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PERRON CONDITIONS FOR EXPONENTIAL EXPANSIVENESS OF ONE-PARAMETER SEMIGROUPS

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We present a new approach for the theorems of Perron type for exponential expansiveness of one-parameter semigroups in terms of $l^p(\mathbf{N}, X)$ spaces. We prove that an exponentially bounded semigroup is exponentially expansive if and only if the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is completely admissible relative to a discrete equation associated to the semigroup, where $p, q \in [1, \infty)$, $p \geq q$. We apply our results in order to obtain very general characterizations for exponential expansiveness of C_0-semigroups in terms of the complete admissibility of the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ and for exponential dichotomy, respectively, in terms of the admissibility of the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$.

1. Introduction.

In the last decades, the asymptotic theory of one-parameter semigroups became one of the domains with a spectacular development (see [2], [5], [16], [17]). Important results in the theory of evolution equations were based on the relatively recent studies on the properties of the so-called evolution semigroups associated to evolution operators or to linear skew-product flows on diverse function spaces (see [2], [7], [8], [15]).

Among of the most important techniques in the study of the asymptotic behavior of evolution equations, we refer the input-output conditions relative to

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functional equations associated to them. These methods have the origin in the classical work of Perron and they were intensively used in papers concerning exponential stability, exponential expansiveness and exponential dichotomy (see [2]-[4], [6]-[9], [11]-[15], [20], [21]).

The purpose of the present paper is to give very general conditions for the exponential expansiveness of one-parameter semigroups using discrete-time methods. Our main tools are based on the input-output conditions or on the so-called "theorems of Perron type". We associate to an exponentially bounded semigroup on a Banach space X a discrete-time equation and we discuss its expansiveness relative to the unique solvability of this equation between two $l^p(\mathbf{N}, X)$ spaces. We prove that the complete admissibility of the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ implies the exponential expansiveness of the semigroup, and the converse implication is valid if and only if $p \geq q$. Next, we apply our results for the study of the exponential expansiveness and of the exponential dichotomy of C_0-semigroups. Thus, we obtain necessary and sufficient conditions for exponential expansiveness of a C_0-semigroup in terms of the complete admissibility of the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ relative to an integral equation associated to it. As an application, we deduce a characterization for exponential dichotomy of C_0-semigroups, using the admissibility of the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$.

2. Discrete admissibility and exponential expansiveness.

In what follows we establish the connection between the exponential expansiveness of exponentially bounded semigroups and the complete admissibility of the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$.

Let X be a real or a complex Banach space. Throughout this paper, the norm on X and on $\mathcal{B}(X)$-the Banach algebra of all bounded linear operators on X, will be denoted by $||\cdot||$.

Definition 2.1. A family $\mathbf{T} = \{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be a *semigroup* on X, if $T(0) = I$ and $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$.

Definition 2.2. A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be:

(i) *exponentially bounded* if there are $M \geq 1$ and $\omega > 0$ such that $||T(t)|| \leq Me^{\omega t}$, for all $t \geq 0$;

(ii) C_0*-semigroup* if $\lim\limits_{t\searrow 0} T(t)x = x$, for all $x \in X$.

Remark 2.1. If $\mathbf{T}$ is a C_0-semigroup, then it is exponentially bounded (see [17], Theorem 2.2, p. 4).

Definition 2.3. A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be

(i) *exponentially unstable* if there are two constants $N, \nu > 0$ such that

$$||T(t)x|| \geq N\, e^{\nu t}||x||, \quad \forall (t,x) \in \mathbf{R}_+ \times X;$$

(ii) *exponentially expansive* if it is unstable and for every $t \geq 0$, $T(t)$ is invertible.

Remark 2.2. If $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a semigroup such that there is $\tau > 0$ such that $T(\tau)$ is invertible, then $T(t)$ is invertible for all $t \geq 0$.

