the previous paper [8] that presented an introduction to the L_p-based Sobolev spaces of integer order using Muramatu's integral formula. Compared with the paper [9] due to Muramatu who had derived an advanced-form integral formula [9, Theorem 1] from the basic integral formula [9, Corollary 1], and developed the theory of the Sobolev and Besov spaces of fractional order, the previous paper [8] put a priority to simplicity and showed usefulness of the basic integral formula [9, Corollary 1], which we call Muramatu's integral formula, for the study of the Sobolev spaces of integer order.

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In this paper, we will apply Muramatu's integral formula to improve the proof given in [8] for the embedding into the Lebesgue space and to prove the embedding into the BMO or VMO space as well as Trudinger's inequality.

For a positive integer m and $1 \le p < \infty$ we consider the embedding theorems for the Sobolev space $W_p^m(\mathbb{R}^n)$ on the n-dimensional Euclidean space $\mathbb{R}^n$. The embedding theorems take different forms according to the sign of $m - n/p$. If $m - n/p > 0$, then $W_p^m(\mathbb{R}^n)$ is embedded into the the Hölder-Zygmund space of order $m - n/p$. As shown in [9, 10] (see also [8]), Muramatu's integral formula enables us to prove this embedding theorem easily.

This paper deals with the two remaining cases. First let $m - n/p < 0$ with $1 < p < \infty$. Take q so that $m - n/p = -n/q$, which implies $1 < p < q < \infty$. Then $W_p^m(\mathbb{R}^n)$ is embedded into $L_q(\mathbb{R}^n)$. In [8] we proved this embedding theorem by deriving the inequality

$$|f(x)| \le C \int_{\mathbb{R}^n} |x-y|^{m-n} |\nabla^m f(y)|\, dy \tag{1.1}$$

from Muramatu's integral formula, and invoking the Hardy-Littlewood-Sobolev (HLS) inequality for the integral operator related to the Riesz potential; the idea comes from Muramatu [9] and goes back to Sobolev. In Section 2 we prove this embedding theorem without relying on the HLS inequality by combining Muramatu's formula with the techniques used in the proof of HLS inequality. Inspection of our proof shows that Muramatu's integral formula is more adjustable than the integral involved with the Riesz potential, when applying the techniques used in the proof of HLS inequality.

Next let $m - n/p = 0$ with $1 \le p < \infty$. This case has two types of embedding theorems. One theorem states that $W_p^m(\mathbb{R}^n)$ is embedded into $\mathrm{BMO}(\mathbb{R}^n)$, the space of functions of bounded mean oscillation, and more strongly into $\mathrm{VMO}(\mathbb{R}^n)$, the space of functions of vanishing mean oscillation. The usual way to prove the BMO or VMO embedding (see e.g. [3, 4]) is based on Poincaré's inequality

$$\|f - f_\Omega\|_{L_n(\Omega)} \le C(n, \Omega) \|\nabla f\|_{L_n(\Omega)}$$

for $f \in W_n^1(\Omega)$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$, and f_Ω denotes the mean of f over Ω. In Section 3 we prove this theorem by Muramatu's integral formula. Another theorem states that Trudinger's inequality holds except for the case $p = 1$, namely that a function f in $W_p^m(\mathbb{R}^n)$ belongs to the Orlicz space described by the function $\exp(t^{p/(p-1)})$. Unlike the BMO embedding theorem, Trudinger's inequality involves not only $\|\nabla^m f\|_{L_p}$ but also $\|f\|_{L_p}$. As in the usual proof (see e.g. [2, 12, 15] and the references therein), the key lies in evaluating the L_q norms of f for $q \ge p$. We do so by Muramatu's integral formula in Section 4. We note that there is another type of embedding theorem (see [13]), which is outside the aim of this paper.

We quickly review Muramatu's integral formula. Choose a function $\rho \in C_0^\infty(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$ and $\operatorname{supp}\rho \subset \{x \in \mathbb{R}^n : |x| < 1\}$. For a positive integer N we set

$$\varphi(x) = \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_x^\alpha\{x^\alpha \rho(x)\},$$
$$M(x) = \sum_{|\alpha|=N} \partial_x^\alpha M_\alpha, \quad M_\alpha = \frac{N}{\alpha!}x^\alpha \rho(x).$$

Let $f \in W_p^m(\mathbb{R}^n)$ with $m \in \mathbb{N}$, the set of positive integers, and $1 \le p < \infty$. Let p' denote the conjugate exponent of p, i.e.

