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INFINITELY MANY SOLUTIONS TO THE NEUMAN PROBLEM FOR QUASILINEAR ELLIPTIC SYSTEMS

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In this paper we deal with the existence of weak solutions for the following Neumann problem

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \alpha(x)f(u,v) \text{ in } \Omega \\ -\Delta_q v + \mu(x)|v|^{q-2}v = \alpha(x)g(u,v) \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\,. \end{cases}$$

where ν is the outward unit normal to the boundary $\partial\Omega$ of the bounded open set $\Omega \subset \mathbb{R}^N$.The existence of solutions is proved by applying a critical point theorem obtained by B. Ricceri as consequence of a more general variational principle.

1. Introduction.

Here and in the sequel:
$\Omega \subset \mathbb{R}^N$ is a bounded open set with boundary of class C^1;
$N \geq 1$; $p > N$; $q > N$;
$\lambda, \mu \in L^\infty(\Omega)$, such that $\operatorname{essinf}_\Omega \lambda > 0$, $\operatorname{essinf}_\Omega \mu > 0$;

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$\alpha \in C^0(\overline{\Omega})$ nonnegative;
$f, g \in C^0(\mathbb{R}^2)$ such that the differential form $f(u, v)du + g(u, v)dv$ be exact.

In this paper we are interested in the following problem:

$$(P) \qquad \begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \alpha(x)f(u, v) \text{ in } \Omega \\ -\Delta_q v + \mu(x)|v|^{q-2}v = \alpha(x)g(u, v) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \\ \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \,. \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$. More precisely we are interested in the existence of infinitely many weak solutions to such a problem.

Whereas many results are available in the case of Dirichlet boundary conditions when $p = q$ (see e.g. [5] and [4]), it seems that nothing is known in the case of Neumann boundary conditions.

The existence of solutions to Problem (P) is proved by applying the following critical point theorem ([2] and [3]) obtained by B. Ricceri as a consequence of a more general variational principle.

Theorem 1. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gateaux differentiable functionals. Assume also that Ψ is strongly continuous and satisfies $\lim_{\|x\|\to\infty} \Psi(x) = +\infty$. For each $\rho > inf_X \Psi$, put*

$$\varphi(\rho) = \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - \inf_{\overline{(\Psi^{-1}(]-\infty,\rho[))}_w} \Phi}{\rho - \Psi(x)},$$

where $\overline{(\Psi^{-1}(]-\infty, \rho[))}_w$ is the closure of $\Psi^{-1}(]-\infty, \rho[)$ in the weak topology. Furthermore, set

$$\gamma = \liminf_{\rho\to\infty} \varphi(\rho)$$

and

$$\delta = \liminf_{\rho\to(\inf_X \Psi)^+} \varphi(\rho).$$

Then, the following conclusions hold:

(a) For each $\rho > inf_X \Psi$ and each $\beta > \varphi(\rho)$, the functional $\Phi + \beta\Psi$ has a critical point which lies in $\Psi^{-1}(]-\infty, \rho[)$.
(b) If $\gamma < +\infty$, then, for each $\beta > \gamma$ the following alternative holds: either $\Phi + \beta\Psi$ has a global minimum, or there exists a sequence $\{x_n\}$ of critical points of $\Phi + \beta\Psi$ such that $\lim_{n\to\infty} \Psi(x_n) = +\infty$.

(c) *If $\delta < +\infty$, then, for each $\beta > \delta$ the following alternative holds: either there exists a global minimum of Ψ which is a local minimum of $\Phi + \beta\Psi$, or there exists a sequence $\{x_n\}$ of pairwise distinct critical points of $\Phi + \beta\Psi$, with $\lim_{n\to\infty} \Psi(x_n) = \inf_X \Psi$, which weakly converges to a global minimum of Ψ.*

Let $G : \mathbb{R}^2 \to \mathbb{R}$ be the differentiable function such that $G_u(u, v) = f(u, v)$, $G_v(u, v) = g(u, v)$, $G(0, 0) = 0$ and let $F(x, u, v) = \alpha(x)G(u, v)$, then $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a differentiable function with respect to u and v and $F_u(x, u, v) = \alpha(x)f(u, v)$, $F_v(x, u, v) = \alpha(x)g(u, v)$. Then (P) can be written in the form

