LE MATEMATICHE
Vol. LXXIV (2019) – Issue II, pp. 301–320
doi: 10.4418/2019.74.2.6

ON CLASSICAL n-ABSORBING SUBMODULES

R. NIKANDISH - M. J. NIKMEHR - A. YASSINE

In this paper, we introduce the notion of classical n-absorbing submodules of a module M over a commutative ring R with identity, which is a generalization of classical prime submodules. A proper submodule N of M is said to be classical n-absorbing if whenever $a_1a_2\cdots a_{n+1}m \in N$ for $a_1,a_2,\ldots,a_{n+1} \in R$ and $m \in M$, then there are n of the a_i's whose product with m is in N. We give some basic results concerning classical n-absorbing submodules. Then the classical n-absorbing avoidance theorem for submodules is proved. Finally, classical n-absorbing submodules in several classes of modules are studied.

1. Introduction

We assume throughout this paper that all rings are commutative with identity. The notion of prime ideals has been generalized and studied in several directions. For instance, in 2007, Badawi introduced the notion of 2-absorbing ideals [7]. Let R be a ring. A nonzero proper ideal I of R is called a 2-absorbing ideal if whenever $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Later, in 2011, Anderson and Badawi generalized this concept to n-absorbing ideals for some integer n [4], that is an ideal I of R is said to be n-absorbing ideal if whenever $a_1a_2\cdots a_{n+1} \in I$ for $a_1,a_2,\ldots,a_{n+1} \in R$, then there are n of the a_i's whose product is in N. In [11], n-absorbing ideals were extended by Yousefian

Received on May 14, 2019

AMS 2010 Subject Classification: 13A15, 13C99, 13F05
Keywords: Classical n-absorbing submodule, n-absorbing ideal, Classical prime submodule.
Corresponding author: Reza Nikandish

Darani and Soheilnia to n-absorbing submodules and studied later by Dubey and Aggarwal in [12]. Let M be an R-module and N a proper submodule of M. Then N is called n-absorbing submodule of M if whenever $a_1a_2\cdots a_nm \in N$ for $a_1,a_2,\ldots,a_n \in R$ and $m \in M$, then there are $n-1$ of the a_i's whose product with m is in N or $a_1a_2\cdots a_n \in (N :_R M) = \{a \in R : aM \subseteq N\}$. A proper submodule N of M is said to be a prime (p-primary) submodule, if whenever $am \in N$ for $a \in R$ and $m \in M$, then either $m \in N$ or $a \in (N :_R M)$ ($a \in \sqrt{(N :_R M)} = p$). Classical prime submodule which is a generalization of prime submodule was studied by Behboodi in [8, 9]. A proper submodule N of M is called a classical prime submodule, if whenever $abm \in N$ for $a,b \in R$ and $m \in M$, then either $am \in N$ or $bm \in N$.

In this paper, we extend the notion of n-absorbing ideal to classical n-absorbing submodule which is a generalization of classical prime submodule of an R-module M. A proper submodule N of M is said to be classical n-absorbing if whenever $a_1a_2\cdots a_{n+1}m \in N$ for $a_1,a_2,\ldots,a_{n+1} \in R$ and $m \in M$, then there are n of the a_i's whose product with m is in N. We will transfer some results parallel to n-absorbing ideals in commutative ring introduced in [4].

In Section 2, we introduce the notion of classical n-absorbing submodules (see Definition 2.1) and we give some of their basic properties. For example, in Theorem 2.6, we characterize classical n-absorbing submodules in um-rings. In Theorem 2.12, it is shown that if M is Noetherian, then M contains a finite number of minimal classical n-absorbing submodules. In Theorem 2.14 classical n-absorbing submodules of a finite direct product of modules are studied. In Section 3, we continue the study of properties of classical n-absorbing submodules. We show that in Theorem 3.1 if N is a classical n-absorbing submodule of M, then $(N :_R M)$ should be n-absorbing ideal of R, and then we prove by using the technique of efficient covering of submodules the classical n-absorbing avoidance theorem and an application of it (Theorem 3.6) is given. In the final section, we study classical n-absorbing submodules in various classes of modules over commutative rings. Most of the results discussed in [4] which are related to n-absorbing ideals have an extension to modules, we develop them in this section for classical n-absorbing submodules. For instance, in Lemma 4.1 it is shown that if I is an n-absorbing ideal of R such that R is a valuation ring, and M is a faithful multiplication R-module, then IM is a classical n-absorbing submodule of M. If M is a finitely generated faithful multiplication module over the valuation Noetherian integral domain R and N a submodule of M, then M is a Dedekind module if and only if for every classical n-absorbing submodule N of M, we have $N = N_1\cdots N_m$ where N_i's are maximal submodules of M and $1 \leq m \leq n$ (Theorem 4.2). We close this paper by discussing on the relationship between primary submodules and classical n-absorbing submodules (Theorem

4.5).