$$p^{-1} + (p')^{-1} = 1.$$

For $t > 0$ and a function $u(x)$ we set

$$u_t(x) = t^{-n}u(x/t).$$

Using the relation $\partial_t\{\varphi_t(x)\} = -t^{-1}M_t(x)$ and the fact that the convolution

$$\varphi_t * f(x) = \int_{\mathbb{R}^n} \varphi_t(x-y)f(y)\,dy$$

converges to $f(x)$ as $t \to 0^+$ in the $L_p(\mathbb{R}^n)$ norm, and also for a.e. $x \in \mathbb{R}^n$, we have

$$f = \int_0^R M_t * f\,\frac{dt}{t} + \varphi_R * f, \quad R > 0.$$

Noting that $(\partial^\alpha M_\alpha)_t * f = t^{|\beta|}(\partial^{\alpha-\beta}M_\alpha)_t * (\partial^\beta f)$ for $0 < \beta \le \alpha$, and taking N sufficiently large, we find that there exist C^∞ functions K_j supported on the unit ball $|x| < 1$ with $j = 1, \ldots, n$ such that

$$f = \sum_{j=1}^n \int_0^R t^m (K_j)_t * (\partial_j^m f)\,\frac{dt}{t} + \varphi_R * f. \tag{1.2}$$

Since $\|\varphi_R * f\|_{L_\infty} \le R^{-n/p}\|\varphi\|_{L_{p'}}\|f\|_{L_p}$ by Hölder's inequality, letting $R \to \infty$ in (1.2) gives

$$f = \sum_{j=1}^n \int_0^\infty t^m (K_j)_t * (\partial_j^m f)\,\frac{dt}{t} \tag{1.3}$$

for a.e. $x \in \mathbb{R}^n$. We use (1.2) or (1.3) as Muramatu's integral formula.

Throughout this paper we often omit $\mathbb{R}^n$ in the symbol $L_p(\mathbb{R}^n)$. The symbol C stands for constants which may differ from line to line. For pure derivatives we use the symbols

$$\nabla^m f = (\partial_1^m f, \ldots, \partial_n^m f), \qquad \|\nabla^m f\|_{L_p} = \sum_{j=1}^n \|\partial_j^m f\|_{L_p}.$$

2. Sobolev inequality for $m - n/p < 0$

When $m - n/p < 0$, the Sobolev embedding theorem is formulated as follows.

Theorem 2.1. *Let $m \in \mathbb{N}$ and $1 < p < \infty$ satisfy $m - n/p < 0$. Choose q so that $m - n/p = -n/q$, which implies $p < q < \infty$. Then $W_p^m(\mathbb{R}^n)$ is embedded into $L_q(\mathbb{R}^n)$, and the inequality*

$$\|f\|_{L_q(\mathbb{R}^n)} \le C(m,n,p)\|\nabla^m f\|_{L_p(\mathbb{R}^n)}$$

holds for $f \in W_p^m(\mathbb{R}^n)$.

Remark 2.2. As is well known, Theorem 2.1 also holds for $p = 1$ (see e.g. [14]), which is outside the scope of our method.

As stated in the Introduction, we proved this theorem in [8] relying on the HLS inequality, which asserts that the L_q norm of the right-hand side in (1.1) is estimated by $C\|\nabla^m f\|_{L_p}$. There are several methods to prove the HLS inequality. We will incorporate two of them separately into the proof of Theorem 2.1.

We first take up Hedberg's method [7] of using the Hardy-Littlewood maximal function. Remember that for a locally integrable function f the maximal function Mf is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|\,dy,$$

where the supremum is taken over all balls B centered at x, and $|B|$ denotes the volume of B, and that M is a bounded operator on L_p (see e.g. [14, Chapter 1, Theorem 1]).

Proof of Theorem 2.1 by Hedberg's method. We use Muramatu's formula (1.3). Let b be the volume of the unit ball. For simplicity we temporarily set $K = K_j$ and $u = \partial_j^m f$. By the definition of the maximal function we have

$$|K_t * u(x)| \le \|K\|_{L_\infty} t^{-n}\int_{|y|\le t} |u(x-y)|\,dy \le b\|K\|_{L_\infty} Mu(x).$$

On the other hand, Hölder's inequality gives

$$|K_t * u(x)| \le \|K_t\|_{L_{p'}}\|u\|_{L_p} = t^{-n/p}\|K\|_{L_{p'}}\|u\|_{L_p}.$$

Setting $g = |\nabla^m f|$ and using the above inequalities, we have

$$\begin{aligned}|f(x)| &\le C\left(\int_0^R t^m Mg(x)\,\frac{dt}{t} + \int_R^\infty t^{m-n/p}\|g\|_{L_p}\frac{dt}{t}\right)\\ &\le C\left(R^m Mg(x) + R^{m-n/p}\|g\|_{L_p}\right).\end{aligned}$$