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = F_u(x, u, v) \text{ in } \Omega \\ -\Delta_q v + \mu(x)|v|^{q-2}v = F_v(x, u, v) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \\ \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\,. \end{cases}$$

and also in the form

$$\begin{cases} -\Delta_p u = \frac{\partial}{\partial u}\big(\alpha(x)G(u, v) - \frac{1}{p}\lambda(x)|u|^p - \frac{1}{q}\mu(x)|v|^q\big) \\ -\Delta_q v = \frac{\partial}{\partial v}\big(\alpha(x)G(u, v) - \frac{1}{p}\lambda(x)|u|^p - \frac{1}{q}\mu(x)|v|^q\big) \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \\ \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\,. \end{cases}$$

and therefore it is a gradient system [1]. Following [3] we first consider the space $W^{1,p}(\Omega)$ with the norm

$$\|u\|_\lambda = \left(\int_\Omega \lambda(x)|u(x)|^p dx + \int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}}$$

and the space $W^{1,q}(\Omega)$ with the norm

$$\|v\|_\mu = \left(\int_\Omega \mu(x)|v(x)|^q dx + \int_\Omega |\nabla v(x)|^q dx\right)^{\frac{1}{q}}.$$

Since by hypotheses $p > N$ and $q > N$, $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ are both compactly embedded in $C^0(\overline{\Omega})$. So, if we put

$$c_1 = c(\lambda) = \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{\|u\|_\lambda}$$

and

$$c_2 = c(\mu) = \sup_{u \in W^{1,q}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{\|u\|_\mu}$$

then both c_1 and c_2 are finite.

Then we take $X = W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ with the norm $\|(u, v)\|_X = \sqrt{\|u\|_\lambda^2 + \|v\|_\mu^2}$ and $Y = C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ with the norm $\|(u, v)\|_Y = \sqrt{\|u\|^2_{C^0(\overline{\Omega})} + \|v\|^2_{C^0(\overline{\Omega})}}$. Of course the space X is compactly embedded in Y and if we put

$$c = \sup_{(u,v)\in X\setminus\{(0,0)\}} \frac{\|(u, v)\|_Y}{\|(u, v)\|_X}$$

we have $c = \max\{c_1, c_2\}$. In order to apply theorem 1 we set

$$\Psi(u, v) = \frac{1}{p}\|u\|_\lambda^p + \frac{1}{q}\|v\|_\mu^q$$

and

$$\Phi(u, v) = -\int_\Omega F(x, u(x), v(x))dx$$

for all $(u, v) \in X$. Since X is compactly embedded in Y, not only the constant c is finite, but also the functionals Φ and Ψ are (well defined and) sequentially weakly lower semicontinuous and Gateaux differentiable in X, the critical points of $\Phi + \Psi$ being precisely the weak solutions to Problem (P). Moreover Ψ is coercive (and strongly continuous as well).

The sets $A(r)$, $B(r)$, $r > 0$, below specified, play an important role in our exposition:

$$A(r) = \left\{(\xi, \eta) \in \mathbb{R}^2 \text{ such that } \frac{1}{pc_1^p}|\xi|^p + \frac{1}{qc_2^q}|\eta|^q \le r\right\}$$

$$B(r) = \left\{(\xi, \eta) \in \mathbb{R}^2 \text{ such that} \frac{\int_\Omega \lambda(x)dx}{p}|\xi|^p + \frac{\int_\Omega \mu(x)dx}{q}|\eta|^q \le r\right\}.$$

The following inclusion holds:

$$B(r) \subseteq A(r).$$

To see this, we observe that by the definition of c_1 we have

$$\|u\|_{C^0(\overline{\Omega})} \le c_1\|u\|_\lambda$$

for every $u \in W^{1,p}(\Omega)$, hence (taking $u \equiv 1$)

$$1 \leq c_1^p \int_\Omega \lambda(x)dx.$$

Analogously we have

$$1 \leq c_2^q \int_\Omega \mu(x)dx.$$

Thus, the inequality

$$\frac{1}{pc_1^p}|\xi|^p + \frac{1}{qc_2^q}|\eta|^q \leq \frac{\int_\Omega \lambda(x)dx}{p}|\xi|^p + \frac{\int_\Omega \mu(x)dx}{q}|\eta|^q$$

holds for every $(\xi, \eta) \in \mathbb{R}^2$ and therefore the inclusion $B(r) \subseteq A(r)$ holds.