Now we define some concepts that will be used in this paper. Let M be an R-module, N a submodule of M, S the set of non-zero divisors of R and let $T = \{t \in S : tm = 0 \text{ for some } m \in M \text{ implies } m = 0\}$ be a multiplicatively closed subset of S. Then M is said to be *multiplication* module if for every submodule N of M there exists an ideal I of R such that $N = IM$. We can see by [1] that $N = (N :_R M)M$. The product of two submodules $N = IM$ and $K = JM$ of a multiplication module M denoted by NK is defined by $NK = IJM$ where I, J are ideals of R. The submodule N is called *invertible* if $N^{-1}N = M$ where $N^{-1} = \{x \in R_T : xN \subseteq M\}$. Dedekind modules were introduced by Naoum and Al-Alwan in [16]. The module M is called *Dedekind*, if every non-zero submodule of M is invertible, but if every non-zero finitely generated submodule of M is invertible, then M is called *Prüfer module*. An R module M is said to be *valuation* if for any submodules N and K of M, either $N \subseteq K$ or $K \subseteq N$ [15]. An R module M is said to be *Bézout* if every finitely generated submodule of M is a principal submodule [2]. If P is a maximal submodule of M, then M is called *P-cyclic* if there exists $x \in P$ and $m \in M$ such that $(1-x)M \subseteq Rm$ [1]. For any undefined notation or terminology in commutative ring theory, we refer the reader to [4, 17, 18].

2. Classical n-Absorbing Submodules

In this section, the notion of classical n-absorbing submodules is introduced and some of their basic properties are given.

Definition 2.1. Let M be an R-module and N be a proper submodule of M. Then N is called classical n-absorbing submodule if whenever $a_1a_2\cdots a_{n+1}m \in N$ for $a_1, a_2, \ldots, a_{n+1} \in R$ and $m \in M$, then there are n of the a_i's whose product with m is in N.

It is easy to see that every n-absorbing submodule is classical n-absorbing but the converse need not be true. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p_1} \bigoplus \mathbb{Z}_{p_2} \bigoplus \cdots \bigoplus \mathbb{Z}_{p_n} \bigoplus \mathbb{Q}$ where p_i's are distinct primes and take the zero submodule N of M. It is easy to see that N is classical n-absorbing submodule of M, but since $p_1p_2\cdots p_n(1,1,\ldots,1,0) \in N$ and if $\widehat{p_i} = p_1p_2\cdots p_{i-1}p_{i+1}\ldots p_n$ for every $i \in \{1,\ldots,n\}$, then we can easily see that $\widehat{p_i}(1,1,\ldots,1,0) \notin N$ for every $i \in \{1,\ldots,n\}$ and $p_1p_2\ldots p_n(1,1,\ldots,1) \notin N$. Thus N is not an n-absorbing submodule of M. We can see that there is a module which has no classical n-absorbing submodule. Consider the $\mathbb{Z}$-module $M = \mathbb{Q}/\mathbb{Z}$ and take a submodule $E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = \frac{r}{p^n} + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \text{ and } n \in \mathbb{N}_0\}$ of M where p is a fixed prime number (see [17, Example 7.10]). Each proper submodule of $E(p)$ is equal to $G_t = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = \frac{r}{p^t} + \mathbb{Z}$ for some $r \in \mathbb{Z}\}$ for some $t \in \mathbb{N}_0$. It is easy to see that G_t is not classical n-absorbing submodule of M for each $t \in \mathbb{N}_0$, since $p^{n+1}(\frac{1}{p^{t+n+1}} + \mathbb{Z}) = \frac{1}{p^t} + \mathbb{Z} \in G_t$ and $p^n(\frac{1}{p^{t+n+1}} + \mathbb{Z}) = \frac{1}{p^{t+1}} + \mathbb{Z} \notin G_t$, hence M has no classical n-absorbing submodule.

Proposition 2.2. *Let N be a submodule of the R-module M and let $N_r = (N :_M r) = \{m \in M : rm \in N\}$, for every $r \in R$. Then the following statements hold.*

(1) *If M is a cyclic R-module, then N is a classical n-absorbing submodule of M if and only if $(N :_R M)$ is an n-absorbing ideal of R.*

(2) *If N is a classical n-absorbing submodule of M, then N_r is a classical n-absorbing submodule of M containing N for all $r \in R \setminus (N :_R M)$.*

Proof. **(1)** $\Rightarrow$) Let M be a cyclic multiplication R-module and N be a classical n-absorbing submodule of M. Then $M = Rm$ for some $m \in M$. Let $a_1, \ldots, a_{n+1} \in R$ such that $a_1 \cdots a_{n+1} \in (N :_R M)$ and suppose that $\widehat{a_i} = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{n+1}$ for each $i \in \{1, \ldots, n+1\}$. If $\widehat{a_i} a_{n+1} m \notin N$ for each $i \in \{1, \ldots, n\}$, then $a_1 \cdots a_n m \in N$, since N is a classical n-absorbing submodule of M. Therefore $(N :_R M)$ is an n-absorbing ideal of R.