Choosing R so that the two terms in the parentheses are equal, we get

$$|f(x)| \le C\|g\|_{L_p}^{mp/n} Mg(x)^{1-mp/n} = C\|g\|_{L_p}^{1-p/q} Mg(x)^{p/q}.$$

The theorem follows by the L_p boundedness of the maximal operator M. □

We next take up the method of using the weak Lebesgue space and the Marcinkiewicz interpolation theorem (see e.g. [14, Appendix B], [5, Section 6.28]), which is essentially the same as the method used to prove the theorem on a certan integral operator [5, Theorem 6.36]. Application of this method to the HLS inequality can be found in [6, Theorem 6.2.6] (see also [16, Proposition II.2.6]), where the Riesz potential is expressed by the integral representation with the Gauss kernel, which is very similar to Muramatu's integral formula (1.3).

Let μ denote the Lebesgue measure on $\mathbb{R}^n$. For $\lambda > 0$ and a measurable function f we simply write $\mu(f > \lambda)$ for $\mu(\{x \in \mathbb{R}^n : f(x) > \lambda\})$.

Lemma 2.3. *Let m, p, q be as in Theorem 2.1, and let χ be the characteristic function of the unit ball $|x| < 1$. For $g \in L_p(\mathbb{R}^n)$ set*

$$Tg = \int_0^\infty t^m \chi_t * g \,\frac{dt}{t}.$$

Then the operator T is of weak type (p,q). Namely,

$$\mu(|Tg| > \lambda) \le C(m,n,p)\lambda^{-q}\|g\|_{L_p(\mathbb{R}^n)}^q. \tag{2.1}$$

Moreover, T is a bounded operator from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$:

$$\|Tg\|_{L_q(\mathbb{R}^n)} \le C(m,n,p)\|g\|_{L_p(\mathbb{R}^n)}. \tag{2.2}$$

Proof. Without loss of generality we may assume that $\|g\|_{L_p} = 1$. For $R > 0$ we set

$$Tg = G_0 + G_1$$

with

$$G_0 = \int_0^R t^m \chi_t * g \,\frac{dt}{t}, \quad G_1 = \int_R^\infty t^m \chi_t * g \,\frac{dt}{t}.$$

Minkowski's inequality and Young's inequality give

$$\|G_0\|_{L_p} \le \int_0^R t^m \|\chi\|_{L_1}\|g\|_{L_p}\frac{dt}{t} = C_0 R^m.$$

Similarly,

$$\|G_1\|_{L_\infty} \le \int_R^\infty t^{m-n/p}\|\chi\|_{L_{p'}}\|g\|_{L_p}\frac{dt}{t} = C_1 R^{m-n/p}.$$

Given $\lambda > 0$, we choose R so that $C_1 R^{m-n/p} = \lambda$. Since $|Tg| > 2\lambda$ implies $|G_0| > \lambda$, Chebyshev's inequality gives

$$\begin{aligned}\mu(|Tg| > 2\lambda) \le \mu(|G_0| > \lambda) &\le \lambda^{-p}(C_0 R^m)^p \\ &= (C_0 C_1^{-1} R^{n/p})^p = C_0^p C_1^{q-p} \lambda^{-q}.\end{aligned}$$

Thus we get (2.1).

To show (2.2) we note that $1 < p < n/m$ and take p_0, p_1, q_0, q_1 so that

$$\begin{gathered} 1 < p_0 < p < p_1 < n/m, \\ m - n/p_0 = -n/q_0, \quad m - n/p_1 = -n/q_1. \end{gathered}$$

Observe that $1 < p_l < q_l < \infty$ with $l = 0,1$ and that there exists $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Since T is of weak-type (p_0, q_0), and of weak-type (p_1, q_1) by (2.1), we get (2.2) by the Marcinkiewicz interpolation theorem. □

Proof of Theorem 2.1 by the weak L_q space. Since $|(K_j)_t * (\partial_j^m f)| \le \|K_j\|_{L_\infty} \chi_t * |\partial_j^m f|$, Muramatu's formula (1.3) and Lemma 2.3 yield the theorem. □

3. Sobolev inequality with the BMO or VMO space for $m - n/p = 0$

In order to formulate the embedding theorem for $m - n/p = 0$ we remember the definition of the BMO and VMO spaces. For a locally integrable function f on $\mathbb{R}^n$ and a ball B in $\mathbb{R}^n$ we denote by f_B the mean of f over B, and by $I_B(f)$ the mean of $|f - f_B|$ over B:

$$f_B = \frac{1}{|B|}\int_B f(x)\,dx, \quad I_B(f) = \frac{1}{|B|}\int_B |f(x) - f_B|\,dx.$$

