INFINITELY MANY SOLUTIONS TO THE
NEUMANN PROBLEM FOR QUASILINEAR
ELLiptic SYSTEMS

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In this paper we deal with the existence of weak solutions for the following Neumann problem

\[
\begin{align*}
-\Delta u + \lambda(x)|u|^{p-2}u &= \alpha(x)f(u, v) \quad \text{in } \Omega \\
-\Delta v + \mu(x)|v|^{q-2}v &= \alpha(x)g(u, v) \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \\
\frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

where \(\nu\) is the outward unit normal to the boundary \(\partial \Omega\) of the bounded open set \(\Omega \subset \mathbb{R}^N\). The existence of solutions is proved by applying a critical point theorem obtained by B. Ricceri as consequence of a more general variational principle.

1. Introduction.

Here and in the sequel:

\(\Omega \subset \mathbb{R}^N\) is a bounded open set with boundary of class \(C^1\);

\(N \geq 1; \ p > N; \ q > N; \)

\(\lambda, \mu \in L^\infty(\Omega), \text{ such that } \text{essinf}_\Omega \lambda > 0, \text{essinf}_\Omega \mu > 0; \)

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\( \alpha \in C^0(\Omega) \) nonnegative;
\( f, g \in C^0(\mathbb{R}^2) \) such that the differential form \( f(u, v)du + g(u, v)dv \) be exact.

In this paper we are interested in the following problem:

\[
\begin{aligned}
-\Delta_p u + \lambda(x)|u|^{p-2}u &= \alpha(x)f(u, v) \text{ in } \Omega \\
-\Delta_q v + \mu(x)|v|^{q-2}v &= \alpha(x)g(u, v) \text{ in } \Omega \\
\frac{\partial u}{\partial v} &= 0 \text{ on } \partial \Omega \\
\frac{\partial v}{\partial v} &= 0 \text{ on } \partial \Omega.
\end{aligned}
\]

(P)

where \( v \) is the outward unit normal to \( \partial \Omega \). More precisely we are interested in the existence of infinitely many weak solutions to such a problem.

Whereas many results are available in the case of Dirichlet boundary conditions when \( p = q \) (see e.g. [5] and [4]), it seems that nothing is known in the case of Neumann boundary conditions.

The existence of solutions to Problem (P) is proved by applying the following critical point theorem ([2] and [3]) obtained by B. Ricceri as a consequence of a more general variational principle.

**Theorem 1.** Let \( X \) be a reflexive real Banach space, and let \( \Phi, \Psi : X \to \mathbb{R} \) be two sequentially weakly lower semicontinuous and Gateaux differentiable functionals. Assume also that \( \Psi \) is strongly continuous and satisfies \( \lim_{\|x\| \to \infty} \Psi(x) = +\infty \). For each \( \rho > \inf_X \Psi \), put

\[
\varphi(\rho) = \inf_{\|x\| \leq \rho} \left\{ \Phi(x) - \frac{\inf_{\Psi^{-1}(\|x\| - \rho]} \Phi}{\|x\| - \rho} \right\}
\]

where \( \Psi^{-1}(\|x\| - \rho] \) is the closure of \( \Psi^{-1}(\|x\| - \rho] \) in the weak topology. Furthermore, set

\[
\gamma = \lim_{\rho \to \infty} \inf \varphi(\rho)
\]

and

\[
\delta = \lim_{\rho \to \inf_X \Psi^+} \inf \varphi(\rho).
\]

Then, the following conclusions hold:

(a) For each \( \rho > \inf_X \Psi \) and each \( \beta > \varphi(\rho) \), the functional \( \Phi + \beta \Psi \) has a critical point which lies in \( \Psi^{-1}(\|x\| - \rho] \).

(b) If \( \gamma < +\infty \), then, for each \( \beta > \gamma \) the following alternative holds: either \( \Phi + \beta \Psi \) has a global minimum, or there exists a sequence \( \{x_n\} \) of critical points of \( \Phi + \beta \Psi \) such that \( \lim_{n \to \infty} \Psi(x_n) = +\infty \).
(c) If $\delta < +\infty$, then, for each $\beta > \delta$ the following alternative holds: either there exists a global minimum of $\Psi$ which is a local minimum of $\Phi + \beta \Psi$, or there exists a sequence $\{x_n\}$ of pairwise distinct critical points of $\Phi + \beta \Psi$, with $\lim_{n \to +\infty} \Psi(x_n) = \inf_X \Psi$, which weakly converges to a global minimum of $\Psi$.