Proposition 2.1. *An exponentially bounded semigroup* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *is exponentially unstable if and only if there are* $t_0 > 0$ *and* $\delta > 1$ *such that*

$$||T(t_0)x|| \geq \delta ||x||, \quad \forall x \in X.$$

Proof. Necessity is obvious. To prove sufficiency, let $\nu > 0$ be such that $\delta = e^{\nu t_0}$. Let M, ω be given by Definition 2.2 (i). If $t \geq 0$, there are $n \in \mathbf{N}$ and $s \in [0, t_0)$ such that $t = nt_0 + s$. It follows that

$$e^{\nu(n+1)t_0}||x|| \leq ||T((n+1)t_0)x|| \leq Me^{\omega t_0}||T(t)x||, \quad \forall x \in X.$$

If $N = 1/Me^{\omega t_0}$, then $||T(t)x|| \geq Ne^{\nu t}||x||$, for all $t \geq 0$ and all $x \in X$. □

If $p \in [1, \infty)$, we denote by

$$l^p(\mathbf{N}, X) = \{s : \mathbf{N} \to X \mid \sum_{k=0}^{\infty} ||s(k)||^p < \infty\}$$

which is Banach space with respect to the norm

$$||s||_p := \left(\sum_{k=0}^{\infty} ||s(k)||^p \right)^{1/p}.$$

Definition 2.4. Let $p, q \in [1, \infty)$. The pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is said to be *completely admissible* for the semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ if for every $s \in l^q(\mathbf{N}, X)$ there exists a unique $\gamma_s \in l^p(\mathbf{N}, X)$ such that

$$(E^d_{\mathbf{T}}) \qquad \gamma_s(n+1) = T(1)\gamma_s(n) + T(1)s(n), \quad \forall n \in \mathbf{N}.$$

Remark 2.3. If the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is completely admissible for $\mathbf{T}$, then it makes sense to consider the operator

$$Q : l^q(\mathbf{N}, X) \to l^p(\mathbf{N}, X), \quad Q(s) = \gamma_s.$$

We immediately observe that Q is a closed linear operator, so it is bounded.

If $A \subset \mathbf{R}$ we denote by χ_A the characteristic function of the set A.

Theorem 2.1. *If the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is completely admissible for the semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$, then $T(t)$ is invertible, for all $t \geq 0$.*

Proof. It is sufficient to prove that $T(1)$ is invertible.

Injectivity. Let $x \in X$ with $T(1)x = 0$ and $\gamma(n) = 0$, $\gamma'(n) = T(n)x$, for all $n \in \mathbf{N}$. Then the pairs $(\gamma, 0)$, $(\gamma', 0)$ verify the equation $(E^d_{\mathbf{T}})$. Since the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is completely admissible for $\mathbf{T}$, we deduce that $\gamma = \gamma'$, and hence $x = \gamma(0) = 0$. So $T(1)$ is injective.

Surjectivity. Let $x \in X$ and let

$$s : \mathbf{N} \to X, \quad s(n) = -\chi_{\{1\}}(n)x.$$

From hypothesis there is $\gamma \in l^p(\mathbf{N}, X)$ such that the pair (γ, s) verifies the equation $(E^d_{\mathbf{T}})$. Then we have that

$$\gamma(m + 1) = T(1)\gamma(m), \quad \forall m \geq 2$$

and

$$\gamma(2) = T(1)\gamma(1) - T(1)x.$$

Let

$$\varphi : \mathbf{N} \to X, \quad \varphi(n) = \begin{cases} \gamma(1) - x\,, & n = 0 \\ \gamma(n + 1)\,, & n \geq 1. \end{cases}$$

Then $\varphi \in l^p(\mathbf{N}, X)$ and

$$\varphi(n + 1) = T(1)\varphi(n), \quad \forall n \in \mathbf{N}.$$

Taking into account that the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is completely admissible for $\mathbf{T}$ we deduce that $\varphi = 0$. In particular, it follows that $\gamma(1) - x = 0$, so $x = \gamma(1)$. But $\gamma(1) = T(1)\gamma(0)$, so we obtain that $x = T(1)\gamma(0)$. This shows that $T(1)$ is surjective and the proof is complete. $\square$

The first main result of this section is:

Theorem 2.2. *Let $p, q \in [1, \infty)$ and let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be an exponentially bounded semigroup on X. If the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is completely admissible for $\mathbf{T}$, then $\mathbf{T}$ is exponentially expansive.*

Proof. From Theorem 2.1 we have that $T(t)$ is invertible, for all $t \geq 0$.