2. Results.

Theorem 2. *Assume that there are $r > 0$ and $\xi_0 \in \mathbb{R}$, $\eta_0 \in \mathbb{R}$ such that*

$$\frac{1}{p}|\xi_0|^p \int_\Omega \lambda(x)dx + \frac{1}{q}|\eta_0|^q \int_\Omega \mu(x)dx < r$$

and

$$\max_{A(r)} G(\xi, \eta) = G(\xi_0, \eta_0).$$

Then Problem (P) admits a weak solution (u, v) satisfying $\Psi(u, v) < r$

Proof. We apply Theorem 1 (part(a)) showing that $\varphi(r) = 0$.
Since $\overline{\Psi^{-1}(]-\infty, r[)}^w = \Psi^{-1}(]-\infty, r])$ it follows that for all $(u, v) \in \Psi^{-1}(]-\infty, r[)$

$$0 \leq \varphi(r) = \inf_{(u,v)\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(u, v) - \inf_{\overline{(\Psi^{-1}(]-\infty,r[))}_w} \Phi}{r - \Psi(u, v)} \leq$$

$$\leq \frac{\Phi(u, v) - \inf_{\overline{(\Psi^{-1}(]-\infty,r[))}_w} \Phi}{r - \Psi(u, v)}$$

Let $u_0(x) = \xi_0$, $v_0(x) = \eta_0$ for all $x \in \Omega$. Then $\nabla u_0 = 0$, $\nabla v_0 = 0$,

$$\Psi(u_0, v_0) = \frac{1}{p}\left(\int_\Omega \lambda(x)|\xi_0|^p dx\right) + \frac{1}{q}\left(\int_\Omega \mu(x)|\eta_0|^q dx\right) =$$

$$= \frac{1}{p}|\xi_0|^p \int_\Omega \lambda(x)dx + \frac{1}{q}|\eta_0|^q \int_\Omega \mu(x)dx < r$$

whence $(u_0, v_0) \in \Psi^{-1}(]-\infty, r[) \subseteq \overline{\Psi^{-1}(]-\infty, r[)}^w$. Moreover, for each $x \in \overline{\Omega}$ and for each $(u, v) \in \overline{\Psi^{-1}(]-\infty, r[)}^w$, since

$$\frac{1}{pc_1^p}|u(x)|^p + \frac{1}{qc_2^q}|v(x)|^q \le \frac{1}{pc_1^p}\|u\|_{C^0}^p + \frac{1}{qc_2^q}\|v\|_{C^0}^q \le$$

$$\le \frac{1}{pc_1^p}c_1^p\|u\|_\lambda^p + \frac{1}{qc_2^q}c_2^q\|v\|_\mu^q = \frac{1}{p}\|u\|_\lambda^p + \frac{1}{q}\|v\|_\mu^q \le r$$

one has $(u(x), v(x)) \in A(r)$; therefore $G(u(x), v(x)) \le G(\xi_0, \eta_0)$ whence $\alpha(x)G(u(x), v(x)) \le \alpha(x)G(\xi_0, \eta_0)$ whence

$$\int_\Omega \alpha(x)G(u(x), v(x))dx \le \int_\Omega \alpha(x)G(\xi_0, \eta_0)dx$$

i.e. $-\Phi(u, v) \le -\Phi(u_0, v_0)$ for all $(u, v) \in \overline{\Psi^{-1}(]-\infty, r[)}^w$

$$-\Phi(u_0, v_0) = \sup_{\overline{\Psi^{-1}(]-\infty,r[)}^w}(-\Phi(u, v)) = -\inf_{\overline{\Psi^{-1}(]-\infty,r[)}^w}(\Phi(u, v)).$$

From $\Psi(u_0, v_0) < r$ it follows that

$$\Phi(u_0, v_0) - \inf_{\overline{\Psi^{-1}(]-\infty,r[)}^w}(\Phi(u, v)) = \Phi(u_0, v_0) - \Phi(u_0, v_0) = 0$$

whence $\varphi(r) = 0$. □

Theorem 3. *Assume that there are sequences $\{r_n\}$ in $\mathbb{R}^+$ with $\lim_{n\to\infty} r_n = +\infty$, and $\{\xi_n\}$, $\{\eta_n\}$ in $\mathbb{R}$ such that for all $n \in \mathbb{N}$, one has*

$$\frac{1}{p}|\xi_n|^p \int_\Omega \lambda(x)dx + \frac{1}{q}|\eta_n|^q \int_\Omega \mu(x)dx < r_n$$

and

$$\max_{(\xi,\eta)\in A(r_n)} G(\xi, \eta) = G(\xi_n, \eta_n).$$

Finally assume that

$$\limsup_{(\xi,\eta)\to\infty} \frac{G(\xi, \eta)\int_\Omega \alpha(x)dx}{|\xi|^p \int_\Omega \lambda(x)dx + |\eta|^q \int_\Omega \mu(x)dx} > \max\left(\frac{1}{p}, \frac{1}{q}\right).$$

Then, Problem (P) admits an unbounded sequence of weak solutions in X.