Let $G : \mathbb{R}^2 \to \mathbb{R}$ be the differentiable function such that $G_u(u, v) = f(u, v)$, $G_v(u, v) = g(u, v)$, $G(0, 0) = 0$ and let $F(x, u, v) = \alpha(x) G(u, v)$, then $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a differentiable function with respect to $u$ and $v$ and $F_u(x, u, v) = \alpha(x) f(u, v)$, $F_v(x, u, v) = \alpha(x) g(u, v)$. Then (P) can be written in the form

$$
\begin{aligned}
-\Delta_p u + \lambda(x)|u|^{p-2} u &= F_u(x, u, v) \text{ in } \Omega \\
-\Delta_q v + \mu(x)|v|^{q-2} v &= F_v(x, u, v) \text{ in } \Omega \\
\frac{\partial u}{\partial v} &= 0 \text{ on } \partial \Omega \\
\frac{\partial v}{\partial v} &= 0 \text{ on } \partial \Omega .
\end{aligned}
$$

and also in the form

$$
\begin{aligned}
-\Delta_p u &= \frac{\partial}{\partial x} \left( \alpha(x) G(u, v) - \frac{1}{p} \lambda(x)|u|^p - \frac{1}{q} \mu(x)|v|^q \right) \\
-\Delta_q v &= \frac{\partial}{\partial x} \left( \alpha(x) G(u, v) - \frac{1}{p} \lambda(x)|u|^p - \frac{1}{q} \mu(x)|v|^q \right) \\
\frac{\partial u}{\partial v} &= 0 \text{ on } \partial \Omega \\
\frac{\partial v}{\partial v} &= 0 \text{ on } \partial \Omega .
\end{aligned}
$$

and therefore it is a gradient system [1]. Following [3] we first consider the space $W^{1,p}(\Omega)$ with the norm

$$
\|u\|_\lambda = \left( \int_\Omega \lambda(x)|u(x)|^p dx + \int_\Omega |\nabla u(x)|^p dx \right)^{\frac{1}{p}}
$$

and the space $W^{1,q}(\Omega)$ with the norm

$$
\|v\|_\mu = \left( \int_\Omega \mu(x)|v(x)|^q dx + \int_\Omega |\nabla v(x)|^q dx \right)^{\frac{1}{q}}.
$$

Since by hypotheses $p > N$ and $q > N$, $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ are both compactly embedded in $C^0(\overline{\Omega})$. So, if we put

$$
c_1 = c(\lambda) = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\sup_{x \in \Omega} |u(x)|}{\|u\|_\lambda}
$$
and
\[
c_2 = c(\mu) = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\sup_{x \in \Omega} |u(x)|}{\|u\|_\mu}
\]
then both \(c_1\) and \(c_2\) are finite.

Then we take \(X = W^{1,p}(\Omega) \times W^{1,q}(\Omega)\) with the norm \(\|(u, v)\|_X = \sqrt{\|u\|_X^p + \|v\|_\mu^q}\) and \(Y = C^0(\overline{\Omega}) \times C^0(\overline{\Omega})\) with the norm \(\|(u, v)\|_Y = \sqrt{\|u\|_{C^0(\overline{\Omega})}^2 + \|v\|_{C^0(\overline{\Omega})}^2}\) of course the space \(X\) is compactly embedded in \(Y\) and if we put
\[
c = \sup_{(u, v) \in X \setminus \{(0, 0)\}} \frac{\|(u, v)\|_Y}{\|(u, v)\|_X}
\]
we have \(c = \max\{c_1, c_2\}\). In order to apply theorem 1 we set
\[
\Psi(u, v) = \frac{1}{p} \|u\|_X^p + \frac{1}{q} \|v\|_\mu^q
\]
and
\[
\Phi(u, v) = -\int_\Omega F(x, u(x), v(x))dx
\]
for all \((u, v) \in X\). Since \(X\) is compactly embedded in \(Y\), not only the constant \(c\) is finite, but also the functionals \(\Phi\) and \(\Psi\) are (well defined and) sequentially weakly lower semicontinuous and Gateaux differentiable in \(X\), the critical points of \(\Phi + \Psi\) being precisely the weak solutions to Problem (P). Moreover \(\Psi\) is coercive (and strongly continuous as well).