Let $x \in X \setminus \{0\}$. Then $T(n)x \neq 0$, for all $n \in \mathbf{N}$. If Q is the operator from Remark 2.3, let $L = ||Q|| + 1$ and $k = [L^{2p}] + 1$. For every $n \in \mathbf{N}$, we consider the sequences:

$$s : \mathbf{N} \to X, \quad s(i) = -\chi_{\{(n+1)k,\dots,(n+2)k-1\}}(i)\frac{T(i)x}{||T(i)x||}$$

$$\gamma : \mathbf{N} \to X, \quad \gamma(i) = \sum_{j=i}^{\infty} \frac{\chi_{\{(n+1)k,\dots,(n+2)k-1\}}(j)}{||T(j)x||}\, T(i)x.$$

We have that $\gamma \in l^p(\mathbf{N}, X)$, $s \in l^q(\mathbf{N}, X)$ and the pair (γ, s) verifies the equation $(E^d_{\mathbf{T}})$. It follows that $Qs = \gamma$, so $||\gamma||_p \leq ||Q||\, ||s||_q \leq Lk^{1/q}$. In particular, we deduce that

$$\left(\sum_{j=nk}^{(n+1)k-1} ||\gamma(j)||^p\right)^{1/p} \leq Lk^{1/q}.$$

Let $\alpha : \mathbf{N} \to \mathbf{R}_+$, $\alpha(j) = ||T(j)x||$. Denoting by

$$\lambda = \sum_{j=(n+1)k}^{(n+2)k-1} \frac{1}{\alpha(j)}$$

from above, it follows that

$$\left(\sum_{j=nk}^{(n+1)k-1} \alpha(j)^p\right)^{1/p} \leq \frac{Lk^{1/q}}{\lambda}. \tag{2.1}$$

Let

$$h = \begin{cases} 1, & p = 1 \\ k^{1/p'}, & p \in (1, \infty) \text{ and } p' = \frac{p}{p-1}. \end{cases}$$

Since

$$\sum_{j=nk}^{(n+1)k-1} \alpha(j) \leq h\left(\sum_{j=nk}^{(n+1)k-1} \alpha(j)^p\right)^{1/p}$$

by relation (2.1) we deduce that

$$\sum_{j=nk}^{(n+1)k-1} \alpha(j) \leq \frac{Lhk^{1/q}}{\lambda}. \tag{2.2}$$

Moreover

$$k^2 \leq \sum_{j=(n+1)k}^{(n+2)k-1} \frac{1}{\alpha(j)} \sum_{j=(n+1)k}^{(n+2)k-1} \alpha(j) = \lambda \sum_{j=(n+1)k}^{(n+2)k-1} \alpha(j)$$

and by relation (2.2) we obtain that

$$\sum_{j=nk}^{(n+1)k-1} \alpha(j) \leq \frac{Lhk^{1/q}}{k^2} \sum_{j=(n+1)k}^{(n+2)k-1} \alpha(j) \leq \frac{L}{k^{1/p}} \sum_{j=(n+1)k}^{(n+2)k-1} \alpha(j).$$

Using the definition of k it follows that

$$\sum_{j=(n+1)k}^{(n+2)k-1} \alpha(j) \geq L \sum_{j=nk}^{(n+1)k-1} \alpha(j), \quad \forall n \in \mathbf{N}$$

which implies that

$$\sum_{j=nk}^{(n+1)k-1} \alpha(j) \geq L^n \sum_{j=0}^{k-1} \alpha(j), \quad \forall n \in \mathbf{N}. \tag{2.3}$$

Let $M \geq 1$ and $\omega > 0$ be such that $||T(t)|| \leq Me^{\omega t}$, for all $t \geq 0$. Since for every $n \in \mathbf{N}^*$

$$\sum_{j=nk}^{(n+1)k-1} \alpha(j) \leq kMe^{\omega k}\alpha(nk) \quad \text{and} \quad k\alpha(k) \leq Me^{\omega k} \sum_{j=0}^{k-1} \alpha(j)$$

by relation (2.3) we deduce that

$$||T(nk)x|| \geq \frac{L^n}{\theta} \, ||T(k)x||, \quad \forall n \in \mathbf{N}^*, \tag{2.4}$$

where $\theta = M^2 e^{2\omega k}$. Since $T(k)$ is invertible, there exists $c > 0$ such that $||T(k)x|| \geq c||x||$, for all $x \in X$. Then, using the relation (2.4) we deduce that

$$||T(nk)x|| \geq \frac{cL^n}{\theta} \, ||x||, \quad \forall n \in \mathbf{N}^*, \forall x \in X.$$