$\Leftarrow$) Let $(N :_R M)$ be an n-absorbing ideal of R and $a_1, \ldots, a_{n+1} \in R$ be such that $a_1 \cdots a_{n+1} x \in N$ for some $x \in M$. Since $M = Rm$, we deduce that $x = rm$, for some $r \in R$. Therefore $a_1 \cdots a_{n+1} rm \in N$. This implies that $a_1 \cdots a_n (a_{n+1} r) \in (N :_R M)$. Since $(N :_R M)$ is n-absorbing ideal, either $a_1 \cdots a_n \in (N :_R M)$ or $\widehat{a_i} a_{n+1} r \in (N :_R M)$ for some $i \in \{1, \ldots, n\}$, which implies that either $a_1 \cdots a_n x \in N$ or $\widehat{a_i} a_{n+1} x \in N$, for some $i \in \{1, \ldots, n\}$. Therefore N is an n-absorbing submodule.

(2) Let $a_1, \ldots, a_{n+1} \in R$ be such that $a_1 \cdots a_{n+1} m \in (N :_M r)$ for some $m \in M$ and $r \in R \setminus (N :_R M)$, and suppose that $\widehat{a_i} = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{n+1}$ for each $i \in \{1, \ldots, n+1\}$. Then $a_1 \cdots a_{n+1}(rm) \in N$. Since N is a classical n-absorbing submodule, we conclude that $\widehat{a_i}(rm) \in N$, for some $i \in \{1, \ldots, n+1\}$, and so $\widehat{a_i} m \in N_r$. Thus N_r is a classical n-absorbing submodule of M. $\square$

We show that if $(N :_R M)$ is an n-absorbing ideal of R, then N need not be a classical n-absorbing submodule of M. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$ and the submodule $N = \{0\} \times 24\mathbb{Z}$ of M. It is easily seen that $(N :_R M) = \{0\}$ is a 3-absorbing ideal of $\mathbb{Z}$, but N is not a classical 3-absorbing submodule, as $2 \cdot 2 \cdot 2 \cdot 3 \cdot (0,1) \in N$ but $8 \cdot (0,1) \notin N$ and $12 \cdot (0,1) \notin N$.
The next result investigates contraction and extension of classical n-absorbing submodule under an R-homomorphism.

Proposition 2.3. *Let M and K be R-modules, N a submodule of M and $f : M \to K$ an R-homomorphism. Then the following statements hold.*

(1) *If N is a classical n-absorbing submodule of K, then $f^{-1}(N)$ is a classical n-absorbing submodule of M.*

(2) *Let L be a submodule of M and f be an epimorphism such that $ker(f) \subseteq L$. If L is classical n-absorbing, then $f(L)$ is a classical n-absorbing submodule of K.*

Proof. **(1)** Let $a_1, \ldots, a_{n+1} \in R$ and $\widehat{a_i} = a_1a_2 \cdots a_{i-1}a_{i+1} \cdots a_{n+1}$ for every $i \in \{1, \ldots, n+1\}$ and let $m \in M$ such that $a_1 \cdots a_{n+1}m \in f^{-1}(N)$. Since f is R-homomorphism, we have $f(a_1 \ldots a_{n+1}m) = a_1 \ldots a_{n+1}f(m) \in N$. But N is a classical n-absorbing submodule of K and so $\widehat{a_j}f(m) \in N$, for some $j \in \{1, \ldots, n+1\}$. Thus $\widehat{a_j}m \in f^{-1}(N)$, for some $j \in \{1, \ldots, n+1\}$. Hence $f^{-1}(N)$ is a classical n-absorbing submodule of M.

(2) Let L be a classical classical n-absorbing submodule of M and $a_1 \ldots a_{n+1}k \in f(L)$ for some $a_1, \ldots, a_{n+1} \in R$ and $k \in K$. Then there exists $l \in L$ such that $a_1 \ldots a_{n+1}k = f(l)$, and $m \in M$ such that $f(m) = k$, as f is an epimorphism and $k \in K$. This implies that $f(a_1 \ldots a_{n+1}m) = f(l)$, and so $a_1 \ldots a_{n+1}m - l \in ker(f) \subseteq L$. Since L is classical n-absorbing, we conclude that $\widehat{a_j}m \in L$, for some $j \in \{1, \ldots, n+1\}$. Therefore $\widehat{a_j}f(m) = \widehat{a_j}k \in f(L)$, for some $j \in \{1, \ldots, n+1\}$. Thus $f(L)$ is a classical n-absorbing submodule of K. □

Proposition 2.4. *Let N be a submodule of the R-module M and S be a multiplicatively closed subset of R. If N is a classical n-absorbing submodule of M, then $S^{-1}N$ is a classical n-absorbing submodule of $S^{-1}M$.*

