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A TWO-PHASE VARIATIONAL PROBLEM WITH CURVATURE

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In this paper we consider a two phases variational problem related to the following functional: $F(u, \Omega) = \int_D | \nabla u|^2 dx + Area\{u = 0\} + \int_D f(x)\chi_{\{u>0\}}\, dx$. In particular we obtain results about the smoothness of the free boundary $\{u = 0\}$.

1. Introduction.

In [4] the authors consider a free boundary problem arising from the minimization of a Dirichlet-area integral related to the Ginzburg-Landau functional. They show in particular the smoothness of the free boundary. A question that could be of some interest in fluiddynamics, and constitues a natural continuation of that paper, is to examine the effect of a volume (gravity) integral. Accordingly, in this paper we consider the following variational problem.

Given a smooth domain $D \subset \Re^n$ and smooth boundary data g on ∂D, we look for a function $v \in H^1(D)$ with $v_{|\partial D} = g$, that minimizes the functional

$$F\,(v, \Omega) = \int_D |\nabla v|^2\, dx + Area\,\{v = 0\} + \int_D f\,(x)\,\chi_{\{v>0\}}\, dx \tag{1}$$

that is, the Dirichlet integral of v, plus the area of the level surface $\Gamma = \{v = 0\}$, plus a volume integral with density f on the positive phase .

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Heuristically, a minimizer u is harmonic in its positive and negative region while on its zero level set (the free boundary) it satisfies the equation

$$|\nabla u^+|^2 - |\nabla u^-|^2 - f = k(\Gamma) \tag{2}$$

where $k(\Gamma)$ denotes the mean curvature of Γ.

If we assume that f is bounded, the key point in proving the smoothness of the free boundary is to prove that u is Lipschitz and that $k(\Gamma)$ is bounded in a weak (viscosity) sense (see section 4). The proofs of this part parallel those of the corresponding theorems in [4], so that we sketch them pointing out the differences.

This allows one to use the theory of *almost minimal surfaces* (see [8]) to deduce that actually the reduced part Γ^* of Γ is locally a graph of a $C^{1,\alpha}$ function, for any $0 < \alpha < 1$, that satisfies (2) in viscosity sense. Further regularity of f implies more regularity of Γ^* and, in particular, f (real) analytic implies that Γ^* is an analytic surface (section 5).

2. Existence of minimizers and main result.

Let D be a bounded, smooth domain in $\Re^n$ and $g \in H^1(D)$.

Definition 1. *The pair (Ω, v) is admissible if Ω is a set of finite perimeter in D, $v \in H^1(D)$, $v - g \in H_0^1(D)$ and*

$$v\,|_{\Omega\cap D} \geq 0 \qquad v\,|_{\Omega^c\cap D} \leq 0 \qquad a.e.$$

We recall that

$$Per(\Omega, D) = \sup\left\{\int_\Omega \operatorname{div} p\, dx : p \in C_0^1(\Omega, \Re^n), |p(x)| \leq 1\right\} < \infty.$$

For convenience we denote $Per(\Omega) = Per(\Omega, D)$.

Our problem is to minimize the functional

$$F(v, \Omega) = \int_D |\nabla v|^2\, dx + Per(\Omega) + \int_D f(x)\, \chi_{\{v>0\}}\, dx$$

among all admissible pairs (v, Ω).

Proposition 1. *If $f \in L^1(D)$, there exists a pair (u, Ω) that minimizes F.*

Proof. Let $\{(u_m, \Omega_m)\}$ be a minimizing sequence, that is, (u_m, Ω_m) is an admissible pair and

$$F\,(u_m, \Omega_m) \longrightarrow \inf F\,(v, \Omega) \qquad m \to +\infty$$

Passing to a subsequence, there exists a pair (u, Ω) such that:

$$\begin{array}{ll} \chi_{|\Omega_m} \longrightarrow \chi_{|\Omega} & \text{strongly in } L^1(D) \\ u_m \rightharpoonup u & \text{weakly in } H^1(D) \\ u_k \chi_{|\Omega_m \cap D} \longrightarrow u \chi_{|\Omega \cap D} & \text{a.e. in } D \\ u_k \chi_{|\Omega_m^c \cap D} \longrightarrow u \chi_{|\Omega^c \cap D} & \text{a.e. in } D \end{array}$$

Then it follows that

$$u_{|\Omega \cap D} \geq 0 \qquad u_{|\Omega^c \cap D} \leq 0 \qquad \text{a.e. in } D,$$

hence (u, Ω) is admissible.

By the lower semicontinuity of F we have

$$\inf F(v, \Omega) \leq F(u, \Omega) \leq \liminf_{m \to \infty} F(u_m, \Omega_m) = \inf F(v, \Omega),$$

therefore $\inf F(v, \Omega) = F(u, \Omega)$, that is (u, Ω) is a minimizer. □

We call $\Gamma(u) = \partial\Omega \cap D$ the *free boundary.* Our purpose if to show optimal regularity for u and $\Gamma(u)$.

The main results are summarized in the following theorems.