Proof. We apply Theorem 1 (part(b)). From the proof of Theorem 2, we know that $\varphi(r_n) = 0$ for all $n \in \mathbb{N}$. Then, since $\lim_{n\to\infty} r_n = +\infty$, we have

$$\gamma = \liminf_{r\to+\infty} \varphi(r) = 0 < 1 = \beta.$$

Now, observe that, by

$$\limsup_{(\xi,\eta)\to\infty} \frac{G(\xi,\eta)\int_\Omega \alpha(x)dx}{|\xi|^p\int_\Omega \lambda(x)dx + |\eta|^q\int_\Omega \mu(x)dx} > \max\left(\frac{1}{p}, \frac{1}{q}\right)$$

we can choose $\tau \in \mathbb{R}$ such that

$$\limsup_{(\xi,\eta)\to\infty} \frac{G(\xi,\eta)\int_\Omega \alpha(x)dx}{|\xi|^p\int_\Omega \lambda(x)dx + |\eta|^q\int_\Omega \mu(x)dx} > \tau > \max\left(\frac{1}{p}, \frac{1}{q}\right)$$

and a sequence $\{(\rho_n, \sigma_n)\}_{n\in\mathbb{N}}$ in $\mathbb{R}^2$, with $\lim_{n\to\infty}\sqrt{|\rho_n|^2 + |\sigma_n|^2} = +\infty$ in such a way that

$$G(\rho_n, \sigma_n)\int_\Omega \alpha(x)dx > \tau\left(|\rho_n|^p\int_\Omega \lambda(x)dx + |\sigma_n|^q\int_\Omega \mu(x)dx\right)$$

for all $n \in \mathbb{N}$. Denote by u_n the constant function on Ω taking the value ρ_n and by v_n the constant function on Ω taking the value σ_n. One has

$$\Psi(u_n, v_n) + \Phi(u_n, v_n) = \Psi(\rho_n, \sigma_n) + \Phi(\rho_n, \sigma_n) =$$
$$= \frac{1}{p}|\rho_n|^p\int_\Omega \lambda(x)dx + \frac{1}{q}|\sigma_n|^q\int_\Omega \mu(x)dx - \int_\Omega F(x, \rho_n, \sigma_n)dx =$$
$$= \frac{1}{p}|\rho_n|^p\int_\Omega \lambda(x)dx + \frac{1}{q}|\sigma_n|^q\int_\Omega \mu(x)dx - \int_\Omega \alpha(x)G(\rho_n, \sigma_n)dx =$$
$$= \frac{1}{p}|\rho_n|^p\int_\Omega \lambda(x)dx + \frac{1}{q}|\sigma_n|^q\int_\Omega \mu(x)dx - G(\rho_n, \sigma_n)\int_\Omega \alpha(x)dx <$$
$$< \int_\Omega \lambda(x)dx\left(\frac{1}{p} - \tau\right)|\rho_n|^p + \int_\Omega \mu(x)dx\left(\frac{1}{q} - \tau\right)|\sigma_n|^q < 0.$$

Consequently, the functional $\Phi + \Psi$ is unbounded below. At this point , Theorem 1 (part(b)) ensures that there exists a sequence $\{(u_n, v_n)\}$ of critical points of $\Phi + \Psi$ such that $\lim_{n\to\infty}\Psi(u_n, v_n) = +\infty$. But, of course, Ψ is bounded on each bounded subset of X, and so the sequence $\{(u_n, v_n)\}$ is unbounded in X. This concludes the proof. □