We say that f belongs to the space $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\sup_B I_B(f) < \infty,$$

where the supremum is taken over all balls B. Let r_B denote the radius of a ball B. A function f in $\mathrm{BMO}(\mathbb{R}^n)$ is said to belong to $\mathrm{VMO}(\mathbb{R}^n)$ if

$$\lim_{\varepsilon \to 0} \sup_{B:\, r_B \le \varepsilon} I_B(f) = 0.$$

As is easily seen, $I_B(f)$ satisfies

$$I_B(f) \le \frac{2}{|B|}\int_B |f(x)|\,dx, \tag{3.1}$$

$$I_B(f) \le \frac{1}{|B|^2}\iint_{B\times B} |f(x)-f(y)|\,dx\,dy \le 2r_B\|\nabla f\|_{L_\infty(B)}. \tag{3.2}$$

Theorem 3.1. *Let $m \in \mathbb{N}$ and $1 \le p < \infty$ satisfy $m - n/p = 0$. Then $W_p^m(\mathbb{R}^n)$ is embedded into* $\mathrm{BMO}(\mathbb{R}^n)$*, and the inequality*

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C(n,p)\|\nabla^m f\|_{L_p(\mathbb{R}^n)} \tag{3.3}$$

holds for $f \in W_p^m(\mathbb{R}^n)$. More strongly, we have $W_p^m(\mathbb{R}^n) \subset$ $\mathrm{VMO}(\mathbb{R}^n)$.

Proof. We use Muramatu's formula (1.3). Interchanging the order of integration gives

$$I_B(f) \le \sum_{j=1}^n \int_0^\infty t^m I_B((K_j)_t * (\partial_j^m f))\,\frac{dt}{t}$$

for a ball B. For simplicity we temporarily set $K = K_j$ and $u = \partial_j^m f$. Using (3.1) with Jensen's inequality and Young's inequality, we have

$$I_B(K_t * u) \le \frac{2}{|B|^{1/p}}\|K_t * u\|_{L_p(B)} \le C r_B^{-n/p}\|K\|_{L_1}\|u\|_{L_p}. \tag{3.4}$$

By (3.2) and Hölder's inequality we have

$$\begin{aligned} I_B(K_t * u) &\le 2r_B\|\nabla(K_t * u)\|_{L_\infty(B)} \le 2r_B\|\nabla K_t\|_{L_{p'}}\|u\|_{L_p} \\ &\le 2r_B t^{-1-n/p}\|\nabla K\|_{L_{p'}}\|u\|_{L_p}. \end{aligned}$$

Combining these inequalities gives, with $g = |\nabla^m f|$,

$$\begin{aligned} I_B(f) &\le C\int_0^R t^m r_B^{-n/p}\|g\|_{L_p}\frac{dt}{t} + C\int_R^\infty t^{m-1-n/p} r_B\|g\|_{L_p}\frac{dt}{t} \\ &\le C\left\{\left(\frac{R}{r_B}\right)^m\|g\|_{L_p} + \frac{r_B}{R}\|g\|_{L_p}\right\} \end{aligned}$$

for $R > 0$, where we used $m - n/p = 0$. Setting $R = r_B$, we get (3.3) for the BMO space.

To show the assertion for $\mathrm{VMO}(\mathbb{R}^n)$ we need only modify the proof for $\mathrm{BMO}(\mathbb{R}^n)$ by elaborating the second inequality in (3.4). Set

$$J_B(u) = \sup_{y\in\mathbb{R}^n}\|u\|_{L_p(B-y)},$$

where $B-y=\{x-y : x\in B\}$. Inspecting the proof of Young's inequality gives

$$I_B(K_t * u) \le C r_B^{-n/p} \|K\|_{L_1} J_B(u).$$

Using this inequality in place of (3.4), and setting $R = l r_B$ with $l > 0$, we get

$$I_B(f) \le C_1\{l^m J_B(g) + l^{-1}\|g\|_{L_p}\}.$$

Given $\varepsilon > 0$, choose l so that $C_1 l^{-1}\|g\|_{L_p(\mathbb{R}^n)} < \varepsilon/2$. In addition, by taking r_B sufficiently small we get $C_1 l^m J_B(g) < \varepsilon/2$, since $g \in L_p(\mathbb{R}^n)$. Hence $I_B(f) < \varepsilon$. This implies $f \in \mathrm{VMO}(\mathbb{R}^n)$. □