The sets \(A(r), B(r), r > 0\), below specified, play an important role in our exposition:
\[
A(r) = \left\{ (\xi, \eta) \in \mathbb{R}^2 \text{ such that } \frac{1}{pc_1^p} |\xi|^p + \frac{1}{qc_2^q} |\eta|^q \leq r \right\}
\]
\[
B(r) = \left\{ (\xi, \eta) \in \mathbb{R}^2 \text{ such that } \frac{\int_\Omega \lambda(x)dx}{p} |\xi|^p + \frac{\int_\Omega \mu(x)dx}{q} |\eta|^q \leq r \right\}.
\]
The following inclusion holds:
\[B(r) \subseteq A(r).\]
To see this, we observe that by the definition of \(c_1\) we have
\[
\|u\|_{C^0(\overline{\Omega})} \leq c_1 \|u\|_\lambda
\]
for every $u \in W^{1,p}(\Omega)$, hence (taking $u \equiv 1$)

$$1 \leq c_1^p \int_{\Omega} \lambda(x)dx.$$  

Analogously we have

$$1 \leq c_2^q \int_{\Omega} \mu(x)dx.$$  

Thus, the inequality

$$\frac{1}{pc_1^p} |\xi|^p + \frac{1}{qc_2^q} |\eta|^q \leq \frac{\int_{\Omega} \lambda(x)dx}{p} |\xi|^p + \frac{\int_{\Omega} \mu(x)dx}{q} |\eta|^q$$

holds for every $(\xi, \eta) \in \mathbb{R}^2$ and therefore the inclusion $B(r) \subseteq A(r)$ holds.

2. Results.

**Theorem 2.** Assume that there are $r > 0$ and $\xi_0 \in \mathbb{R}$, $\eta_0 \in \mathbb{R}$ such that

$$\frac{1}{p} |\xi_0|^p \int_{\Omega} \lambda(x)dx + \frac{1}{q} |\eta_0|^q \int_{\Omega} \mu(x)dx < r$$

and

$$\max_{A(r)} G(\xi, \eta) = G(\xi_0, \eta_0).$$

Then Problem (P) admits a weak solution $(u, v)$ satisfying $\Psi(u, v) < r$

**Proof.** We apply Theorem 1 (part(a)) showing that $\varphi(r) = 0$.

Since $\Psi^{-1}([-\infty, r]) = \Psi^{-1}([-\infty, r])$ it follows that for all $(u, v) \in \Psi^{-1}([-\infty, r])$

$$0 \leq \varphi(r) = \inf_{(u, v) \in \Psi^{-1}([-\infty, r])} \frac{\Phi(u, v) - \inf_{(\Psi^{-1}([-\infty, r]), \Psi)} \Phi}{r - \Psi(u, v)} \leq \frac{\Phi(u, v) - \inf_{(\Psi^{-1}([-\infty, r]), \Psi)} \Phi}{r - \Psi(u, v)}$$

Let $u_0(x) = \xi_0$, $v_0(x) = \eta_0$ for all $x \in \Omega$. Then $\nabla u_0 = 0$, $\nabla v_0 = 0$,

$$\Psi(u_0, v_0) = \frac{1}{p} \left( \int_{\Omega} \lambda(x) |\xi_0|^p dx \right) + \frac{1}{q} \left( \int_{\Omega} \mu(x) |\eta_0|^q dx \right) =$$
\[
\frac{1}{p} |\xi_0|^p \int_{\Omega} \lambda(x)dx + \frac{1}{q} |\eta_0|^q \int_{\Omega} \mu(x)dx < r
\]

whence \((u_0, v_0) \in \Psi^{-1}([-\infty, r]) \subseteq \Psi^{-1}([-\infty, r])^w\). Moreover, for each \(x \in \Omega\) and for each \((u, v) \in \Psi^{-1}([-\infty, r])^w\), since

\[
\frac{1}{pc_1} |u(x)|^p + \frac{1}{qc_2} |v(x)|^q \leq \frac{1}{pc_1} \|u\|_p^p + \frac{1}{qc_2} \|v\|_q^q \leq \frac{1}{pc_1} \|u\|_p^p + \frac{1}{qc_2} \|v\|_q^q \leq r
\]

one has \((u(x), v(x)) \in A(r)\); therefore \(G(u(x), v(x)) \leq \alpha(x)G(u, v) \leq G(\xi_0, \eta_0)\) whence

\[
\int_{\Omega} \alpha(x)G(u(x), v(x))dx \leq \int_{\Omega} \alpha(x)G(\xi_0, \eta_0)dx
\]

i.e. \(-\Phi(u, v) \leq -\Phi(u_0, v_0)\) for all \((u, v) \in \Psi^{-1}([-\infty, r])^w\)

\[-\Phi(u_0, v_0) = \sup_{\Psi^{-1}([-\infty, r])} (-\Phi(u, v)) = -\inf_{\Psi^{-1}([-\infty, r])} (\Phi(u, v)).\]

From \(\Psi(u_0, v_0) < r\) it follows that

\[
\Phi(u_0, v_0) - \inf_{\Psi^{-1}([-\infty, r])} (\Phi(u, v)) = \Phi(u_0, v_0) - \Phi(u_0, v_0) = 0
\]

whence \(\varphi(r) = 0\). \(\square\)