Theorem 4. *Assume that there are sequences* $\{r_n\}$ *in* $\mathbb{R}^+$ *with* $\lim_{n\to\infty} r_n = 0$, *and* $\{\xi_n\}$, $\{\eta_n\}$ *in* $\mathbb{R}$ *such that for all* $n \in \mathbb{N}$, *one has*

$$\frac{1}{p}|\xi_n|^p \int_\Omega \lambda(x)dx + \frac{1}{q}|\eta_n|^q \int_\Omega \mu(x)dx < r_n$$

and

$$\max_{(\xi,\eta)\in A(r_n)} G(\xi,\eta) = G(\xi_n,\eta_n).$$

Finally assume that

$$\limsup_{(\xi,\eta)\to(0,0)} \frac{G(\xi,\eta)\int_\Omega \alpha(x)dx}{|\xi|^p \int_\Omega \lambda(x)dx + |\eta|^q \int_\Omega \mu(x)dx} > \max\left(\frac{1}{p},\frac{1}{q}\right).$$

Then, Problem (P) admits a sequence of non-zero weak solutions which strongly converges to θ_X *in* X.

Proof. After observing that $\inf_X \Psi = \Psi(\theta_X) = 0$, from the proof of Theorem 2, we know that $\varphi(r_n) = 0$ for all $n \in \mathbb{N}$. Then, since $\lim_{n\to\infty} r_n = 0$, we have

$$\delta = \liminf_{r\to 0^+} \varphi(r) = 0 < 1 = \beta.$$

By

$$\limsup_{(\xi,\eta)\to(0,0)} \frac{G(\xi,\eta)\int_\Omega \alpha(x)dx}{|\xi|^p \int_\Omega \lambda(x)dx + |\eta|^q \int_\Omega \mu(x)dx} > \max\left(\frac{1}{p},\frac{1}{q}\right)$$

there exist $\tau \in \mathbb{R}$ such that

$$\limsup_{(\xi,\eta)\to(0,0)} \frac{G(\xi,\eta)\int_\Omega \alpha(x)dx}{|\xi|^p \int_\Omega \lambda(x)dx + |\eta|^q \int_\Omega \mu(x)dx} > \tau > \max\left(\frac{1}{p},\frac{1}{q}\right)$$

and a sequence $\{(\rho_n,\sigma_n)\}_{n\in\mathbb{N}}$ in $\mathbb{R}^2 \setminus \{(0,0)\}$, converging to zero such that

$$G(\rho_n,\sigma_n)\int_\Omega \alpha(x)dx > \tau\left(|\rho_n|^p \int_\Omega \lambda(x)dx + |\sigma_n|^q \int_\Omega \mu(x)dx\right)$$

for all $n \in \mathbb{N}$. If we denote by u_n the constant function on Ω taking the value ρ_n and by v_n the constant function on Ω taking the value σ_n, of course the sequence $\{(u_n, v_n)\}$ strongly converges to θ_X in X, and one has

$$\Psi(u_n,v_n) + \Phi(u_n,v_n) = \Psi(\rho_n,\sigma_n) + \Phi(\rho_n,\sigma_n) =$$

$$= \frac{1}{p}|\rho_n|^p \int_\Omega \lambda(x)dx + \frac{1}{q}|\sigma_n|^q \int_\Omega \mu(x)dx - \int_\Omega F(x, \rho_n, \sigma_n)dx =$$

$$= \frac{1}{p}|\rho_n|^p \int_\Omega \lambda(x)dx + \frac{1}{q}|\sigma_n|^q \int_\Omega \mu(x)dx - \int_\Omega \alpha(x)G(\rho_n, \sigma_n)dx =$$

$$= \frac{1}{p}|\rho_n|^p \int_\Omega \lambda(x)dx + \frac{1}{q}|\sigma_n|^q \int_\Omega \mu(x)dx - G(\rho_n, \sigma_n)\int_\Omega \alpha(x)dx <$$

$$< \int_\Omega \lambda(x)dx \left(\frac{1}{p} - \tau\right)|\rho_n|^p + \int_\Omega \mu(x)dx \left(\frac{1}{q} - \tau\right)|\sigma_n|^q < 0$$

for all $n \in \mathbb{N}$. Since $\Phi(\theta_X) + \Psi(\theta_X) = 0$, this means that θ_X is not a local minimum of $\Phi+\Psi$. Then , since θ_X is the only global minimum of Ψ, Theorem 1 (part(c)) ensures that there exists a sequence $\{(u_n, v_n)\}$ of pairwise distinct critical points of $\Phi + \Psi$ such that $\lim_{n\to\infty} \Psi(u_n, v_n) = 0$. So, *a fortiori*, one has $\lim_{n\to\infty} \|(u_n, v_n)\|_X = 0$, and the proof is complete. □

A more general consequence of theorem 3 is as follows.