Proof. Let $\frac{a_1}{s_1} \cdots \frac{a_{n+1}}{s_{n+1}} \frac{m}{s} \in S^{-1}N$ such that $\frac{a_i}{s_i} \in S^{-1}R$ for every $i \in \{1, \ldots, n+1\}$ and $\frac{m}{s} \in S^{-1}M$. Then there exists $a \in N$ and $x \in S$ such that $\frac{a_1}{s_1} \cdots \frac{a_{n+1}}{s_{n+1}} \frac{m}{s} = \frac{a}{x}$, and so $a_1 \ldots a_{n+1}xym = s_1 \ldots s_{n+1}sya \in N$ for some $y \in S$. Since N is classical n-absorbing, $\widehat{a_j}xym = a_1a_2 \cdots a_{j-1}a_{j+1} \ldots a_{n+1}xym \in N$ for some $j \in \{1, \ldots, n+1\}$. Therefore $\frac{a_1}{s_1} \cdots \frac{a_{j-1}}{s_{j-1}} \frac{a_{j+1}}{s_{j+1}} \cdots \frac{a_{n+1}}{s_{n+1}} \frac{m}{s} = (\frac{a_1}{s_1} \cdots \frac{a_{j-1}}{s_{j-1}} \frac{a_{j+1}}{s_{j+1}} \cdots \frac{a_{n+1}}{s_{n+1}} \frac{x}{x} \frac{y}{y}) \frac{m}{s} \in S^{-1}N$. Thus $S^{-1}N$ is a classical n-absorbing submodule of $S^{-1}M$. □

Theorem 2.5. *Let N be a proper submodule of the R-module M. Then the following statements are equivalent:*

(1) *N is a classical n-absorbing submodule of M.*

(2) $(N :_M a_1 \cdots a_{n+1}) = (N :_M \widehat{a_1}) \cup (N :_M \widehat{a_2}) \cup \cdots \cup (N :_M \widehat{a_{n+1}})$ *whenever $a_1, \ldots, a_{n+1} \in R$ such that $\widehat{a_i} = a_1a_2 \cdots a_{i-1}a_{i+1} \cdots a_{n+1}$ for each $i \in \{1, \ldots, n+1\}$.*

(3) $(N :_R a_1a_2 \cdots a_nm) = (N :_R \widehat{a_1}m) \cup (N :_R \widehat{a_2}m) \cup \cdots \cup (N :_R \widehat{a_n}m)$ *whenever $a_1, \ldots, a_n \in R$ and $m \in M$ such that $a_1a_2 \cdots a_nm \notin N$.*

Proof. (1) ⇒ (2) Assume that N is a classical n-absorbing submodule of M. Let $a_1, \ldots, a_{n+1} \in R$ such that $\widehat{a_i} = a_1a_2 \cdots a_{i-1}a_{i+1} \cdots a_{n+1}$ for each $i \in \{1, \ldots, n+1\}$ and let $m \in (N :_M a_1 \cdots a_{n+1})$. Then $a_1 \cdots a_{n+1}m \in N$. Since N is a classical n-absorbing submodule, we get $\widehat{a_i}m \in N$ for some $i \in \{1, \ldots, n+1\}$. This means that $m \in (N :_M \widehat{a_1}) \cup (N :_M \widehat{a_2}) \cup \cdots \cup (N :_M \widehat{a_{n+1}})$. Thus $(N :_M a_1 \cdots a_{n+1}) = (N :_M \widehat{a_1}) \cup (N :_M \widehat{a_2}) \cup \cdots \cup (N :_M \widehat{a_{n+1}})$.

(2) $\Rightarrow$ (3) Let $x \in (N :_M a_1 \cdots a_n m)$ for some $x \in R$ and $m \in M$. Then we have $a_1 \cdots a_n x m \in N$. Since N is a classical n-absorbing submodule and $a_1 a_2 \cdots a_n m \notin N$, we infer $\widehat{a_i} x m \in N$ for some $i \in \{1, \ldots, n\}$ where $\widehat{a_i} = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n$. This means that $x \in (N :_M \widehat{a_1} m) \cup (N :_M \widehat{a_2} m) \cup \cdots \cup (N :_M \widehat{a_n} m)$. Thus $(N :_R a_1 a_2 \cdots a_n m) = (N :_R \widehat{a_1} m) \cup (N :_R \widehat{a_2} m) \cup \cdots \cup (N :_R \widehat{a_n} m)$.

(3) $\Rightarrow$ (1) It is clear. □

um-ring was defined in [10] by Quartararo and Butts. A *um*-ring is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them.

Theorem 2.6. *Let R be a um-ring, M an R-module and N be a proper submodule of M. Then the following statements are equivalent:*

(1) *N is a classical n-absorbing submodule of M.*

(2) *$(N :_M a_1 \cdots a_{n+1}) = (N :_M \widehat{a_i})$, for some $i \in \{1, \ldots, n+1\}$ whenever $a_1, \ldots, a_{n+1} \in R$ such that $\widehat{a_j} = a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_{n+1}$ for each $j \in \{1, \ldots, n+1\}$.*

(3) *$(N :_R a_1 a_2 \cdots a_n m) = (N :_R \widehat{a_i} m)$, for some $i \in \{1, \ldots, n\}$ whenever $a_1, \ldots, a_n \in R$ and $m \in M$ such that $a_1 a_2 \cdots a_n m \notin N$.*