Let $n_0 \in \mathbf{N}^*$, with $L^{n_0} c/\theta > 1$. Denoting by $t_0 = n_0 k$ and by $\delta = L^{n_0} c/\theta$, it follows that $||T(t_0)x|| \geq \delta ||x||$, for all $x \in X$. Applying Proposition 2.1 we obtain that $\mathbf{T}$ is exponentially expansive. $\square$

For the next proof, we use the following technical lemma.

Lemma 2.1. *Let* $q \geq 1$ *and* $\nu > 0$. *If* $s \in l^q(\mathbf{N}, \mathbf{R}_+)$ *and* $\gamma_s : \mathbf{N} \to \mathbf{R}_+$, $\gamma_s(n) = \sum_{j=n}^{\infty} e^{-\nu(j-n)} s(j)$, *then* $\gamma_s \in l^q(\mathbf{N}, \mathbf{R}_+)$.

Proof. It follows using Hölder's inequality. □

The second main result of this section is given by:

Theorem 2.3. *Let* $p, q \in [1, \infty)$, $p \geq q$ *and let* $\mathbf{T} = \{T(t)\}_{t \geq 0}$ *be an exponentially bounded semigroup on* X. *Then* $\mathbf{T}$ *is exponentially expansive if and only if the pair* $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ *is completely admissible for it.*

Proof. Necessity. Let $s \in l^q(\mathbf{N}, X)$. We consider the sequence

$$\gamma : \mathbf{N} \to X, \quad \gamma(n) = -\sum_{j=n}^{\infty} T(j-n)^{-1} s(j).$$

By Lemma 2.1 it follows that $\gamma \in l^q(\mathbf{N}, X)$. Because $p \geq q$, $l^q(\mathbf{N}, X) \subset l^p(\mathbf{N}, X)$, so $\gamma \in l^p(\mathbf{N}, X)$.

To prove the uniqueness, it is sufficient to show that if $\gamma \in l^p(\mathbf{N}, X)$ and

$$\gamma(n+1) = T(1)\gamma(n), \quad \forall n \in \mathbf{N} \tag{2.5}$$

then $\gamma = 0$. Indeed, by relation (2.5) we have that $\gamma(n) = T(n)\gamma(0)$, for all $n \in \mathbf{N}$. If $N, \nu > 0$ are given by Definition 2.3, it follows that

$$||\gamma(n)|| \geq N e^{\nu n} ||\gamma(0)||, \quad \forall n \in \mathbf{N}.$$

Since $\gamma \in l^p(\mathbf{N}, X)$, we obtain that $\gamma(0) = 0$, so $\gamma \equiv 0$.

In conclusion, we deduce that the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is completely admissible for $\mathbf{T}$.

Sufficiency follows from Theorem 2.2. □

Remark 2.4. Generally, if $p < q$ and the exponentially bounded semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is exponentially expansive, it does not result that the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is completely admissible for $\mathbf{T}$.

Example 2.1. Let $X = \mathbf{R}$ and $T(t)x = e^t x$, for all $(t, x) \in \mathbf{R} \times \mathbf{R}$. Then $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is exponentially expansive. Let $p, q \in [1, \infty)$, $p < q$, $\delta \in (p, q)$ and $s : \mathbf{N} \to \mathbf{R}$, $s(n) = 1/(n+1)^{1/\delta}$. Then $s \in l^q(\mathbf{N}, \mathbf{R}) \setminus l^p(\mathbf{N}, \mathbf{R})$.