Theorem 2. *Let (u, Ω) be a minimizer in the unit ball $B_1 = B_1(0)$, with $0 \in \Gamma(u)$. If $f \in L^\infty(B_1)$ then, in $B_{1/2}$:*

a) u is Lipschitz continuous ;

b) the curvature $k(\Gamma(u))$ is bounded in the viscosity sense

Corollary 3. *The reduced part $\Gamma^*(u)$ of the free boundary is (locally) a graph of a $C^{1,\alpha}$ function, for any $0 < \alpha < 1$.*

Theorem 4. *If $f \in C^{m,\beta}(B_1)$, $0 < \beta < 1$, then, in $B_{1/2}$ the reduced part $\Gamma^*(u)$ of the free boundary is (locally) a graph of a $C^{m+2,\beta}$ function; if f is (real) analytic, then the reduced part $\Gamma^*(u)$ of the free boundary is analytic.*

In particular, the free boundary relation

$$|\nabla u^+|^2 - |\nabla u^-|^2 - f = k(\Gamma^*)$$

holds in classical sense.

3. Hölder continuity.

The first step in the proof of theorem 2 is to show that u is $\frac{1}{2}$−Hölder continuous and that Ω has positive uniform density at every point of the free boundary. An important role is played by the quantities

$$I_r^{\pm} = \int_{B_r} \frac{\left|\nabla u^{\pm}\right|^2}{|x|^{n-2}}\, dx.$$

Notice that if we set $u_r(y) = \frac{1}{\sqrt{r}} u(ry)$, we have $I_r(u) = r I_1(u_r)$ and (see [4])

$$\left(u^{\pm}\right)^2 \le I_r\left(u^{\pm}\right).$$

Therefore, in order to prove that $u \in C^{\frac{1}{2}}$, using the Monotonicity Formula (see [3]), it's enough to show that

$$I_r\left(u^{\pm}\right) \le cr\, ||u||_{L^\infty}^2. \tag{3}$$

Also notice that if (u, Ω) is a minimizer of F in B_1 and

$$\Omega_r = \{y : ry = x, x \in \Omega\},$$

then (u_r, Ω_r) is a minimizer of $r^{n-1}F$, and therefore of F, in B_1.
For an admissible pair (v, Ω), we define $\Omega^- = B_1 - \overline{\Omega}^+$. We have:

Theorem 3. *Let (u, Ω) be minimizer in B_1 and $f \in L^\infty(B_1)$. Then u is $C^{\frac{1}{2}}$ $H\ddot{o}lder - continuous$ in $B_{\frac{1}{2}}$ and*

$$||u||_{C^{\frac{1}{2}}\left(B_{\frac{1}{2}}\right)} \le c\left(n, \|f\|_\infty\right) ||u||_{L^\infty(B_1)}$$

and, for every $x \in \Gamma(u)$, if $r \le \frac{1}{8}$,

$$\left|B_r(x) \cap \Omega^{\pm}\right| \ge c_0\left(n, \|f\|_\infty\right) r^n.$$

Moreover, $u^{\pm}$ are harmonic in their positivity set.

We now recall the notion of harmonic replacement.

Let K a measurable subset of D and a function $g \in H^1(D)$. We say that g is supported in K if $g = 0$ a.e in $D - K$. Define

$$S = \left\{g : g \in H^1(D), g\{\text{supported in } K\right\}$$

a closed convex set in $H^1(D)$.

Definition 2. *The function* h_0 *is the harmonic replacement of* $f \in H^1(D)$ *in* D *if:*

(i) $h_0 \in S$,
(ii) $h_0 - f \in H^1(D)$,
(iii) h_0 *minimizes the Dirichlet integral in* $S \cap \left\{ f + H_0^1(D) \right\}$.

The main properties of harmonic replacements are summarized in the following two lemmas (see [4]):

Lemma 6. *The harmonic replacement* h_0 *is unique, and if* f *is nonnegative, then* h_0 *is nonnegative and subharmonic; in particular it can be defined everywhere in* D *as a u.s.c. function by limit of solid averanges. Also, in the sense of measures,*

$$\Delta (h_0)^2 = 2 |\nabla h_0|^2$$

Lemma 7. *Let* h_0 *be the harmonic replacement of* $f \geq 0$ *in* D*. Assume* $B_1 \subset D$ *and* $h_0(0) = 0$*. Then*

$$\sup_{B_{(1-s)r}} (h_0)^2 \leq \frac{c(n)}{s^n} \int_{B_r} \frac{|\nabla h_0|^2}{|x|^{n-2}} dx \tag{4}$$

for any $0 < s < 1$ *and* $0 < r \leq 1$*, and*

$$\int_{B_r} \frac{|\nabla h_0|^2}{|x|^{n-2}} dx \leq c(n) r^{-n} \int_{B_{2r} - B_r} (h_0)^2 \, dx \tag{5}$$

for $0 < r < \frac{1}{4}$.