**Theorem 3.** Assume that there are sequences \(\{r_n\}\) in \(\mathbb{R}^+\) with \(\lim_{n \to \infty} r_n = +\infty\), and \(\{\xi_n\}, \{\eta_n\}\) in \(\mathbb{R}\) such that for all \(n \in \mathbb{N}\), one has

\[
\frac{1}{p} |\xi_n|^p \int_{\Omega} \lambda(x)dx + \frac{1}{q} |\eta_n|^q \int_{\Omega} \mu(x)dx < r_n
\]

and

\[
\max_{(\xi, \eta) \in M(r_n)} G(\xi, \eta) = G(\xi_n, \eta_n).
\]

Finally assume that

\[
\limsup_{(\xi, \eta) \to \infty} \frac{G(\xi, \eta) \int_{\Omega} \alpha(x)dx}{\|\xi\|^p \int_{\Omega} \lambda(x)dx + \|\eta\|^q \int_{\Omega} \mu(x)dx} \geq \max \left(\frac{1}{p}, \frac{1}{q}\right)
\]

Then, Problem (P) admits an unbounded sequence of weak solutions in \(X\).
Proof. We apply Theorem 1 (part (b)). From the proof of Theorem 2, we know that \( \psi(r_n) = 0 \) for all \( n \in \mathbb{N} \). Then, since \( \lim_{n \to \infty} r_n = +\infty \), we have

\[
\gamma = \lim \inf_{r \to +\infty} \psi(r) = 0 < 1 = \beta.
\]

Now, observe that, by

\[
\lim_{\xi, \eta \to \infty} \sup_{\xi, \eta} G(\xi, \eta) \int_{\Omega} \alpha(x) dx \geq \max \left( \frac{1}{p}, \frac{1}{q} \right)
\]

we can choose \( \tau \in \mathbb{R} \) such that

\[
\lim_{\xi, \eta \to \infty} \sup_{\xi, \eta} G(\xi, \eta) \int_{\Omega} \alpha(x) dx > \tau > \max \left( \frac{1}{p}, \frac{1}{q} \right)
\]

and a sequence \( \{(\rho_n, \sigma_n)\}_{n \in \mathbb{N}} \) in \( \mathbb{R}^2 \), with \( \lim_{n \to \infty} \sqrt{|\rho_n|^2 + |\sigma_n|^2} = +\infty \) in such a way that

\[
G(\rho_n, \sigma_n) \int_{\Omega} \alpha(x) dx \geq \tau \left( |\rho_n|^p \int_{\Omega} \lambda(x) dx + |\sigma_n|^q \int_{\Omega} \mu(x) dx \right)
\]

for all \( n \in \mathbb{N} \). Denote by \( u_n \) the constant function on \( \Omega \) taking the value \( \rho_n \) and by \( v_n \) the constant function on \( \Omega \) taking the value \( \sigma_n \). One has

\[
\Psi(u_n, v_n) + \Phi(u_n, v_n) = \Psi(\rho_n, \sigma_n) + \Phi(\rho_n, \sigma_n) =
\]

\[
= \frac{1}{p} |\rho_n|^p \int_{\Omega} \lambda(x) dx + \frac{1}{q} |\sigma_n|^q \int_{\Omega} \mu(x) dx - \int_{\Omega} F(x, \rho_n, \sigma_n) dx =
\]

\[
= \frac{1}{p} |\rho_n|^p \int_{\Omega} \lambda(x) dx + \frac{1}{q} |\sigma_n|^q \int_{\Omega} \mu(x) dx - \int_{\Omega} \alpha(x) G(\rho_n, \sigma_n) dx =
\]

\[
= \frac{1}{p} |\rho_n|^p \int_{\Omega} \lambda(x) dx + \frac{1}{q} |\sigma_n|^q \int_{\Omega} \mu(x) dx - \int_{\Omega} \alpha(x) dx <
\]

\[
< \int_{\Omega} \lambda(x) dx \left( \frac{1}{p} - \tau \right) |\rho_n|^p + \int_{\Omega} \mu(x) dx \left( \frac{1}{q} - \tau \right) |\sigma_n|^q < 0.
\]