(4) *For every $a_1, \ldots, a_n \in R$ and $m \in M$ and every ideal I of R such that $a_1 \cdots a_n I m \subseteq N$, either $a_1 \cdots a_n m \in N$ or $\widehat{a_i} I m \subseteq N$, for some $i \in \{1, \ldots, n\}$.*

(5) *$(N :_R a_1 \cdots a_{n-1} I m) = (N :_R a_1 \cdots a_{n-1} m)$ or $(N :_R a_1 \cdots a_{n-1} I m) = (N :_R \widehat{a_i} I m)$ for some $i \in \{1, \ldots, n-1\}$ whenever $a_1, \ldots, a_{n-1} \in R$ and $m \in M$ such that $a_1 a_2 \cdots a_{n-1} I m \nsubseteq N$.*

(6) *For every $a \in R$ and every ideals $I_1, I_2, \ldots, I_n$ of R and $m \in M$ such that $a I_1 I_2 \cdots I_n m \subseteq N$, we have either $I_1 I_2 \cdots I_n m \subseteq N$ or $n-1$ of the I_i's whose product with am is in N.*

(7) *For every ideals $I_1, I_2, \ldots, I_n$ of R and $m \in M$ such that $I_1 I_2 \cdots I_n m \nsubseteq N$, we have $(N :_R I_1 \cdots I_n m) = (N :_R \widehat{I_i} m)$, for some $i \in \{1, \ldots, n\}$ such that $\widehat{I_j} = I_1 I_2 \cdots I_{j-1} I_{j+1} \cdots I_n$ for each $j \in \{1, \ldots, n\}$.*

(8) *For every ideals $J, I_1, I_2, \ldots, I_n$ of R and $m \in M$ such that $I_1 I_2 \cdots I_n J m \subseteq N$, either $n-1$ of the I_i's whose product with Jm is in N or $I_1 I_2 \cdots I_n m \subseteq N$.*

(9) *For every $m \in M \setminus N$, $(N :_R m)$ is a strongly n-absorbing ideal of R.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) The results follow from Theorem 2.5, as R is a *um*-ring.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) The proofs are similar to that of the Theorem 2.5.

(8) $\Rightarrow$ (1) It is obvious.

(8) $\Leftrightarrow$ (9) It follows from [4, p. 1668]. □

Corollary 2.7. *Let I be a proper ideal of the um-ring R. Then the following statements hold:*

(1) *By taking the ring R as an R-module, the submodule $_RI$ of R is a classical n-absorbing submodule if and only if I is a strongly n-absorbing ideal of R.*

(2) *For every R-module M, every proper submodule N of M is classical n-absorbing if and only if every proper ideal of R is strongly n-absorbing ideal.*

(3) *Let M be an R-module and N a classical n-absorbing submodule of M. Suppose that $m \in M \setminus N$. Then for every $x \in \sqrt{(N :_R m)}$ such that $x^{n-1} \notin (N :_R m)$, $(N :_R x^{n-1}m)$ is a prime ideal of R containing every minimal prime ideal over $\sqrt{(N :_R m)}$. Furthermore, if $(N :_R m)$ is a P-primary ideal and n is the least positive integer such that $x^n \in (N :_R m)$, then $(N :_R x^{n-1}m) = P$.*

Proof. **(1)** First, suppose that $_RI$ is a classical n-absorbing submodule of R. It is easy to see that $(_RI :_R 1) = {_RI}$. Hence by Theorem 2.6, I is a strongly n-absorbing ideal of R. Conversely, $_RI$ is a classical n-absorbing submodule of R, since every strongly n-absorbing ideal of R is also an n-absorbing ideal of R and it is easy to see that every n-absorbing submodule is a classical n-absorbing submodule of R.

(2) By taking the ring R as an R-module, by part (1), every proper ideal of R is strongly n-absorbing. Conversely, assume that every proper ideal of R is strongly n-absorbing and let N be a proper submodule of an R-module M. Hence by Theorem 2.6, N is a classical n-absorbing submodule, as $(N :_R m)$ is a proper ideal of R, for every $m \in M$. Thus it is a strongly n-absorbing ideal of R.

(3) It follows from Theorem 2.6, [4, Corollary 3.6] and [4, Corollary 3.7]. □

Recall that an R-module M is said to be multiplication module if every submodule N of M has the form IM for some ideal I of R (see [1]). In the following we characterize modules that their classical n-absorbing submodules are n-absorbing.

Corollary 2.8. *Let M be a cyclic multiplication R-module such that R is a um-ring and let N be a submodule of M. Then the following conditions are equivalent:*

(1) *N is classical n-absorbing.*

(2) *For every submodules $N_1, N_2, \ldots, N_{n+2}$ of M such that $N_1N_2\cdots N_{n+2} \subseteq N$, we have n of the $N_1, \ldots, N_{n+1}$ whose product with N_{n+2} is in N.*

(3) *For every submodules $N_1, N_2, \ldots, N_{n+1}$ of M such that $N_1N_2\cdots N_{n+1} \subseteq N$, we have n of N_i's whose product is in N.*

(4) *N is n-absorbing.*

(5) *$(N :_R M)$ is an n-absorbing ideal of R.*

Proof. (1) $\Rightarrow$ (2) This follows directly from Theorem 2.6.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (4) Suppose that $M = \langle m \rangle$, for some $m \in M$. Let I_1, I_2, I_n be ideals of R such that $I_1 I_2 \cdots I_n m \in N$. Set $N_i := I_i M$, for every $i \in \{1, \ldots, n\}$. Now, the result follows from Part (3).