Let now (u, Ω) be a minimizer. Then

(a) $u^+ = \max\{0, u\}$ is supported in $\Omega \cap D$ and $u^- = \max\{-u, 0\}$ is supported in $\Omega^c \cap D$, moreover u^+ and u^- are harmonic replacement of u,
(b) u^+ and u^- are subharmonic,
(c) at any point x of Lebesgue differentiability of the free boundary, u^+ and u^- vanish; moreover, the monotonicity formula and the estimates (4) and (5) hold in a sufficiently small ball centered at x for u^+ and u^-,
(d) $\Delta \left(u^\pm\right)^2 = 2 \left|\nabla u^\pm\right|^2$ hold in the sense of measures.

For an admissible pair (v, Ω), we define $\Omega^- = B_1 - \overline{\Omega}^+$

Proof. of Theorem 2. Sketch. Consider I_r^+. By rescaling, we may suppose $r = 1$. For $0 < h < 1$ we perturb the free boundary defining

$$\Omega_*^- = \Omega^- \cup B_{1-h} \qquad \Omega_* = \Omega \backslash \overline{B}_{1-h},$$

The new free boundary is:

$$\Gamma^* = \left[\Omega \cap \partial B_{1-h}\right] \cup \left[\Gamma \cap \left(B_1 \backslash \overline{B}_{1-h}\right)\right]$$

Let u_*^- be the harmonic extention of u^- in Ω_*^-, that is:

$$\begin{cases} \Delta u_*^- = 0 & \text{in } \Omega_*^- \\ u_*^- = u^- & \text{su } \partial \Omega_*^- \end{cases}$$

Let

$$U_h = \sup\ u \ \text{ in } B_{1-\frac{h}{4}}$$

and let G be such that:

$$\begin{cases} \Delta G = 0 & \text{in } R_h = B_{1-\frac{h}{4}} - B_{1-h} \\ G = 1 & \text{su } \partial B_{1-\frac{h}{4}} \\ G = 0 & \text{su } \partial B_{1-h} \end{cases}$$

We now define a perturbation u_* of u as follows:

$$u_* = \begin{cases} -u_*^- & \text{in} \quad \Omega_*^- \\ \min\left\{u^+, U_h G\right\} & \text{in} \quad \Omega_*^+ \cap R_h \\ u^+ & \text{in} \quad \Omega_*^+ - R_h \end{cases}$$

The couple $\left(u_*, \Omega_*^+\right)$ is admissible and

$$F\left(u_*, \Omega_*^+\right) \geq F\left(u, \Omega^+\right).$$

We compute the variation of the various terms in the functional F.

Since $\Omega^- \subseteq \Omega_*^-$ we have:

$$\int_{B_1} \left|\nabla u_*^-\right|^2 dx \leq \int_{B_1} \left|\nabla u^-\right|^2 dx. \tag{6}$$

Moreover

$$\int_{B_1} \left|\nabla u_*^+\right|^2 dx \leq \int_{B_1} \left|\nabla u^+\right|^2 dx + cU_h^2 h^{-1}. \tag{7}$$

From the properties of the perimeter,

$$Per\left(\Omega_*^+\right) - Per\left(\Omega^+\right) \leq H_{n-1}\left(\partial B_{1-h} \cap \Omega^+\right) - Per\left(\Omega^+, B_{1-h}\right). \tag{8}$$

Moreover ($b = \|f\|_\infty$),
(9)
$$\int_{B_1} f(x)\,\chi_{\{u_*>0\}}\,dx - \int_{B_1} f(x)\,\chi_{\{u>0\}}\,dx = -\int_{\Omega^+\cap B_{1-h}} f(x)\;dx \le b\left|\Omega^*\cap B_{1-h}\right|$$

From (6), (7), (8), (9) and minimality condition, we conclude that

$$(10)\quad -b\left|\Omega^+\cap B_{1-h}\right| + Per\left(\Omega^+, B_{1-h}\right) \le H_{n-1}\left(\partial B_{1-h}\cap\Omega^+\right) + cU_h^2h^{-1}$$

Let now $\rho_0 = \frac{1}{2}$ and $c \le \frac{1}{4}$, we define

$$\rho_{m+1} = \rho_m - c2^{-m}.$$

Put:

$$I_m = I^+_{\rho_m} \qquad \text{and} \qquad V_m = \left|\Omega^+\cap\left(B_{\rho_m} - \overline{B}_{\rho_{m+1}}\right)\right|,$$

we show the following inequality:

$$(11)\qquad I_{m+1} \le C^m I_m V_m.$$

In fact, from Lemma 7, we have

$$I_{m+1} \le C2^{2nm}\sup_{B_{\rho_m'}}\left(u^+\right)^2 V_m \le C2^{4nm} I_m V_m \le C_1^m I_m V_m$$

where $\rho_m^{'} = \rho_{m+1} + c2^{-(m+1)}$.