Consequently, the functional \( \Phi + \Psi \) is unbounded below. At this point, Theorem 1 (part (b)) ensures that there exists a sequence \( \{(u_n, v_n)\} \) of critical points of \( \Phi + \Psi \) such that \( \lim_{n \to \infty} \Psi(u_n, v_n) = +\infty \). But, of course, \( \Psi \) is bounded on each bounded subset of \( X \), and so the sequence \( \{(u_n, v_n)\} \) is unbounded in \( X \). This concludes the proof.
Theorem 4. Assume that there are sequences \( \{ r_n \} \) in \( \mathbb{R}^+ \) with \( \lim_{n \to \infty} r_n = 0 \), and \( \{ \xi_n \}, \{ \eta_n \} \) in \( \mathbb{R} \) such that for all \( n \in \mathbb{N} \), one has

\[
\frac{1}{p} |\xi_n|^p \int_\Omega \lambda(x)dx + \frac{1}{q} |\eta_n|^q \int_\Omega \mu(x)dx < r_n
\]

and

\[
\max_{(\xi, \eta) \in \mathcal{M}(r_n)} G(\xi, \eta) = G(\xi_n, \eta_n).
\]

Finally assume that

\[
\limsup_{(\xi, \eta) \to (0,0)} \frac{G(\xi, \eta) \int_\Omega \alpha(x)dx}{|\xi|^p \int_\Omega \lambda(x)dx + |\eta|^q \int_\Omega \mu(x)dx} > \max \left( \frac{1}{p}, \frac{1}{q} \right).
\]

Then, Problem (P) admits a sequence of non-zero weak solutions which strongly converges to \( \theta_X \) in \( X \).

Proof. After observing that \( \inf_X \Psi = \Psi(\theta_X) = 0 \), from the proof of Theorem 2, we know that \( \varphi(r_n) = 0 \) for all \( n \in \mathbb{N} \). Then, since \( \lim_{n \to \infty} r_n = 0 \), we have

\[
\delta = \liminf_{r \to 0^+} \varphi(r) = 0 < 1 = \beta.
\]

By

\[
\limsup_{(\xi, \eta) \to (0,0)} \frac{G(\xi, \eta) \int_\Omega \alpha(x)dx}{|\xi|^p \int_\Omega \lambda(x)dx + |\eta|^q \int_\Omega \mu(x)dx} > \max \left( \frac{1}{p}, \frac{1}{q} \right)
\]

there exist \( \tau \in \mathbb{R} \) such that

\[
\limsup_{(\xi, \eta) \to (0,0)} \frac{G(\xi, \eta) \int_\Omega \alpha(x)dx}{|\xi|^p \int_\Omega \lambda(x)dx + |\eta|^q \int_\Omega \mu(x)dx} > \tau > \max \left( \frac{1}{p}, \frac{1}{q} \right)
\]

and a sequence \( \{(\rho_n, \sigma_n)\}_{n \in \mathbb{N}} \) in \( \mathbb{R}^2 \setminus \{(0,0)\} \), converging to zero such that

\[
G(\rho_n, \sigma_n) \int_\Omega \alpha(x)dx > \tau \left( |\rho_n|^p \int_\Omega \lambda(x)dx + |\sigma_n|^q \int_\Omega \mu(x)dx \right)
\]

for all \( n \in \mathbb{N} \). If we denote by \( u_n \) the constant function on \( \Omega \) taking the value \( \rho_n \) and by \( v_n \) the constant function on \( \Omega \) taking the value \( \sigma_n \), of course the sequence \( \{(u_n, v_n)\} \) strongly converges to \( \theta_X \) in \( X \), and one has

\[
\Psi(u_n, v_n) + \Phi(u_n, v_n) = \Psi(\rho_n, \sigma_n) + \Phi(\rho_n, \sigma_n) =
\]
\[ \frac{1}{p} |\rho_n|^p \int_\Omega \lambda(x) dx + \frac{1}{q} |\sigma_n|^q \int_\Omega \mu(x) dx - \int_\Omega F(x, \rho_n, \sigma_n) dx = \\
= \frac{1}{p} |\rho_n|^p \int_\Omega \lambda(x) dx + \frac{1}{q} |\sigma_n|^q \int_\Omega \mu(x) dx - \int_\Omega \alpha(x) G(\rho_n, \sigma_n) dx = \\
= \frac{1}{p} |\rho_n|^p \int_\Omega \lambda(x) dx + \frac{1}{q} |\sigma_n|^q \int_\Omega \mu(x) dx - G(\rho_n, \sigma_n) \int_\Omega \alpha(x) dx < \\
< \int_\Omega \lambda(x) dx \left( \frac{1}{p} - \tau \right) |\rho_n|^p + \int_\Omega \mu(x) dx \left( \frac{1}{q} - \tau \right) |\sigma_n|^q < 0 \]

for all \( n \in \mathbb{N} \). Since \( \Phi(\theta_X) + \Psi(\theta_X) = 0 \), this means that \( \theta_X \) is not a local minimum of \( \Phi + \Psi \). Then, since \( \theta_X \) is the only global minimum of \( \Psi \), Theorem 1 (part(c)) ensures that there exists a sequence \( \{(u_n, v_n)\} \) of pairwise distinct critical points of \( \Phi + \Psi \) such that \( \lim_{n \to \infty} \Psi(u_n, v_n) = 0 \). So, \( a \text{ fortiori} \), one has \( \lim_{n \to \infty} \|(u_n, v_n)\|_X = 0 \), and the proof is complete. \( \square \)

A more general consequence of theorem 3 is as follows.