(4) $\Leftrightarrow$ (5) By [11, Proposition 2].

(4) $\Rightarrow$ (1) It is straightforward. □

Theorem 2.9. *Let N be a submodule of the R-module M such that R is a um-ring. Then the following statements hold:*

(1) *If N is a classical n-absorbing submodule of M and F is a flat R-module such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a classical n-absorbing submodule of $F \otimes M$.*

(2) *Let F be a faithfully flat R-module. Then N is a classical n-absorbing submodule of M if and only if $F \otimes N$ is a classical n-absorbing submodule of $F \otimes M$.*

Proof. (1) Let N be a classical n-absorbing submodule of M and $a_1, a_2, \ldots, a_{n+1} \in R$. By Theorem 2.6, $(N :_M a_1 \cdots a_{n+1}) = (N :_M \widehat{a_i})$, for some $i \in \{1, \ldots, n+1\}$ such that $\widehat{a_j} = a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_{n+1}$ for each $j \in \{1, \ldots, n+1\}$. Therefore by [5, Lemma 3.2], $(F \otimes N :_{F\otimes M} a_1 \cdots a_{n+1}) = F \otimes (N :_M a_1 \cdots a_{n+1}) = F \otimes (N :_M \widehat{a_i}) = (F \otimes N :_{F\otimes M} \widehat{a_i})$, and so, by Theorem 2.6, $F \otimes N$ is a classical n-absorbing submodule of $F \otimes M$.

(2) Suppose that N is a classical n-absorbing submodule of M. If $F \otimes N = F \otimes M$, then we have the exact sequence $0 \longrightarrow F \otimes N \xrightarrow{\subseteq} F \otimes M \longrightarrow 0$, and so $0 \longrightarrow N \xrightarrow{\subseteq} M \longrightarrow 0$ is exact, since F is a faithfully flat module. Hence $N = M$, a contradiction. This implies that $F \otimes N \neq F \otimes M$. Consequently, $F \otimes N$ is a classical n-absorbing submodule of M, by Part (1). Now let $F \otimes N$ be a classical n-absorbing submodule of M. Then $F \otimes N \neq F \otimes M$ and so $N \neq M$. We show that N is a classical n-absorbing submodule. Let $a_1, a_2, \ldots, a_{n+1} \in R$. Since $F \otimes N$ is a classical n-absorbing submodule, by Theorem 2.6, $(F \otimes N : a_1 \cdots a_{n+1}) = (F \otimes N : \widehat{a_i})$, for some $i \in \{1, \ldots, n+1\}$ where $\widehat{a_i}$ as above. Therefore by [5, Lemma 3.2], $F \otimes (N :_M a_1 \cdots a_{n+1}) = (F \otimes N :_{F\otimes M} a_1 \cdots a_{n+1}) = (F \otimes N :_{F\otimes M} \widehat{a_i}) = F \otimes (N :_M \widehat{a_i})$. Hence the sequence $0 \longrightarrow F \otimes (N :_M a_1 \cdots a_{n+1}) \xrightarrow{\subseteq} F \otimes (N :_M \widehat{a_i}) \longrightarrow 0$ is exact. Since F is a faithfully flat module, $0 \longrightarrow (N : ab) \xrightarrow{\subseteq} (N : \widehat{a_i}) \longrightarrow 0$ is also an exact sequence which implies that $(N :_M \widehat{a_i}) = (N :_M a_1 \cdots a_{n+1})$ for some $i \in \{1, \ldots, n+1\}$. Thus by Theorem 2.6, N is a classical n-absorbing submodule of M. □

In light of Theorem 2.9, we state the following corollary.

Corollary 2.10. *Let R be a um-ring, M an R-module and let X be an indeterminate. If N is a classical n-absorbing submodule of M, then $N[X]$ is a classical n-absorbing submodule of $M[X]$.*

Proof. Let N be a classical n-absorbing submodule of M. Then by Theorem 2.9, $N[X] \simeq R[X] \otimes N$ is a classical n-absorbing submodule of $M[X] \simeq R[X] \otimes M$, as $R[X]$ is a flat R-module. □

For a classical n-absorbing submodule N of an R-module M, we say that N is minimal, if for any classical n-absorbing submodule K of M such that $K \subseteq N$, we have $K = N$.