Theorem 5. *Let* $\{\varepsilon_n\}$ *and* $\{\delta_n\}$ *be two sequences in* $\mathbb{R}^+$ *satisfying*

$$\delta_n < \varepsilon_n \quad \forall \quad n \in \mathbb{N}, \quad \lim_{n\to\infty} \delta_n = +\infty, \quad \lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n} = +\infty$$

$$A_n = \{|\xi|^p + |\eta|^q \le \varepsilon_n\} \quad B_n = \{|\xi|^p + |\eta|^q \le \delta_n\} \quad \sup_{A_n \setminus B_n} G \le 0$$

Finally assume that

$$\limsup_{(\xi,\eta)\to\infty} \frac{G(\xi, \eta)\int_\Omega \alpha(x)dx}{|\xi|^p \int_\Omega \lambda(x)dx + |\eta|^q \int_\Omega \mu(x)dx} > \max\left(\frac{1}{p}, \frac{1}{q}\right).$$

Then, Problem (P) admits an unbounded sequence of weak solutions in X.

Proof. From $\delta_n < \varepsilon_n$ it follows that $B_n \subseteq A_n$. Let

$$\gamma' = \min\left\{\frac{1}{pc_1^p}, \frac{1}{qc_2^q}\right\} > 0$$

$$\delta' = \max\left\{\frac{\int_\Omega \lambda(x)dx}{p}, \frac{\int_\Omega \mu(x)dx}{q}\right\} > 0$$

Since $\frac{\delta'}{\gamma'} > 0$ and $\lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n} = +\infty$ we can suppose

$$\frac{\delta'}{\gamma'} < \frac{\varepsilon_n}{\delta_n} \quad \forall \quad n \in \mathbb{N}.$$

Let $r_n = \gamma'\varepsilon_n$. We have $\{r_n\}$ in $\mathbb{R}^+$ and $\lim_{n\to\infty} r_n = +\infty$ and $B_n \subseteq B(r_n) \subseteq A(r_n) \subseteq A_n$. Since $G \leq 0$ in $A_n \setminus B_n$ we have $\max_{B_n} G = \max_{A_n} G$, then $\max_{B_n} G = \max_{A(r_n)} G$ and so there is $(\xi_n, \eta_n) \in B_n$ such that

$$\max_{(\xi,\eta)\in A(r_n)} = G(\xi_n, \eta_n).$$

Moreover

$$\frac{\int_\Omega \lambda(x)dx}{p}|\xi_n|^p + \frac{\int_\Omega \mu(x)dx}{q}|\eta_n|^q \leq \delta'(|\xi_n|^p + |\eta_n|^q) \leq \delta'\delta_n < r_n$$

and so the sequences $\{\xi_n\}$, $\{\eta_n\}$ and $\{r_n\}$ have the properties required in theorem 3 from which the conclusion follows directly. □

Likewise, applying Theorem 4, we get the following theorem:

Theorem 6. *Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be two sequences in $\mathbb{R}^+$ satisfying*

$$\delta_n < \varepsilon_n \quad \forall \quad n \in \mathbb{N}, \quad \lim_{n\to\infty} \varepsilon_n = 0, \quad \lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n} = +\infty$$

$$A_n = \{|\xi|^p + |\eta|^q \leq \varepsilon_n\} \quad B_n = \{|\xi|^p + |\eta|^q \leq \delta_n\} \quad \sup_{A_n\setminus B_n} G \leq 0$$

Finally assume that

$$\limsup_{(\xi,\eta)\to(0,0)} \frac{G(\xi,\eta)\int_\Omega \alpha(x)dx}{|\xi|^p\int_\Omega \lambda(x)dx + |\eta|^q \int_\Omega \mu(x)dx} > \max\left(\frac{1}{p}, \frac{1}{q}\right).$$

Then, Problem (P) admits a sequence of non-zero weak solutions which strongly converges to θ_X in X.