**Theorem 5.** Let \( \{\varepsilon_n\} \) and \( \{\delta_n\} \) be two sequences in \( \mathbb{R}^+ \) satisfying

\[ \delta_n < \varepsilon_n \quad \forall \quad n \in \mathbb{N}, \quad \lim_{n \to \infty} \delta_n = +\infty, \quad \lim_{n \to \infty} \frac{\varepsilon_n}{\delta_n} = +\infty \]

\[ A_n = \{ |\xi|^p + |\eta|^q \leq \varepsilon_n \} \quad B_n = \{ |\xi|^p + |\eta|^q \leq \delta_n \} \quad \sup_{A_n \setminus B_n} G \leq 0 \]

Finally assume that

\[ \limsup_{(\xi, \eta) \to \infty} \frac{G(\xi, \eta) \int_\Omega \alpha(x) dx}{|\xi|^p \int_\Omega \lambda(x) dx + |\eta|^q \int_\Omega \mu(x) dx} > \max \left( \frac{1}{p}, \frac{1}{q} \right) \]

Then, Problem \( (P) \) admits an unbounded sequence of weak solutions in \( X \).

**Proof.** From \( \delta_n < \varepsilon_n \) it follows that \( B_n \subseteq A_n \). Let

\[ \gamma' = \min \left\{ \frac{1}{pc_1^p}, \frac{1}{q c_2^q} \right\} > 0 \]

\[ \delta' = \max \left\{ \frac{\int_\Omega \lambda(x) dx}{p}, \frac{\int_\Omega \mu(x) dx}{q} \right\} > 0 \]
Since $\frac{\gamma'}{\gamma'} > 0$ and $\lim_{n \to \infty} \frac{\varepsilon_n}{\delta_n} = +\infty$ we can suppose that
\[
\frac{\delta'}{\gamma'} < \frac{\varepsilon_n}{\delta_n} \quad \forall \quad n \in \mathbb{N}.
\]

Let $r_n = \gamma' \varepsilon_n$. We have $\{r_n\}$ in $\mathbb{R}^+$ and $\lim_{n \to \infty} r_n = +\infty$ and $B_n \subseteq B(r_n) \subseteq A(r_n) \subseteq A_n$. Since $G \leq 0$ in $A_n \setminus B_n$ we have $\max_{B_n} G = \max_{A_n} G$, then $\max_{B_n} G = \max_{A(r_n)} G$ and so there is $(\xi_n, \eta_n) \in B_n$ such that
\[
\max_{(\xi, \eta) \in A(r_n)} G = G(\xi_n, \eta_n).
\]

Moreover
\[
\int_{\Omega} \lambda(x) dx \frac{|\xi_n|^p}{p} + \int_{\Omega} \mu(x) dx \frac{|\eta_n|^q}{q} \leq \delta'(\xi_n|^p + |\eta_n|^q) \leq \delta' \delta_n < r_n
\]
and so the sequences $\{\xi_n\}, \{\eta_n\}$ and $\{r_n\}$ have the properties required in theorem 3 from which the conclusion follows directly. \hfill \Box

Likewise, applying Theorem 4, we get the following theorem:

**Theorem 6.** Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be two sequences in $\mathbb{R}^+$ satisfying
\[
\delta_n < \varepsilon_n \quad \forall \quad n \in \mathbb{N}, \quad \lim_{n \to \infty} \varepsilon_n = 0, \quad \lim_{n \to \infty} \varepsilon_n = +\infty
\]
\[
A_n = \{|\xi|^p + |\eta|^q \leq \varepsilon_n\} \quad B_n = \{|\xi|^p + |\eta|^q \leq \delta_n\} \quad \sup_{A_n \setminus B_n} G \leq 0
\]

Finally assume that
\[
\limsup_{(\xi, \eta) \to (0,0)} \frac{G(\xi, \eta) \int_{\Omega} \alpha(x) dx}{\int_{\Omega} \lambda(x) dx + |\xi|^p \int_{\Omega} \lambda(x) dx + |\eta|^q \int_{\Omega} \mu(x) dx} > \max \left( \frac{1}{p}, \frac{1}{q} \right).
\]

Then, Problem (P) admits a sequence of non-zero weak solutions which strongly converges to $\theta_X$ in $X$. 
3. Examples.