Let $\overline{\rho} > 0$, be a positive number with $\rho_{m+1} < \overline{\rho} < \rho_m^{'}$. From isoperimetric inequality we have:

$$-rb\left|\Omega^+\cap B_{1-h}\right| + c(n)\left|\Omega^+\cap B_{1-h}\right|^{\frac{n-1}{n}} \le H_{n-1}\left(\partial B_{1-h}\cap\Omega^+\right) + cU_h^2h^{-1}$$

therefore

$$\left|\Omega^+\cap B_{1-h}\right|^{\frac{n-1}{n}} \le c_1(n,b)\left(H_{n-1}\left(\partial B_{1-h}\cap\Omega^+\right) + cU_h^2h^{-1}\right).$$

In particular, from (11), we conclude that

$$\left(V_{m+1}\right)^{\frac{n-1}{n}} \le c_1(n)\left(H_{n-1}\left(\partial B_r\cap\Omega^+\right) + C_1^m I_m\right).$$

Integrating with respect to $\overline{\rho}$ over the interval $\left(\rho_{m+1}, \rho_m^{'}\right)$, we get:

$$\left(V_{m+1}\right) \le c_2(n)\,C_1^m\left(V_m + I_m\right)^{\frac{n}{n-1}}.$$

From (11) we have

$$V_{m+1} + I_{m+1} \leq c_2 C_1^m \, (I_m + V_m)^{\frac{n}{n-1}} \tag{12}$$

As consequence (12) there exists a constant δ such that if $V_0 + I_0 \leq \delta < 1$, then $V_m + I_m \to 0$, when $m \to \infty$. But that is no possible because $\rho_m \to \rho_\infty > 0$. Hence, we have:

$$V_0 + I_0 > \delta.$$

In particular, put $I_0 \leq \delta_0 = \frac{\delta}{2}$, we have

$$\frac{\delta}{2} < V_0 < \left| \Omega^+ \cap B_{\frac{1}{2}} \right| .$$

If $0 < \overline{\rho} < 1$ then the conclusion follows from the scaling properties of the minimizers.

We recall the following Lemma (see [4]):

Lemma 8. *a) If* $I_1^+ I_1^- \leq \Lambda$. *Then*

$$I_{\frac{1}{8}}^+ \leq c\,(n, \Lambda)\,, \qquad I_{\frac{1}{8}}^- \leq c\,(n, \Lambda)\,.$$

b) If $I_1^\pm \leq \Lambda^\pm$. *Then*

$$\left|\Omega^\pm \cap B_{\frac{1}{2}}\right| \geq C\left(n, \Lambda^\pm\right) > 0.$$

We are now ready to prove Theorem 5.

Proof. From the Monotonicity Formula, for $r \leq \frac{1}{8}$,

$$I_r^+ I_r^- \leq c r^4 \, ||u||_{L^\infty(B_1)}$$

By rescaling, for the function $u_r\,(y) = r^{-\frac{1}{2}} u\,(ry)$, we have

$$I_1^+\,(u_r)\, I_1^-\,(u_r) \leq c r^2\, ||u||^4_{L^\infty(B_1)}$$

For Lemma 8 we deduce that:

$$I_{\frac{1}{2}}^+\,(u_r) \leq C\, ||u||^2_{L^\infty(B_1)}\,, \qquad I_{\frac{1}{2}}^-\,(u_r) \leq C\, ||u||^2_{L^\infty(B_1)}$$

Recalling (5) (with $h = \frac{1}{2}$) and Lemma 8, we conclude that $u \in C^{\frac{1}{2}}$.

4. Lipschitz Continuity of a minimizer.

We now show the following theorem:

Theorem 9. *Let (u, Ω^{+}) be a minimizer in B_1 and $f \in L^{\infty}(B_1)$. Then u is lipschitz continuous on $B_{\frac{1}{2}}$.*

Proof. Let φ be a cutoff function, $0 \le \varphi \le 1$, $\varphi = 1$ in $B_{\frac{1}{4}}$, $\varphi = 0$ outside $B_{\frac{1}{2}}$. For $\varepsilon > 0$, define $\vartheta = [u - \varepsilon]^{+}$ and let M be the smallest constant such that $Md(x) \ge \vartheta(x)\varphi(x)$ for each $x \in B_1$, where $d(x) = dist(x, \Gamma(u))$.

Suppose x_0 is a point such that $Md(x_0) = \vartheta(x_0)\varphi(x_0)$ and that $d(x_0) = dist(x_0, y_0)$ with $y_0 \in \Gamma$. By a rotation and translation we may suppose that $y_0 = 0$ and $x_0 = d(x_0)e_1$.