Suppose by contrary that the pair $(l^p(\mathbf{N}, \mathbf{R}),\ l^q(\mathbf{N}, \mathbf{R}))$ is completely admissible for $\mathbf{T}$. Then there is a unique $\gamma \in l^p(\mathbf{N}, \mathbf{R})$ such that $\gamma(n+1) = T(1)\gamma(n) + T(1)s(n)$, for all $n \in \mathbf{N}$. It follows that

$$\gamma(n) = e^n \left[\gamma(0) + \sum_{j=0}^{n-1} e^{-j} s(j) \right], \quad \forall n \in \mathbf{N}^*.$$

Taking into account that $\lim\limits_{n\to\infty} \gamma(n) = 0$, we deduce that $\gamma(0) = -\sum\limits_{j=0}^{\infty} e^{-j} s(j)$, so

$$\gamma(n) = -e^n \sum_{j=n}^{\infty} e^{-j} s(j), \quad \forall n \in \mathbf{N}.$$

A simple calculus shows that $\lim\limits_{n\to\infty} \frac{|\gamma(n)|}{s(n)} = \frac{e}{e-1}$. Then, since $s \notin l^p(\mathbf{N}, \mathbf{R})$ we obtain that $\gamma \notin l^p(\mathbf{N}, \mathbf{R})$, which is a contradiction.

In conclusion, the pair $(l^p(\mathbf{N}, \mathbf{R}), l^q(\mathbf{N}, \mathbf{R}))$ is not completely admissible for $\mathbf{T}$.

3. Application 1: exponential expansiveness of C_0-semigroups.

In this section, as a consequence of the above results, we establish characterizations in terms of $L^p(\mathbf{R}, X)$ spaces for exponential expansiveness of C_0-semigroups.

Let X be a real or a complex Banach space and let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X. Let $p \in [1, \infty)$. We denote by $L^p(\mathbf{R}_+, X)$ the linear space of all Bochner measurable functions $v : \mathbf{R}_+ \to X$ with $\int\limits_0^\infty ||v(\tau)||^p \, d\tau < \infty$. This is a Banach space with respect to the norm

$$||v||_p := (\int_0^\infty ||v(\tau)||^p \, d\tau)^{1/p}.$$

Definition 3.1. Let $p, q \in [1, \infty)$. The pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ is said to be *completely admissible* for the C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ if for every $v \in L^q(\mathbf{R}_+, X)$ there exists a unique continuous function $f \in L^p(\mathbf{R}_+, X)$ such that the pair (f, v) verifies the equation

$$(E_{\mathbf{T}}) \qquad f(t) = T(t-s)f(s) + \int_s^t T(t-\tau)v(\tau) \, d\tau, \quad \forall t \geq s \geq 0.$$

Theorem 3.1. *Let $p, q \in [1, \infty)$ and let $\mathbf{T} = \{T(t)\}_{t \geq 0}$ be a C_0-semigroup on X. If the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ is completely admissible for $\mathbf{T}$, then $\mathbf{T}$ is exponentially expansive.*

Proof. Let $\alpha : [0, 1] \to [0, 2]$ be a continuous function with the support contained in $(0, 1)$ and $\int_0^1 \alpha(\tau)\, d\tau = 1$. Let $s \in l^1(\mathbf{N}, X)$. Then $s \in l^q(\mathbf{N}, X)$ and the function

$$v : \mathbf{R}_+ \to X, \quad v(t) = T(t - [t])s([t])\alpha(t - [t])$$

is continuous with $v \in L^q(\mathbf{R}_+, X)$. By hypothesis, there exists a unique continuous function $f \in L^p(\mathbf{R}_+, X)$ such that

$$f(t) = T(t - s)f(s) + \int_s^t T(t - \tau)v(\tau)\, d\tau, \quad \forall t \geq s \geq 0. \tag{3.1}$$

We consider the sequence $\gamma : \mathbf{N} \to X$, $\gamma(n) = f(n)$. By relation (3.1) we deduce that

$$\gamma(n + 1) = T(1)\gamma(n) + T(1)s(n), \quad \forall n \in \mathbf{N}$$

so the pair (γ, s) verifies the equation $(E^d_{\mathbf{T}})$. Let $M, \omega \in (0, \infty)$ be such that $||T(t)|| \leq Me^{\omega t}$, for all $t \geq 0$. By (3.1) we have that

$$||\gamma(n + 1)|| \leq Me^{\omega}||f(t)|| + Me^{\omega}||s(n)||, \quad \forall t \in [n, n + 1)$$

so

$$||\gamma(n + 1)|| \leq Me^{\omega}\left(\int_n^{n+1} ||f(t)||^p\, dt\right)^{1/p} + Me^{\omega}||s(n)||,\ \forall n \in \mathbf{N}. \tag{3.2}$$

Since $s \in l^p(\mathbf{N}, X)$ and $f \in L^p(\mathbf{R}_+, X)$, by relation (3.2) it follows that $\gamma \in l^p(\mathbf{N}, X)$.