Here is an example of application of theorem 3

**Example 1.** Let \( N = 1, \ p = q = 2, \ \lambda \equiv 1, \ \mu \equiv 1, \ \Omega \equiv ]0, 1[, \ f(u, v) = G_u(u, v) \) and \( g(u, v) = G_v(u, v) \) where \( G : \mathbb{R}^2 \to \mathbb{R} \) is the function defined by setting

\[
G(u, v) = \frac{1}{2}[(u^2 + v^2)\sin \log(u^2 + v^2 + 1)]
\]

Then for each \( \alpha \in C^0(\bar{\Omega}) \) with \( \alpha(t) \geq 0 \) in \( \Omega \) and \( j_0^1 \alpha(t) dt > 1 = m(\Omega) \), the following problem

\[
\begin{align*}
-u'' + u &= \alpha(t) f(u, v) \\
-v'' + v &= \alpha(t) g(u, v) \\
u'(0) &= u'(1) = 0 \\
v'(0) &= v'(1) = 0.
\end{align*}
\]

admits an unbounded sequence of weak solutions in \( X = H^1(\Omega) \times H^1(\Omega) \).

**Proof.** To prove this we apply Theorem 3. For each \( n \in \mathbb{N} \) put

\[
a_n = \sqrt{e^{(2n-1)\pi}} - 1 \\
b_n = \sqrt{e^{2n\pi}} - 1 \\
r_n = \frac{1}{2} \left( \frac{b_n}{c} \right)^2.
\]

Hence

\[
b_n = c\sqrt{2r_n}.
\]

Moreover, since \( p = q = 2 \) and \( c_1 = c_2 = c \), we have

\[
\frac{1}{p c_1^p} |\xi|^p + \frac{1}{q c_2^q} |\eta|^q = \frac{1}{2c^2} |\xi|^2 + \frac{1}{2c^2} |\eta|^2
\]

and

\[
A(r_n) = \quad \left\{ (\xi, \eta) \in \mathbb{R}^2 \text{such that} \quad \frac{1}{2c^2} |\xi|^2 + \frac{1}{2c^2} |\eta|^2 \leq r_n \right\} = \\
= \left\{ (\xi, \eta) \in \mathbb{R}^2 \text{such that} \quad \sqrt{\xi^2 + \eta^2} \leq c\sqrt{2r_n} \right\}.
\]
Observe that if \( a_n \leq \sqrt{|\xi|^2 + |\eta|^2} \leq b_n \), then
\[
(2n - 1)\pi \leq \log(|\xi|^2 + |\eta|^2 + 1) \leq 2n\pi,
\]
and so
\[
G(\xi, \eta) = \frac{1}{2} \left( (\xi^2 + \eta^2) \sin \log(\xi^2 + \eta^2 + 1) \right) \leq 0.
\]
Consequently, since \( G(0, 0) = 0 \), we have
\[
\max_{\sqrt{\xi^2 + \eta^2} \leq b_n} G(\xi, \eta) = \max_{\sqrt{\xi^2 + \eta^2} \leq a_n} G(\xi, \eta).
\]
Therefore we can fix \((\xi_n, \eta_n)\), with \(\sqrt{\xi_n^2 + \eta_n^2} \leq a_n\), such that
\[
G(\xi_n, \eta_n) = \max_{\sqrt{\xi^2 + \eta^2} \leq b_n} G(\xi, \eta).
\]
From [3] it follows that \( c \leq \sqrt{2} \). Then, since \( e^{(2n-1)\pi} - 1 < 2r_n \), we have
\[
0 \leq \xi_n^2 + \eta_n^2 \leq a_n^2 = e^{(2n-1)\pi} - 1 < 2r_n, \quad \text{whence} \quad 0 \leq \xi_n^2 + \eta_n^2 < 2r_n,
\]
therefore
\[
\frac{\xi_n^2 + \eta_n^2}{2} < r_n.
\]
Moreover \( \lim_{n \to \infty} r_n = +\infty \) and finally
\[
\limsup_{(\xi, \eta) \to \infty} \frac{\int_0^1 \alpha(t)dt \, G(\xi, \eta)}{|\xi|^2 + |\eta|^2} = \limsup_{(\xi, \eta) \to \infty} \frac{\int_0^1 \alpha(t)dt \, \frac{1}{2} \left( (\xi^2 + \eta^2) \sin \log(\xi^2 + \eta^2 + 1) \right)}{|\xi|^2 + |\eta|^2} = \limsup_{(\xi, \eta) \to \infty} \frac{1}{2} \int_0^1 \alpha(t)dt \left[ \sin \log(\xi^2 + \eta^2 + 1) \right] = \frac{1}{2} \int_0^1 \alpha(t)dt \limsup_{(\xi, \eta) \to \infty} \sin(\xi^2 + \eta^2 + 1) = \frac{1}{2} \int_0^1 \alpha(t)dt > \frac{1}{2}.
\]