4. Trudinger's inequality for $m - n/p = 0$

Following the papers [1, 11], we formulate Trudinger's inequality so that it is scale-invariant; the inequality is stable if $f(x)$ is replaced by $f(\lambda x)$ with a parameter $\lambda > 0$. For $1 < p < \infty$ we define the function Φ_p by

$$\Phi_p(t) = \exp(t) - \sum_{k\in\mathbb{N}\cup\{0\},\,k<p-1} \frac{1}{k!}t^k.$$

Theorem 4.1. *Let $m \in \mathbb{N}$ and $1 < p < \infty$ satisfy $m - n/p = 0$. Then there exist positive constants c and C depending only on n, p such that for $f \in W_p^m(\mathbb{R}^n)$ with $f \ne 0$ we have*

$$\int_{\mathbb{R}^n} \Phi_p\left(c\left(\frac{|f(x)|}{\|\nabla^m f\|_{L_p(\mathbb{R}^n)}}\right)^{p/(p-1)}\right) dx \le C\left(\frac{\|f\|_{L_p(\mathbb{R}^n)}}{\|\nabla^m f\|_{L_p(\mathbb{R}^n)}}\right)^p.$$

Proof. We may assume that $\|\nabla^m f\|_{L_p} = 1$ by replacing f by $f/\|\nabla^m f\|_{L_p}$. We use Muramatu's formula (1.2) and write $f = f_0 + \varphi_R * f$ with

$$f_0 = \sum_{j=1}^n \int_0^R t^m (K_j)_t * (\partial_j^m f)\,\frac{dt}{t}$$

for $R > 0$. Let χ be the characteristic function of the unit ball $|x| < 1$, and set $g = \sum_{j=1}^n |\partial_j^m f|$. By Fubini's theorem

$$|f_0(x)| \le \int_{\mathbb{R}^n} H(x-y)g(y)\,dy$$

with

$$H(x) = \max_{1\le j\le n} \|K_j\|_{L_\infty} \int_0^R t^{m-n}\chi(x/t)\,\frac{dt}{t}.$$

Noting that $|x| \le t \le R$ in the above integral, and making the change of variables $|x|/t = s$, we know that $H(x) = 0$ for $|x| > R$, and $H(x) \le C_0|x|^{m-n}$ for $|x| < R$ with $C_0 = C_0(n,p)$.

Let q, r satisfy

$$p \le q < \infty, \quad p^{-1} + r^{-1} = 1 + q^{-1}.$$

Note that $r^{-1} = (p')^{-1} + q^{-1}$. Let b be the volume of the unit ball. Since $\|g\|_{L_p} \le 1$, a calculation with Young's inequality shows that

$$\begin{aligned} \|f_0\|_{L_q} \le \|H\|_{L_r} = C_0 \left(\frac{nbR^{(m-n)r+n}}{(m-n)r+n} \right)^{1/r} &\le C_0 \left(\frac{bq}{r} \right)^{1/r} R^{n/q} \\ &\le C_1 \left(1 + \frac{q}{p'} \right)^{(p')^{-1}+q^{-1}} R^{n/q}. \end{aligned} \tag{4.1}$$

On the other hand, Young's inequality gives

$$\|\varphi_R * f\|_{L_q} \le R^{-n(1-1/r)} \|\varphi\|_{L_r} \|f\|_{L_p} \le C_2 R^{-m+n/q} \|f\|_{L_p}. \tag{4.2}$$

Combining (4.1) and (4.2), and choosing R so that $R^{n/q} = R^{-m+n/q}\|f\|_{L_p}$, we have

$$\|f\|_{L_q}^q \le C_3^q \left(1 + \frac{q}{p'} \right)^{1+q/p'} \|f\|_{L_p}^p$$

with $C_3 = 2\max\{C_1, C_2\}$. We apply this inequality with $q = kp'$ for all integers k satisfying $kp' \ge p$, i.e. $k \ge p-1$. Given $c > 0$, we set

$$C_4 = \sum_{k \ge p-1} (cC_3^{p'})^k (1+k)^{k+1} / k!.$$

Then we get

$$\int_{\mathbb{R}^n} \Phi_p(c|f(x)|^{p'})\, dx \le C_4 \|f\|_{L_p}^p.$$

Here we find by the ratio test that C_4 is finite if $cC_3^{p'} e < 1$, since the ratio is dominated by

$$\frac{cC_3^{p'}}{k+1} \frac{(k+2)^{k+2}}{(k+1)^{k+1}} \le cC_3^{p'} e \left(1 + \frac{1}{k+1} \right).$$

Thus we complete the proof. □

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