Proposition 2.11. *Let $\{N_i : i \in I\}$ be a chain of classical n-absorbing submodules of the R-module M. Then $\bigcap_{i \in I} N_i$ is a classical n-absorbing submodule of M. In particular, every classical n-absorbing submodule N of M contains a minimal classical n-absorbing submodule of M.*

Proof. Let $a_1, \dots, a_{n+1} \in R$ and $\widehat{a_i} = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n$ for every $i \in \{1, \dots, n\}$ and let $m \in M$ such that $a_1 \cdots a_{n+1} m \in \bigcap_{i \in I} N_i$. Suppose that $\widehat{a_i} a_{n+1} m \notin \bigcap_{i \in I} N_i$ for every $i \in \{1, \dots, n\}$. Therefore, one may assume that $\widehat{a_i} a_{n+1} m \notin N_i$, for every $i \in \{1, \dots, n\}$, and so $\widehat{a_i} a_{n+1} m \notin N_j$, for every submodule $N_j \subseteq N_i$ and for each $i \in \{1, \dots, n\}$. Hence, for every submodule N_t such that $N_t \subseteq \bigcap_{i=1}^{n} N_i$ we get $a_1 \cdots a_n m \in N_t$. Thus $a_1 \cdots a_n m \in \bigcap_{i \in I} N_i$. For the"in particular" statement, let N be a classical n-absorbing submodule of M and let

$$\Theta = \{L : L \text{ is a classical } n\text{-absorbing submodule of } M \text{ and } L \subseteq N\}.$$

We show that N contains a minimal classical n-absorbing submodule of M. Let $\{N_i : i \in I\}$ be a chain in Θ. It therefore follows from the above discussion that $\bigcap_{i \in I} N_i$ is a classical n-absorbing submodule of M that is contained in N, and so it is in Θ. Hence, by Zorn's Lemma, Θ has at least one minimal element which is clearly a minimal classical n-absorbing submodule of M. Thus, N contains a minimal classical n-absorbing submodule of M. Therefore, every classical n-absorbing submodule of M contains a minimal classical n-absorbing submodule of M. □

Next we show that every Noetherian module contains finitely many minimal classical n-absorbing submodules.

Theorem 2.12. *Let M be a Noetherian R-module. Then M contains a finite number of minimal classical n-absorbing submodules.*

Proof. Let M be a Noetherian R-module containing infinitely many minimal classical n-absorbing submodules. Let

$$\Theta = \{N : N \text{ is a submodule of } M \text{ such that the module } M/N \text{ has an infinite number of minimal classical } n\text{-absorbing submodules}\}.$$

Clearly, $\Theta \neq \emptyset$ ($0 \in \Theta$). It follows from the maximal condition that Θ has a maximal member K with respect to inclusion, as M is a Noetherian R-module. If K is n-absorbing, then $0_{M/K}$ is n-absorbing and so M/K has a finite number of minimal classical n-absorbing submodules, a contradiction. Hence, K is not a classical n-absorbing submodule. Thus, there exist ideals $I_1, I_2, \ldots, I_{n+1}$ in R and $m \in M$ such that $I_1 I_2 \cdots I_{n+1} m \subseteq K$, and $\widehat{I_i} m \nsubseteq K$) for every $i \in \{1, \ldots, n+1\}$ such that $\widehat{I_j} = I_1 I_2 \cdots I_{j-1} I_{j+1} \cdots I_{n+1}$ for each $j \in \{1, \ldots, n+1\}$. Since K is maximal in Θ, $K + \widehat{I_i} m \notin \Theta$, and so $M/(K + \widehat{I_i} m)$ has a finitely minimal classical n-absorbing submodules, for every $i \in \{1, \ldots, n+1\}$. If L/K is a minimal classical n-absorbing submodule of M/K, then $I_1 I_2 \cdots I_{n+1} m \subseteq K \subseteq L$. This implies that $\widehat{I_i} m \subseteq L$ for some $i \in \{1, \ldots, n+1\}$. Therefore $L/(K + \widehat{I_i} m)$ is a minimal classical n-absorbing submodule of $M/(K + \widehat{I_i} m)$ for some $i \in \{1, \ldots, n+1\}$. Hence, there are finite number of possibilities for the submodule L. This implies that M/K has a finite number of minimal classical n-absorbing submodules, a contradiction. □

In the following theorem, we determine the classical n-absorbing submodules in the R-module M where $R = R_1 \times R_2 \times \cdots \times R_k$ is decomposable, $M = M_1 \times M_2 \times \cdots \times M_k$ and M_i is an R_i-module for every $i \in \{1, \ldots, k\}$. First we need the following lemma.