Here is an example of application of theorem 4:
Example 2. Let \( N = 1, \ p = q = 2, \ \lambda \equiv 1, \ \mu \equiv 1, \ \Omega = ]0, 1[, \ f(u, v) = G_u(u, v) \) and \( g(u, v) = G_v(u, v) \) where \( G : \mathbb{R}^2 \rightarrow \mathbb{R} \) is the function defined by setting

\[
G(\xi, \eta) = \begin{cases} 
\frac{1}{2}(\xi^2 + \eta^2)\cos\log\frac{\xi^2 + \eta^2}{2} & \text{if } (\xi, \eta) \neq (0, 0) \\
0 & \text{if } (\xi, \eta) = (0, 0).
\end{cases}
\]

Then for each \( \alpha \in C^0(\overline{\Omega}) \) with \( \alpha(t) \geq 0 \) in \( \Omega \) and \( \int_0^1 \alpha(t)dt > 1 = m(\Omega) \), the following problem

\[
\begin{align*}
-u'' + u &= \alpha(t)f(u, v) \\
-v'' + v &= \alpha(t)g(u, v) \\
u'(0) &= u'(1) = 0 \\
v'(0) &= v'(1) = 0
\end{align*}
\]

admits a sequence of nonzero weak solutions which strongly converges to \( \theta_X \) in \( X = H^1(\Omega) \times H^1(\Omega) \).

Proof. To prove this, we apply Theorem 4. Put

\[
d_n = \left( \frac{1}{e^{\frac{2n\pi}{2}} + 2n\pi} \right)^\frac{1}{2},
\]

\[
b_n = \left( \frac{1}{e^{\frac{2n\pi}{2}} + 2n\pi} \right)^\frac{1}{2},
\]

\[
r_n = \frac{1}{2} \left( \frac{b_n}{c} \right)^2 = \frac{1}{2} \frac{b_n^2}{c^2} = \frac{1}{2c^2} b_n^2
\]

for each \( n \in \mathbb{N} \). Hence \( b_n = c\sqrt{2r_n} \). Again we have

\[
A(r_n) = \left\{ (\xi, \eta) \in \mathbb{R}^2 \text{ such that } \sqrt{\xi^2 + \eta^2} \leq c\sqrt{2r_n} \right\}.
\]

Observe that if \( a_n \leq \sqrt{\xi^2 + \eta^2} \leq b_n \), then \( G(\xi, \eta) \leq 0 \). Consequently, since \( G(0, 0) = 0 \), we have

\[
\max_{\sqrt{\xi^2 + \eta^2} \leq a_n} G(\xi, \eta) = \max_{\sqrt{\xi^2 + \eta^2} \leq b_n} G(\xi, \eta)
\]

Therefore we can fix \( (\xi_n, \eta_n) \), with \( \sqrt{\xi_n^2 + \eta_n^2} \leq a_n \), such that

\[
G(\xi_n, \eta_n) = \max_{\sqrt{\xi^2 + \eta^2} \leq b_n} G(\xi, \eta).
\]
Observe that \( c \leq \sqrt{2} \). Then, since \( \frac{\xi^2}{2} + \frac{\eta^2}{2} \leq a_n \), from \( e^x > 2 \) and
\[
2r_n = \frac{b_n^2}{c^2} = \frac{1}{c^2} \frac{1}{e^{\frac{b_n^2}{c^2} + 2n\pi}} \geq \frac{1}{2} \frac{1}{e^{\frac{1}{e^{\frac{b_n^2}{c^2} + 2n\pi}}} \frac{1}{e^{\frac{b_n^2}{c^2} + 2n\pi}}} = \frac{1}{e^{\frac{b_n^2}{c^2} + 2n\pi}} = a_n^2
\]
it follows that
\[
\Psi(\xi_n, \eta_n) = \frac{\xi_n^2 + \eta_n^2}{2} < r_n.
\]
Moreover, \( \lim_{n \to \infty} r_n = 0 \) and finally
\[
\limsup_{(\xi, \eta) \to (0,0)} \frac{G(\xi, \eta) \int_0^1 \alpha(t) dt}{\xi^2 + \eta^2} = \limsup_{(\xi, \eta) \to (0,0)} \frac{\frac{1}{2} (\xi^2 + \eta^2) \cos \log \frac{1}{\xi^2 + \eta^2} \int_0^1 \alpha(t) dt}{\xi^2 + \eta^2} = \frac{1}{2} \int_0^1 \alpha(t) dt > \frac{1}{2}. \quad \square
\]

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**REFERENCES**


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