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ON THE PROOF OF A MINIMAX PRINCIPLE

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The aim of this note is to point out that the basic argument in the proof of Theorem 2 in [5] does not work. Comments on this topic are given.

This short paper deals with a minimax principle in the nonsmooth critical point theory for functionals $I : X \to (-\infty, +\infty]$ on a real Banach space X which have the following structure

(H) $I = \Phi + \Psi$, with $\Phi : X \to \mathbb{R}$ locally Lipschitz and $\Psi : X \to (-\infty, +\infty]$ proper (i.e., $\not\equiv +\infty$), convex and lower semicontinuous.

In Chapter 3 of the book [8] a critical point theory has been developed for the class of nonsmooth functionals verifying (H). A preliminary version of it has been given in [4]. In the setting of this nonsmooth critical point theory the main concepts are the following.

Definition 1. ([8], page 64). An element $u \in X$ is called a critical point of the functional I in (H) if

$$\Phi^0(u; v - u) + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in X.$$

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Definition 2. ([8], page 64). The functional I in (H) is said to satisfy the Palais-Smale condition at the level $c \in \mathbb{R}$ if every sequence $\{u_n\} \subset X$ verifying $I(u_n) \to c$ and

$$\Phi^0(u_n; v - u_n) + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$, contains a convergent subsequence. If the Palais-Smale condition is fulfilled for all $c \in \mathbb{R}$, I is said to satisfy the Palais-Smale condition (for short, (PS)).

Here the notation Φ^0 stands for the generalized directional derivative of Φ in the sense of Clarke [2], i.e.,

$$\Phi^0(u; v) = \limsup_{\substack{w \to u \\ t \to 0^+}} \frac{1}{t}(\Phi(w + tv) - \Phi(w)), \quad \forall u, v \in X.$$

In order to see the area of applicability of the approach related to Definitions 1 and 2, we briefly discuss some significant situations.

Example 1. If in (H) one has $\Phi \in C^1(X)$, then Definitions 1 and 2 reduce to the corresponding definitions in the nonsmooth critical point theory of Szulkin [9]. If in (H) one has $\Psi = 0$, then Definitions 1 and 2 coincide with the corresponding ones in the nonsmooth critical point theory of Chang [1]. For $\Phi \in C^1(X)$ and $\Psi = 0$ in (H), one obtains the basic concepts in the smooth critical point theory.

Example 2. Every local extremum (minimum or maximum) $u \in X$ with $I(u) < +\infty$ of a nonsmooth functional $I : X \to (-\infty, +\infty]$ satisfying (H) is a critical point in the sense of Definition 1. Indeed, if $u \in X$ with $I(u) < +\infty$ is a local minimum of I, then for any $v \in X$ and a small $t > 0$ we have

$$0 \leq I((1-t)u + tv) - I(u) \leq \Phi(u + t(v - u)) - \Phi(u) + t(\Psi(v) - \Psi(u)),$$

where the convexity of Ψ has been used. Dividing by t and letting $t \to 0^+$ we deduce that u is a critical point of I as required in Definition 1. Suppose now that $u \in X$ is a local maximum of I satisfying (H) with $I(u) < +\infty$. Then u is in the interior of the effective domain of Ψ, and thus Ψ is Lipschitz near u. Then the calculus with generalized gradients (see [2]) yields

$$0 \in \partial I(u) \subset \partial \Phi(u) + \partial \Psi(u),$$

where $\partial\Phi(u)$ is the generalized gradient of Φ and $\partial\Psi(u)$ is the subdifferential of Ψ in the sense of convex analysis, so $0 = z+w$, with $z \in \partial\Phi(u)$ and $w \in \partial\Psi(u)$. By the definition of the generalized gradient and using the convexity of Ψ we infer that

$$\Phi^0(u; v-u) + \Psi(v) - \Psi(u) \geq \langle z, v-u\rangle + \langle w, v-u\rangle = 0, \quad \forall v \in X.$$

Thus u is a critical point like in Definition 1.

The minimax principle for nonsmooth functionals with the structure (H) formulated in Theorem 3.2 of [8] provides critical points in the sense of Definition 1 which generally are not local extrema, thus being of saddle point type. This minimax principle makes use of the following notion of linking (see, e.g., [3]).

Definition 3. Let S be a closed nonempty subset of the Banach space X and let Q be a compact topological submanifold of X with nonempty boundary ∂Q (in the sense of manifolds with boundary). We say that S and Q link if $S \cap \partial Q = \emptyset$ and $f(Q) \cap S \neq \emptyset$ whenever $f \in \Gamma$, for

$$\Gamma := \{f \in C(Q, X) : \; f|_{\partial Q} = \mathrm{id}_{\partial Q}\}.$$

We now recall from [8] the minimax principle for nonsmooth functionals of type (H).

Theorem 1. ([8], Theorem 3.2, page 74). *Let the functional $I : X \to (-\infty, +\infty]$ on the Banach space X satisfy assumptions* (H) *and* (PS) *(see Definition 2). Let S and Q link in the sense of Definition 3. Assume further that*

$$\sup_Q I \in \mathbb{R}, \; b := \inf_S I \in \mathbb{R}, \; a := \sup_{\partial Q} I < b.$$

Then the number

$$c := \inf_{f \in \Gamma} \sup_{x \in Q} I(f(x)),$$

for Γ in Definition 3, is a critical value of I with $c \geq b$. In particular, there exists a critical point u of I in the sense of Definition 1 and $I(u) = c$.

Remark 1. The so-called limiting case $c = a$ is treated in [7].

In the paper [5], Theorem 1 is presented (under the label Theorem 2 therein) with a different proof. It seems that the reason of its presence in [5] is to produce a simplification of the initial proof given in [8]. As shown in the sequel, the proof written in [5] is not correct.

The central argument of the proof given in [5] for Theorem 1 consists of the following claim:

" First of all we prove that $I \circ f$ is continuous on Q
for every $f \in \Gamma$ such that $\Lambda(f) < +\infty$ " $\quad (*)$

(see [5], page 195). Here, $\Lambda(f) = \sup_{u \in Q} I(f(u))$.

The claim $(*)$ is wrong as shown in the simple example below.

Example 3. Let $X = \mathbb{R}^2$, $Q = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 - 2y \le 0\}$ and $f = \mathrm{id}_Q \in \Gamma$. Choose $\Phi = 0$ and $\Psi : \mathbb{R}^2 \to (-\infty, +\infty]$ defined for any $(x, y) \in \mathbb{R}^2$ by

$$\Psi(x, y) = \begin{cases} \frac{x^2+y^2}{2y} + 1 & \text{if } x^2 + y^2 - 2y \le 0,\ y \neq 0 \\ 1 & \text{if } (x, y) = (0, 0) \\ +\infty & \text{otherwise.} \end{cases}$$

The function Ψ is proper, convex and lower semicontinuous, so assumption (H) is satisfied. Moreover, it is seen that $\sup_Q \Psi = 2$, which ensures $\Lambda(f) < +\infty$ as required in $(*)$. However, the function $I \circ f = \Psi \circ f = \Psi$ is not continuous at $(0, 0) \in Q$. This establishes that the claim $(*)$ does not hold.

Remark 2. Theorem 1 is stated in [6] as Theorem 8 therein. The proof given in [6] contains the error indicated in $(*)$ too (see [6], page 390).

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