3. Examples.

Here is an example of application of theorem 3

Example 1. *Let* $N = 1$, $p = q = 2$, $\lambda \equiv 1$, $\mu \equiv 1$, $\Omega =]0, 1[$, $f(u, v) = G_u(u, v)$ *and* $g(u, v) = G_v(u, v)$ *where* $G : \mathbb{R}^2 \to \mathbb{R}$ *is the function defined by setting*

$$G(u, v) = \frac{1}{2}\left[(u^2 + v^2)\sin\log(u^2 + v^2 + 1)\right]$$

Then for each $\alpha \in C^0(\overline{\Omega})$ *with* $\alpha(t) \geq 0$ *in* Ω *and* $\int_0^1 \alpha(t)dt > 1 = m(\Omega)$, *the following problem*

$$\begin{cases} -u'' + u = \alpha(t)f(u, v) \\ -v'' + v = \alpha(t)g(u, v) \\ u'(0) = u'(1) = 0 \\ v'(0) = v'(1) = 0. \end{cases}$$

admits an unbounded sequence of weak solutions in $X = H^1(\Omega) \times H^1(\Omega)$.

Proof. To prove this we apply Theorem 3. For each $n \in \mathbb{N}$ put

$$a_n = \sqrt{e^{(2n-1)\pi} - 1}$$

$$b_n = \sqrt{e^{2n\pi} - 1}$$

$$r_n = \frac{1}{2}\left(\frac{b_n}{c}\right)^2.$$

Hence

$$b_n = c\sqrt{2r_n}.$$

Moreover, since $p = q = 2$ and $c_1 = c_2 = c$, we have

$$\frac{1}{pc_1^p}|\xi|^p + \frac{1}{qc_2^q}|\eta|^q = \frac{1}{2c^2}|\xi|^2 + \frac{1}{2c^2}|\eta|^2$$

and

$$A(r_n) =$$

$$= \left\{(\xi, \eta) \in \mathbb{R}^2 \text{such that} \frac{1}{2c^2}|\xi|^2 + \frac{1}{2c^2}|\eta|^2 \leq r_n\right\} =$$

$$= \left\{(\xi, \eta) \in \mathbb{R}^2 \text{such that} \sqrt{\xi^2 + \eta^2} \leq c\sqrt{2r_n}\right\}.$$

Observe that if $a_n \leq \sqrt{|\xi|^2 + |\eta|^2} \leq b_n$, then

$$(2n-1)\pi \leq \log(|\xi|^2 + |\eta|^2 + 1) \leq 2n\pi,$$

and so

$$G(\xi, \eta) = \frac{1}{2}\big[(\xi^2 + \eta^2)\sin\log(\xi^2 + \eta^2 + 1)\big] \leq 0.$$

Consequently, since $G(0, 0) = 0$, we have

$$\max_{\sqrt{\xi^2+\eta^2}\leq b_n} G(\xi, \eta) = \max_{\sqrt{\xi^2+\eta^2}\leq a_n} G(\xi, \eta).$$

Therefore we can fix (ξ_n, η_n), with $\sqrt{\xi_n^2 + \eta_n^2} \leq a_n$, such that

$$G(\xi_n, \eta_n) = \max_{\sqrt{\xi^2+\eta^2}\leq b_n} G(\xi, \eta).$$

From [3] it follows that $c \leq \sqrt{2}$. Then, since $e^{(2n-1)\pi} - 1 < 2r_n$, we have $0 \leq \xi_n^2 + \eta_n^2 \leq a_n^2 = e^{(2n-1)\pi} - 1 < 2r_n$, whence $0 \leq \xi_n^2 + \eta_n^2 < 2r_n$, therefore

$$\frac{\xi_n^2 + \eta_n^2}{2} < r_n.$$

Moreover $\lim_{n\to\infty} r_n = +\infty$ and finally

$$\limsup_{(\xi,\eta)\to\infty} \frac{\int_0^1 \alpha(t)dt\, G(\xi, \eta)}{|\xi|^2 + |\eta|^2} =$$

$$= \limsup_{(\xi,\eta)\to\infty} \frac{\int_0^1 \alpha(t)dt\, \frac{1}{2}\big[(\xi^2 + \eta^2)\sin\log(\xi^2 + \eta^2 + 1)\big]}{|\xi|^2 + |\eta|^2} =$$

$$= \limsup_{(\xi,\eta)\to\infty} \frac{1}{2}\int_0^1 \alpha(t)dt\, \big[\sin\log(\xi^2 + \eta^2 + 1)\big] =$$

$$= \frac{1}{2}\int_0^1 \alpha(t)dt \limsup_{(\xi,\eta)\to\infty} \sin\log(\xi^2 + \eta^2 + 1) = \frac{1}{2}\int_0^1 \alpha(t)dt > \frac{1}{2}.$$