Since ϑ is smooth around x_0, we have

$$d(x) \ge d(x_0) + \frac{\langle \nabla(\vartheta\varphi)(x_0), x - x_0\rangle}{M} + \frac{P(x - x_0)}{M} + O\left(\frac{|x - x_0|^3}{M}\right)$$

where P is a quadratic polynomial satisfyting $\Delta P = \Delta(\vartheta\varphi)$ at $x = x_0$ and $D_{11}P \le 0$. Estimating ΔP as in [4] we have, $\Delta P \ge -\frac{CM}{\varphi(x_0)}$ In particular on the hyperplane $x_1 = d(x_0)$

$$\Delta_{x'}\bar{P}(x') \ge -\frac{CM}{\varphi(x_0)}. \tag{13}$$

which implies

$$d(x) \ge d(x_0) + \frac{\bar{P}(x')}{M} + O\left(\frac{|x'|^3}{M}\right) \tag{14}$$

Therefore the free boundary, near the origin, is below the surface

$$S = \left\{(x_1, x') : x_1 = \psi(x') = -\frac{\bar{P}(x')}{M} + O\left(\frac{|x'|^3}{M}\right)\right\}.$$

If now put $k(S)(x)$ the mean curvature of S, from (13), we have:

$$k(S)(x) = -\frac{1}{n-1}\Delta\psi(x') \le \frac{C}{\varphi(x_0)} + O(|x'|).$$

Near the origin, for $x_1 > \psi(x')$,

$$u^{+}(x) \ge \frac{CM}{\varphi(x_0)}x_1 + o(|x|).$$

while from the Monotonicity Formula,

$$\sup_{B_r(y)} u^- \leq c\frac{\varphi(x_0)}{M}r$$

for small r and y on the free boundary.

We perform a perturbation of the free boundary by means of the family of surfaces

$$S_t^- = \left\{\left(x_1, x^{'}\right) : x_1 = \psi_t^-(x^{'}) = \psi(x^{'}) + \frac{\alpha_0}{\varphi(x_0)}\left|x^{'}\right|^2 - t\right\}$$

and

$$S_t^+ = \left\{\left(x_1, x^{'}\right) : x_1 = \psi_t^+\left(x^{'}\right) = \psi\left(x^{'}\right) + t\right\}.$$

where $t \geq 0$, $\alpha_0 > 0$ and both small. Denote H_t the lens-shaped domain between S_t^+ and S_t^-, that is

$$H_t = \left\{\psi_t^-\left(x^{'}\right) < x_1 < \psi_t^+\left(x^{'}\right)\right\}.$$

Put

$$\Omega_t^+ = \Omega^+ \cup H_t, \quad \Omega_t^- = B_1 - \overline{\Omega_t^+}, \quad W_t = \Omega^- \cap \left\{x_1 > \psi_t^-\left(x^{'}\right)\right\}.$$

Let w_t be the harmonic extension of u^+ in H_t, that is

$$w_t = \begin{cases} \Delta w_t = 0 & \text{in } H_t \\ w_t = u^+ & \text{on } \partial H_t \end{cases}$$

We now define a perturbation u_t of u as follows:

$$u_t = \begin{cases} u^+ & \text{in} & \Omega_t^+ - H_t \\ w_t & \text{in} & H_t \\ -\min\left\{u^-, c\frac{\varphi(x_0)}{M}d_t\right\} & \text{in} & W_{2t} - W_t \\ -u^- & \text{in the rest of} & B_1 \end{cases}$$

The couple $\left(u_t, \Omega_t^+\right)$ is admissible and

$$F\left(u_t, \Omega_t^+\right) \geq F\left(u, \Omega^+\right).$$

We estimate the variation of the various terms in the functional F. From the property of the perimeter,

$$Per\left(\Omega_t^+\right) = Per\left(\Omega^+, B_1 - \overline{H}_t\right) + H_{n-1}\left(S_t^- \cap \Omega^-\right)$$

while

$$Per\left(\Omega^+\right) \geq Per\left(\Omega^+, H_t\right) + Per\left(\Omega^+, B_1 - \overline{H}_t\right).$$

As in [2]:

$$Per\left(\Omega_t^+\right) - Per\left(\Omega^+\right) \leq \frac{C}{\varphi\left(x_0\right)} \left|W_t\right| \tag{15}$$

$$\int_{B_1} \left|\nabla u_t^-\right|^2 dx - \int_{B_1} \left|\nabla u^-\right|^2 dx \leq c\frac{\varphi\left(x_0\right)^2}{M^2}\left(\left|W_{2t} - W_t\right|\right) \tag{16}$$

$$\int_{B_1} \left|\nabla u^+\right|^2 dx - \int_{B_1} \left|\nabla u_t^+\right|^2 dx \geq \frac{CM^2}{\varphi\left(x_0\right)^2}\left|W_{\frac{t}{2}}\right| \tag{17}$$

moreover

$$\int_{B_1} f\left(x\right) \chi_{\{u_t>0\}}\, dx - \int_{B_1} f\left(x\right) \chi_{\{u>0\}}\, dx = \int_{W_t} f\left(x\right)\, dx \tag{18}$$

From (15), (16), (17), (18) and from minimality condition, we have:

$$0 \leq -\frac{CM^2}{\varphi\left(x_0\right)^2}\left|W_{\frac{t}{2}}\right| + c\frac{\varphi\left(x_0\right)^2}{M^2}\left(\left|W_{2t} - W_t\right|\right) + \frac{C}{\varphi\left(x_0\right)}\left|W_t\right| + \int_{W_t} f\left(x\right)\, dx$$

Since W_t has positive density, we conclude that:

$$\frac{CM^2}{\varphi\left(x_0\right)^2} - c\frac{\varphi\left(x_0\right)^2}{M^2} \leq \frac{C}{\varphi\left(x_0\right)} + f\left(0\right).$$

and therefore $M \leq C_0$.