To prove the uniqueness it is sufficient to show that if $\gamma \in l^p(\mathbf{N}, X)$ such that

$$\gamma(n + 1) = T(1)\gamma(n), \quad \forall n \in \mathbf{N} \tag{3.3}$$

then $\gamma \equiv 0$. Indeed, let $\gamma \in l^p(\mathbf{N}, X)$ which verifies the equation (3.3). We consider the function

$$\delta : \mathbf{R}_+ \to X, \quad \delta(t) = T(t - [t])\gamma([t]).$$

By relation (3.3) we have that δ is continuous and $\delta(t) = T(t - s)\delta(s)$, for all $t \geq s \geq 0$. Moreover, since $\gamma \in l^p(\mathbf{N}, X)$ it follows that $\delta \in L^p(\mathbf{R}_+, X)$. By hypothesis, we obtain that $\delta(t) = 0$, for all $t \geq 0$, so $\gamma = 0$. Thus, we deduce that the pair $(l^p(\mathbf{N}, X), l^1(\mathbf{N}, X))$ is completely admissible for $\mathbf{T}$. From Theorem 2.2 it follows that $\mathbf{T}$ is exponentially expansive. □

Lemma 3.1. *Let $p, q \in [1, \infty)$ with $p \geq q$ and let $\nu > 0$. If $v \in L^q(\mathbf{R}_+, \mathbf{R}_+)$, then the function*

$$f : \mathbf{R}_+ \to \mathbf{R}_+, \quad f(t) = \int_t^\infty e^{-\nu(s-t)} v(s)\, ds$$

belongs to $L^p(\mathbf{R}_+, \mathbf{R}_+)$.

Proof. It follows using Hölder's inequality. □

Theorem 3.2. *Let $p, q \in [1, \infty)$, $p \geq q$ and let $\mathbf{T} = \{T(t)\}_{t \geq 0}$ be a C_0-semigroup on X. Then $\mathbf{T}$ is exponentially expansive if and only if the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ is completely admissible for $\mathbf{T}$.*

Proof. Necessity. Let $v \in L^q(\mathbf{R}_+, X)$. We consider the function

$$f : \mathbf{R}_+ \to X, \quad f(t) = -\int_t^\infty T(\tau - t)^{-1} v(\tau)\, d\tau.$$

Then f is continuous and the pair (f, v) verifies the equation $(E_{\mathbf{T}})$. By Lemma 3.1 one obtain that $f \in L^p(\mathbf{R}_+, X)$. The uniqueness of f follows using a similar argument as in the proof of Theorem 2.3. So, the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ is completely admissible for $\mathbf{T}$.

Sufficiency follows from Theorem 3.1. □

Remark 3.1. Generally, if $p < q$ and the C_0-semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is exponentially expansive, it does not result that the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ is completely admissible for $\mathbf{T}$.

Example 3.1. Let $X = \mathbf{R}$ and $T(t)x = e^t x$, for all $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$. Then $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is C_0-semigroup, which is exponentially expansive.

Let $p, q \in [1, \infty)$ with $p < q$ and let $\delta \in (p, q)$. If $v : \mathbf{R}_+ \to \mathbf{R}$, $v(t) = 1/(t+1)^{1/\delta}$, then there is no $f \in L^p(\mathbf{R}_+, \mathbf{R})$ such that the pair (f, v) verifies the equation $(E_{\mathbf{T}})$. In conclusion, the pair $(L^p(\mathbf{R}_+, \mathbf{R}), L^q(\mathbf{R}_+, \mathbf{R}))$ is not completely admissible for $\mathbf{T}$.