□

Here is an example of application of theorem 4 :

Example 2. *Let* $N = 1$, $p = q = 2$, $\lambda \equiv 1$, $\mu \equiv 1$, $\Omega =]0, 1[$, $f(u, v) = G_u(u, v)$ *and* $g(u, v) = G_v(u, v)$ *where* $G : \mathbb{R}^2 \to \mathbb{R}$ *is the function defined by setting*

$$G(\xi, \eta) = \begin{cases} \frac{1}{2}(\xi^2 + \eta^2)\cos\log\frac{1}{\xi^2+\eta^2} & \textit{if } (\xi, \eta) \neq (0, 0) \\ 0 & \textit{if } (\xi, \eta) = (0, 0). \end{cases}$$

Then for each $\alpha \in C^0(\overline{\Omega})$ *with* $\alpha(t) \geq 0$ *in* Ω *and* $\int_0^1 \alpha(t)dt > 1 = m(\Omega)$, *the following problem*

$$\begin{cases} -u'' + u = \alpha(t)f(u, v) \\ -v'' + v = \alpha(t)g(u, v) \\ u'(0) = u'(1) = 0 \\ v'(0) = v'(1) = 0 \end{cases}$$

admits a sequence of nonzero weak solutions which strongly converges to θ_X *in* $X = H^1(\Omega) \times H^1(\Omega)$.

Proof. To prove this, we apply Theorem 4. Put

$$a_n = \left(\frac{1}{e^{\frac{3\pi}{2}+2n\pi}}\right)^{\frac{1}{2}}$$

$$b_n = \left(\frac{1}{e^{\frac{\pi}{2}+2n\pi}}\right)^{\frac{1}{2}}$$

$$r_n = \frac{1}{2}\left(\frac{b_n}{c}\right)^2 = \frac{1}{2}\frac{b_n^2}{c^2} = \frac{1}{2c^2}b_n^2$$

for each $n \in \mathbb{N}$. Hence $b_n = c\sqrt{2r_n}$. Again we have

$$A(r_n) = \left\{(\xi, \eta) \in \mathbb{R}^2 \text{ such that } \sqrt{\xi^2 + \eta^2} \leq c\sqrt{2r_n}\right\}.$$

Observe that if $a_n \leq \sqrt{\xi^2 + \eta^2} \leq b_n$, then $G(\xi, \eta) \leq 0$. Consequently, since $G(0, 0) = 0$, we have

$$\max_{\sqrt{\xi^2+\eta^2} \leq a_n} G(\xi, \eta) = \max_{\sqrt{\xi^2+\eta^2} \leq b_n} G(\xi, \eta)$$

Therefore we can fix (ξ_n, η_n), with $\sqrt{\xi_n^2 + \eta_n^2} \leq a_n$, such that

$$G(\xi_n, \eta_n) = \max_{\sqrt{\xi^2+\eta^2} \leq b_n} G(\xi, \eta).$$

Observe that $c \leq \sqrt{2}$. Then, since $\sqrt{\xi_n^2 + \eta_n^2} \leq a_n$, from $e^{\pi} > 2$ and

$$2r_n = \frac{b_n^2}{c^2} = \frac{1}{c^2}\frac{1}{e^{\frac{\pi}{2}+2n\pi}} \geq \frac{1}{2}\frac{1}{e^{\frac{\pi}{2}+2n\pi}} > \frac{1}{e^{\pi}}\frac{1}{e^{\frac{\pi}{2}+2n\pi}} = \frac{1}{e^{\frac{3\pi}{2}+2n\pi}} = a_n^2$$

it follows that

$$\Psi(\xi_n, \eta_n) = \frac{\xi_n^2 + \eta_n^2}{2} < r_n.$$

Moreover $\lim_{n\to\infty} r_n = 0$ and finally

$$\limsup_{(\xi,\eta)\to(0,0)} \frac{G(\xi,\eta)\int_0^1 \alpha(t)dt}{\xi^2+\eta^2} = \limsup_{(\xi,\eta)\to(0,0)} \frac{\frac{1}{2}(\xi^2+\eta^2)\cos\log\frac{1}{\xi^2+\eta^2}\int_0^1 \alpha(t)dt}{\xi^2+\eta^2} =$$

$$= \frac{1}{2}\int_0^1 \alpha(t)dt > \frac{1}{2}. \qquad \square$$

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