5. Regularity of the Free Boundary.

From the lipschitz continuity of u is now easy to show that the reduced boundary is a $C^{1,\frac{1}{2}}$ surface but for a set of zero $s-$ dimensional Hausdorff measure for any $s > n-8$. Precisely:

Lemma 10. *Let $\left(u, \Omega^{+}\right)$ be a minimizer in B_1. If f is bounded, then:*

1. $\partial^{*}\Omega^{+} \cap B_{\frac{1}{2}}$ is a $C^{1,\frac{1}{2}}$ ipersurface and
2. $H_s\left[\left(\partial\Omega^{+} - \partial^{*}\Omega^{+}\right) \cap B_{\frac{1}{2}}\right] = 0$ for every $s > n-8$

Proof. Let $A \subset B_{\frac{1}{2}}$ and select Ω_1 such that $\Omega_1 \Delta \Omega_2 \subset B_r(x)$, $x \in A$, r small. Let u_r be any perturbation of u inside $B_r(x)$ with the same Lipschitz constant L and such that the pair $\left(u_r, \Omega_1^{+}\right)$ is admissible. Then $F\left(u, \Omega^{+}\right) \leq F\left(u_r, \Omega_1^{+}\right)$, which forces

$$Per\left(\Omega^{+}, B_r(x)\right) - Per\left(\Omega_1^{+}, B_r(x)\right) \leq \left(b + cL^2\right) r^n.$$

Therefore $\partial^{*}\Omega^{+}$ is an almost minimal surface and the conclusion follows from [2] or [8].

From Lemma 11 $\Gamma^{*} = \partial^{*}\Omega^{+} \cap B_{\frac{1}{2}}$ is locally described by the graph of a $C^{1,1/2}$ function. To obtain further regularity, we show that on Γ^{*} the free boundary relation

$$\left|\nabla u^{+}\right|^2 - \left|\nabla u^{-}\right|^2 - f(x) = k\left(\Gamma^{*}\right) \tag{19}$$

is satisfied in the viscosity sense according to the following definition.

Definition 3. *A surface S given be the graph of a continuous function $x_1 = h\left(x'\right)$, defined in an open set $U \subset \Re^{n-1}$, is a weak subsolution (respectively, supersolution) of the equation:*

$$k(S) = g,$$

g continuous on S, if, for $every$ $surface$ S_P, graph of a quadratic polynomial $x_1 = P\left(x'\right)$, and

$$k\left(S_P\right) \leq g \quad (respectively, \ \geq)$$

then $P - h$, cannot we have a local minimum (respectively maximum) in U. S is a weak solution of $k = g$ if it is both a weak-sub- and a supersolution.

Lemma 11. *Let* (u, Ω^+) *be a minimizer in* B_1, *and* f *continuous. Then* Γ^* *is a weak solution of the free boundary equation (19).*

Proof. We show that Γ^* is a weak subsolution (in a similar way one can prove that it is also a supersolution). Assuming the contrary. Let S_P be the graph of a quadratic polinomial touching Γ^* from the Ω^+ side, so that $P - h$ has a minimum at $x_0 \in \Gamma^*$ and $k(S_P) \le g$ where $g(x) = |\nabla u^+|^2 - |\nabla u^-|^2 - f(x)$. By a rotation, translation and rescaling we suppose that:

a) In B_1 the free boundary is given by the graph of a function $x_1 = h(x')$, with $h(0) = 0$, $\nabla h(0) = 0$.
b) By the Hopf maximum principle, at the free bondary, u has a linear behavior

We put

$$H_t = \{P(x') - t < x_1 < P(x') + t\} \qquad \Omega_t^+ = \Omega^+ \cup H_t,$$

$$\Omega_t^- = B_1 - \overline{\Omega}_t^+, \qquad W_t = \Omega^- \cap \{x_1 > P(x') - t\}$$

Let w_t^+ and w_t^- be, respectively, the harmonic extention of u^+ in $\Omega_t^+ = \Omega^+ \cup H_t$ and of u^-in $\Omega_t^- = B_1 - \overline{\Omega}_t^+$:

$$\begin{cases} \Delta w_t^+ = 0 & \text{in} & \Omega_t^+ \\ w_t^+ = u^+ & \text{su} & \partial B_1 \\ w_t^+ = 0 & \text{su} & \partial\Omega_t^+ \cap B_1 \end{cases} \qquad \begin{cases} \Delta w_t^- = 0 & \text{in} & \Omega_t^- \\ w_t^- = u^- & \text{su} & \partial B_1 \\ w_t^- = 0 & \text{su} & \partial\Omega_t^- \cap B_1 \end{cases}$$

Define:

$$u_t = \begin{cases} w_t^+ & \text{in} & \Omega_t^+ \\ -w_t^- & \text{in} & \Omega_t^- \end{cases}$$