4. Application 2: exponential dichotomy of C_0-semigroups.

In this section we present a new example of applicability of the results of this paper. In fact, we will use the results obtained in the previous section in order to obtain characterizations for exponential dichotomy of C_0-semigroups.

Let X be a real or a complex Banach space, let $\mathbf{T} = \{T(t)\}_{t \geq 0}$ be a C_0-semigroup on X and let $p, q \in [1, \infty)$.

Definition 4.1. $\mathbf{T}$ is said to be *exponentially dichotomic* if there exist a projection $P \in \mathcal{B}(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that:

(i) $T(t)P = PT(t)$, for all $t \geq 0$;

(ii) $T(t)_{|} : Ker P \to Ker P$ is an isomorphism, for all $t \geq 0$;

(iii) $||T(t)x|| \leq Ke^{-\nu t}||x||$, for all $x \in Im\ P$ and all $t \geq 0$;

(iv) $||T(t)x|| \geq \frac{1}{K} e^{\nu t}||x||$, for all $x \in Ker\ P$ and all $t \geq 0$.

Definition 4.2. A linear subspace $U \subset X$ is said to be $\mathbf{T}$-*invariant* if $T(t)U \subset U$, for all $t \geq 0$.

Definition 4.3. The pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ is said to be *admissible* for $\mathbf{T}$ if for every $v \in L^q(\mathbf{R}_+, X)$ there is a continuous function $f \in L^p(\mathbf{R}_+, X)$ such that

$$f(t) = T(t-s)f(s) + \int_s^t T(t-\tau)v(\tau)\, d\tau, \quad \forall t \geq s \geq 0.$$

We consider the linear subspace

$$X_1 = \{x \in X : \int_0^\infty ||T(t)x||^p\, dt < \infty\}$$

and we suppose that X_1 is closed and there is a closed $\mathbf{T}$-invariant subspace X_2 such that $X = X_1 \oplus X_2$.

Theorem 4.1. *If the pair* $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ *is admissible for* $\mathbf{T}$, *then the following properties hold:*

(i) *for every* $t \geq 0$ *the restriction* $T(t)_{|} : X_2 \to X_2$ *is an isomorphism;*

(ii) *there are* $K, \nu > 0$ *such that*

$$||T(t)x|| \geq \frac{1}{K} e^{\nu t}||x||, \quad \forall t \geq 0, \forall x \in X_2.$$

Proof. Let $T_2(t) = T(t)_{|X_2}$, for all $t \geq 0$. Then $\mathbf{T}_2 = \{T_2(t)\}_{t\geq 0}$ is a C_0-semigroup on X_2. We prove that the pair $(L^p(\mathbf{R}_+, X_2), L^q(\mathbf{R}_+, X_2))$ is completely admissible for $\mathbf{T}_2$.

Let $v \in L^q(\mathbf{R}_+, X_2)$. Then $v \in L^q(\mathbf{R}_+, X)$, so there is a continuous function $g \in L^p(\mathbf{R}_+, X)$ such that

$$g(t) = T(t-s)g(s) + \int_s^t T(t-\tau)v(\tau)\, d\tau, \quad \forall t \geq s \geq 0.$$

If $g(0) = x_1 + x_2$ with $x_k \in X_k$, $k \in \{1, 2\}$ we consider the function

$$f : \mathbf{R}_+ \to X, \quad f(t) = g(t) - T(t)x_1.$$

Then f is continuous and $f \in L^p(\mathbf{R}_+, X)$. Moreover, from

$$f(t) = T(t)x_2 + \int_0^t T(t-\tau)v(\tau)\, d\tau, \quad \forall t \geq 0$$

we deduce that $f(t) \in X_2$, for all $t \geq 0$, so $f \in L^p(\mathbf{R}_+, X_2)$.

To prove the uniqueness of f, let $\tilde{f} \in L^p(\mathbf{R}_+, X_2)$ be a continuous function such that the pair $(\tilde{f}, v)$ verifies the integral equation associated with $\mathbf{T}$. Setting $\varphi = f - \tilde{f}$ we have that $\varphi \in L^p(\mathbf{R}_+, X_2)$ and

$$\varphi(t) = T(t)\varphi(0), \quad \forall t \geq 0. \tag{4.1}$$

From relation (4.1) it follows that $\varphi(0) \in X_1$. Since $\varphi(0) \in X_2$ we deduce that $\varphi(0) = 0$. This shows that $\varphi = 0$, which proves the uniqueness of f.