The pair $(u_{t,} \Omega_t^+)$ is admissible and:

$$F(u_{t,} \Omega_t^+) \ge F(u_, \Omega^+)$$

We compute the variation of the various terms in the functional F. From [4],

$$\int_{B_1} |\nabla u_t^+|^2 \, dx - \int_{B_1} |\nabla u^+|^2 \, dx \le -\int_{W_t} |\nabla u_t^+|^2 \, dx \tag{20}$$

$$\int_{B_1} |\nabla u^-|^2 \, dx - \int_{B_1} |\nabla u_t^-|^2 \, dx \le \int_{W_t} |\nabla u^-|^2 \, dx + (c(\varepsilon, t) + c(n)\,\varepsilon)\,|W_t| \tag{21}$$

with $c(\delta, t) \to 0$ while $t \to 0$, δ fixed.

Moreover, if d_t denotes the distance from $H_t^- = \{x_1 = P(x') - t\}$
we have

$$\text{(22)} \qquad Per\left(\Omega_t^+\right) - Per\left(\Omega^+\right) \leq -\int_{W_t} \Delta d_t(x)\, dx$$

and finally

$$\text{(23)} \qquad \int_{B_1} f(x)\, \chi_{\{u_*>0\}}\, dx - \int_{B_1} f(x)\, \chi_{\{u>0\}}\, dx = \int_{W_t} f(x)\, dx$$

Collecting (20), (21), (22), (23) from the minimality condition, we have:

$$\int_{W_t} \left|\nabla u_t^+\right|^2 dx - \int_{W_t} \left|\nabla u^-\right|^2 dx \leq$$

$$(c(\delta, t) + c(n)\,\delta)\,|W_t| + \int_{W_t} f(x)\, dx - \int_{W_t} \Delta d_t(x)\, dx.$$

Dividing by $|W_t|$, and letting first $t \to 0$, and $\delta \to 0$, we have:

$$k(S_P)(0) - \left|\nabla u^+(0)\right|^2 + \left|\nabla u^-(0)\right|^2 + f(0) \geq 0.$$

This contradicts the assumption.

6. Analyticity of the free boundary.

We now prove that Γ^* is analytic surface, by using the theory of elliptic coercive systems (see [1]).

We recall briefly the partial hodograph and Legendre transformations. Let $u(x)$ be a function defined in $\Omega \cup \Gamma$ and satisfying on Γ the conditions

$$\partial_n^p u = 0 \qquad \partial_n^{p+1} u \neq 0$$

We suppose that $\partial_n u > 0$ if $p = 0$ and $\partial_n^2 u < 0$ if $p = 1$ (see [7]). The transformation defined by

$$\text{(24)} \qquad \begin{cases} y_\alpha = x_\alpha & \alpha = 1, \ldots, n-1 \\ y_n = u(x) \end{cases}$$

is called a *zeroth order (partial) hodograph transformation*. The associated "partial Legendre transform" (which defines the inverse mapping) is:

$$\begin{cases} x_\alpha = y_\alpha & \alpha = 1, \dots, n-1 \\ x_n = \psi(y) & p = 0 \end{cases} \tag{25}$$

Let us compute the derivatives of u in terms of derivatives of ψ. We have:

$$\psi_\alpha = -\frac{u_\alpha}{u_n} \qquad \psi_n = \frac{1}{u_n} \tag{26}$$

Also, from (26),

$$\frac{\partial}{\partial x_\alpha} = \partial_\alpha - \frac{\psi_\alpha}{\psi_n}\partial_n \qquad \frac{\partial}{\partial x_n} = \frac{1}{\psi_n}\partial_n \tag{27}$$

Moreover, we introduce the *reflection* mapping (from $\Omega \cup \Gamma$ to a neighborhood $\Omega^- \cup \Gamma$ of 0 on the opposite site of Γ)

$$\begin{cases} x_\alpha = y_\alpha & \alpha = 1, \dots, n-1 \\ x_n = \psi(x) - C y_n & p = 0 \end{cases} \tag{28}$$

where C is any constant larger than ψ_n.

For the reflection we have:

$$\frac{\partial}{\partial x_\alpha} = \partial_\alpha - \frac{\psi_\alpha}{\psi_n - C}\partial_n \qquad \frac{\partial}{\partial x_n} = \frac{1}{\psi_n - C}\partial_n \tag{29}$$

Note that given a function $u(x)$ defined in Ω^-, we can pull it back to a function $\phi(y)$ defined in Ω by the rule $\phi(y) = u(x)$, where x and y are related by (28).

The proof of part *b) in theorem 4*, follows from the following lemma where Ω, Γ and Ω^- are as above.

Lemma 12. *Let $\Gamma = \partial\Omega \cap B_1$ be an $(n-1)-$ dimensional C^∞ manifold, with 0 $\in \Gamma$. Suppose f analytic on Γ and $u \in C^2(\Omega \cup \Gamma) \cap C^2(\Omega^- \cup \Gamma)$ satisfies:*

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \cup \Omega^- \\ u = 0 & \text{on} \quad \Gamma \\ \left|\nabla u^+\right|^2 - \left|\nabla u^-\right|^2 - f(x) = k(\Gamma^*) & \text{on} \quad \Gamma \end{cases} \tag{30}$$

Then Γ is analyitc.

Proof. Assume e_n is the normal unit vector to Γ at 0. We apply our zeroth order hodograph transform (24) - (25):

$$y = (x_1, x_2, \ldots, x_{n-1}, u) = \left(x', u\right)$$

$$u_n(0) > 0$$

$$x_n = \psi$$

Since $x_n = \psi(y_1, ...y_{n-1}, 0)$ parametrizes Γ, we have:

$$v = \frac{(-\psi_1, ..., -\psi_{n-1}, 1)}{\sqrt{1 + \sum\limits_{\alpha<n} \psi_\alpha^2}} \quad u_v^+ = \frac{1}{\psi_n}\sqrt{1 + \sum_{\alpha<n} \psi_\alpha^2}$$

$$u_v^- = \frac{\phi_n}{(\psi_n - C)}\sqrt{1 + \sum_{\alpha<n} \psi_\alpha^2}.$$

The mean curvature is

$$k = \frac{1}{n-1}\sum \frac{\left(\left(1 + \sum\limits_{\alpha<n} \psi_\alpha^2\right)\delta_{\alpha\beta} - \psi_\alpha\psi_\beta\right)\psi_{\alpha\beta}}{\left(1 + \sum\limits_{\alpha<n} \psi_\alpha^2\right)^{\frac{3}{2}}}$$

therefore:

$$\left(u_v^+\right)^2 - \left(u_v^-\right)^2 = \left(\frac{1}{\psi_n^2} - \frac{\phi_n^2}{(\psi_n - C)^2}\right)\left(1 + \sum_{\alpha<n} \psi_\alpha^2\right)$$

From (24), the sistem (30) becomes:

$$\begin{cases} -\frac{1}{\psi_n^3}\left(1 + \sum\limits_{\alpha<n} \psi_\alpha^2\right)\psi_{nn} - \frac{1}{\psi_n}\sum\limits_{\alpha<n} \psi_{\alpha\alpha} + \frac{2}{\psi_n^2}\sum\limits_{\alpha<n} \psi_\alpha\psi_{\alpha n} = 0 \ \text{ in } U^+ = y\left(\Omega^+\right) \\ \frac{1}{\psi_n - C}\left(\frac{\phi_n}{\psi_n - C}\right)_n + \sum\limits_{\alpha<n}\left(\left(\phi_\alpha - \frac{\psi_\alpha\phi_n}{\psi_n - C}\right)_\alpha - \frac{\psi_\alpha}{\psi_n - C}\left(\phi_\alpha - \frac{\psi_\alpha\phi_n}{\psi_n - C}\right)_n\right) = 0 \\ \hfill \text{in } U^- = y\left(\Omega^-\right) \\ \left(\frac{1}{\psi_n^2} - \frac{\phi_n^2}{(\psi_n - C)^2}\right)\left(1 + \sum\limits_{\alpha<n} \psi_\alpha^2\right) - \widetilde{f}\left(y', \psi\right) = \frac{1}{n-1}\sum \frac{\left(\left(1 + \sum\limits_{\alpha<n} \psi_\alpha^2\right)\delta_{\alpha\beta} - \psi_\alpha\psi_\beta\right)\psi_{\alpha\beta}}{\left(1 + \sum\limits_{\alpha<n} \psi_\alpha^2\right)^{\frac{3}{2}}} \\ \hfill \text{on } S = y(\Gamma) \end{cases}$$

We linearize the problem, with respect to the variable ψ. Note that by our choice of coordinates, we have

$$\psi_n(0) = \frac{1}{u_n(0)} > 0 \qquad \psi_\alpha(0) = 0 \qquad \alpha < n$$

and putting:

$$\beta = u_v^+(0) = \frac{1}{\psi_n(0)} \qquad \gamma = u_v^-(0) = \frac{\phi_n(0)}{\psi_n(0) - C}$$

we obtain

$$\begin{cases} \beta^2 \overline{\psi}_{nn} + \sum \overline{\psi}_{\alpha\alpha} = 0 & \text{in} \quad U^+ \\ \frac{1}{A^2}\overline{\phi}_{nn} + \sum \overline{\phi}_{\alpha\alpha} - \gamma\left(\frac{1}{A^2}\overline{\psi}_{nn} + \sum \overline{\psi}_{\alpha\alpha}\right) = 0 & \text{in} \quad U^- \\ \sum_{\alpha<n} \overline{\psi}_{\alpha\alpha} = 0 \qquad \overline{\phi} = 0 & \text{on} \quad S \end{cases}$$

where $\overline{\psi}$ and $\overline{\phi}$ are, respectively, the increments of ψ and ϕ .

The sistem is elliptic and coercive (see [7]) and the boundary conditions are equivalent to $\overline{\psi} = 0$ and $\overline{\phi} = 0$. Hence Γ is analytic. □

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