Thus, we have that the pair $(L^p(\mathbf{R}_+, X_2), L^q(\mathbf{R}_+, X_2))$ is completely admissible for $\mathbf{T}$. By applying Theorem 3.1 for the C_0-semigroup $\mathbf{T}_2$ we obtain the conclusion. □

Theorem 4.2. *If the pair* $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ *is admissible for* $\mathbf{T}$ *and the subspace*

$$X_1 := \{x \in X : T(\cdot)x \in L^p(\mathbf{R}_+, X)\}$$

is closed and it has a $\mathbf{T}$*-invariant complement* X_2*, then* $\mathbf{T}$ *is exponentially dichotomic.*

Proof. Let P be the projection with $Im\ P = X_1$ and $Ker\ P = X_2$. Then we have that $T(t)P = PT(t)$, for all $t \geq 0$. Let $T_1(t) = T(t)_{|Im\ P}$, for all $t \geq 0$. Since

$$\int_0^\infty ||T_1(t)x||^p\, dt < \infty, \quad \forall x \in Im\ P$$

from [17] (see Theorem 4.1, pp. 116) it follows that there are $K, \nu > 0$ such that

$$||T(t)|| \leq Ke^{-\nu t}||x||, \quad \forall t \geq 0, \forall x \in Im\ P.$$

From Theorem 4.1 we obtain the conclusion. □

Lemma 4.1. *Let $p, q \in [1, \infty)$ with $p \geq q$ and let $\nu > 0$. If $v \in L^q(\mathbf{R}_+, \mathbf{R}_+)$, then the function*

$$g : \mathbf{R}_+ \to \mathbf{R}_+, \quad g(t) = \int_0^t e^{-\nu(t-s)} v(s)\, ds$$

belongs to $L^p(\mathbf{R}_+, \mathbf{R}_+)$.

Proof. It follows using Hölder's inequality. □

Theorem 4.3. *Let $p, q \in [1, \infty)$ with $p \geq q$ and let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X. Then $\mathbf{T}$ is exponentially dichotomic if and only if the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ is admissible for $\mathbf{T}$ and the subspace*

$$X_1 := \{x \in X : T(\cdot)x \in L^p(\mathbf{R}_+, X)\}$$

is closed and is has a $\mathbf{T}$-invariant complement.

Proof. Necessity. Let P be the projection and let $K, \nu > 0$ be given by Definition 4.1. For $v \in L^q(\mathbf{R}_+, X)$ we define $f : \mathbf{R}_+ \to X$ given by

$$f(t) = \int_0^t T(t-s)Pv(s)\, ds - \int_t^\infty T(s-t)_|^{-1}(I-P)v(s)\, ds$$

where $T(s)_|^{-1}$ denotes the inverse of the operator $T(s)_| : Ker\ P \to Ker\ P$. From Lemma 3.1 and Lemma 4.1 we obtain that $f \in L^p(\mathbf{R}_+, X)$ and an easy computation shows that the pair (f, v) satisfies the integral equation associated with $\mathbf{T}$. So the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ is admissible for $\mathbf{T}$.

Let $x \in X_1$. Then $\sup\limits_{t\geq 0} ||T(t)x|| < \infty$. Taking into account that

$$||x - Px|| \leq Ke^{-\nu t}||T(t)(I-P)x|| \leq$$
$$\leq Ke^{-\nu t}(\sup_{s\geq 0} ||T(s)x|| + Ke^{-\nu t}||Px||), \quad \forall t \geq 0$$

we deduce that $x - Px = 0$, so $x \in Im\ P$. Since $Im\ P \subset X_1$, it follows that $X_1 = Im\ P$, so it is closed and it has a $\mathbf{T}$-invariant complement - $Ker\ P$.

Sufficiency. It follows from Theorem 4.2. □

Remark 4.1. Generally, if $p < q$ and the C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is exponentially dichotomic, it does not result that the pair $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ is admissible for $\mathbf{T}$. This fact immediately results via Example 3.1.

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