Lemma 2.13. *Let M be an R-module and N_i be a classical n_i-absorbing submodule of M for every $i \in \{1, \ldots, k\}$. Then the intersection of N_i's is a classical n-absorbing submodule of M where $n = \sum_{i=1}^{k} n_i$. In particular, the intersection of n classical prime submodules of M is a classical n-absorbing submodule of M.*

Proof. Let $a_1, \ldots, a_{n+1} \in R$, $\widehat{a_i} = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n$ for every $i \in \{1, \ldots, n\}$ and let $m \in M$ such that $a_1 \cdots a_{n+1} m \in N = N_1 \cap \cdots \cap N_k$. Suppose that $\widehat{a_i} a_{n+1} m \notin N$ for each $i \in \{1, \ldots, n\}$. Then one may assume that $\widehat{a_i} a_{n+1} m \notin N_i$ for every $i \in \{1, \ldots, k\}$, and for every $j \in \{k+1, \ldots, n\}$ $\widehat{a_{k+j}} a_{n+1} m \notin N_i$ for some $i \in \{1, \ldots, k\}$. Since every N_i is a classical n_i-absorbing submodule, $a_1 a_2 \cdots a_n m \in N_i$ for every $i \in \{1, \ldots, n\}$. Thus N is a classical n-absorbing submodule of M. The"In particular" statement is clear. □

Theorem 2.14. *Let $M = M_1 \times M_2 \times \cdots \times M_k$ be an R-module where $R = R_1 \times R_2 \times \cdots \times R_k$ is a decomposable ring and $1 < k < \infty$ such that for every $i \in \{1, \ldots, k\}$ M_i is an R_i-module and let N be a proper submodule of M. If $N = N_1 \times N_2 \times \cdots \times N_k$ is a classical n-absorbing submodule of M, then either N_i is a classical n-absorbing submodule of M_i for every $i \in I \subseteq \{1, \ldots, k\}$ and $N_i = M_i$*

for every $i \in \{1,2,\dots,k\} \setminus I$, or N_i is a classical $(n-1)$-absorbing submodule of M_i for every $i \in \{1,\dots,k\}$. In fact, the converse is true if whenever N_i is a classical n_i-absorbing of M_i for $i \in I \subseteq \{1,\dots,k\}$ and $\sum_{i\in I} n_i = n$.

Proof. Let $N = N_1 \times N_2 \times \cdots \times N_k$ be a classical n-absorbing submodule of M. We use induction on k. In case $k = 2$, if $N_2 = M_2$, then $N_1 \neq M_1$, as N is proper. By considering the module $M' = \frac{M}{\{0\}\times M_2}$, it is easy to see that $N = \frac{N}{\{0\}\times M_2}$ is a classical n-absorbing submodule of M', $M' \cong M_1$ and $N' \cong N_1$, and so N_1 is a classical n-absorbing submodule of M_1. Now, if $N_1 \neq M_1$ and $N_2 \neq M_2$, then there exists $x \in M_2 \setminus N_2$. Let $a_1,\dots,a_n \in R_1$ and $\widehat{a_i} = a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_n$ for every $i \in \{1,\dots,n\}$ and let $m \in M$ such that $a_1\cdots a_n m \in N_1$. Hence

$$(a_1,1)(a_2,1)\cdots(a_n,1)(1,0)(m,x) = (a_1\cdots a_n m,0) \in N = N_1 \times N_2.$$

Since N is a classical n-absorbing submodule and $x \notin N_2$, $(\widehat{a_i},1)(1,0)(m,x) = (\widehat{a_i}m,0) \in N$ for some $i \in \{1,\dots,n\}$. Hence $\widehat{a_i}m \in N_1$ for some $i \in \{1,\dots,n\}$. Thus N_1 is a classical $(n-1)$-absorbing submodule of M_1 . Similarly one may show that N_2 is a classical $(n-1)$-absorbing submodule of M_2. Conversely, suppose that $N = N_1 \times M_2$. It is easy to see that N is a classical n-absorbing submodule of M if N_1 is either classical $(n-1)$-absorbing submodule or classical n-absorbing submodule of M_1. Suppose that $N = N_1 \times N_2$, for some classical n_1-absorbing submodule N_1 and classical n_2-absorbing submodule N_2 of M_1 and M_2, respectively, where $n_1 + n_2 = n$. By Lemma 2.13, $(N_1 \times M_2) \cap (M_1 \times N_2) = N_1 \times N_2 = N$ is a classical n-absorbing submodule of M.

We now turn to the inductive step. Assume, inductively, that the results have been proved for smaller values of k. We have shown that if $N = (N_1 \times N_2 \times \cdots \times N_{k-1}) \times N_k$ is a classical n-absorbing submodule of M, then either N_k is a classical n-absorbing submodule of M_k and $N_j = M_j$ for every $j \in \{1,2,\dots,k-1\}$, or $N_k = M_k$ and $L = N_1 \times N_2 \times \cdots \times N_{k-1}$ is a classical n-absorbing submodule of $M_1 \times M_2 \times \cdots \times M_{k-1}$, or $N = L \times N_k$ where L, N_k are classical $(n-1)$-absorbing submodules. Suppose that L is a classical n-absorbing submodule. Apply the inductive assumption to L to see that either N_i's are classical $(n-1)$-absorbing submodules or N_i is a classical n-absorbing submodule of M_i for every $i \in I$ where $I \subseteq \{1,2,\dots,k-1\}$ and $N_i = M_i$ for every $i \in \{1,2,\dots,k-1\} \setminus I$. In fact, the converse statement, that is, if N_i is a classical n_i-absorbing of M_i for $i \in I \subseteq \{1,\dots,k\}$ and $\sum_{i\in I} n_i = n$, then N is a classical n-absorbing submodule of M, is true since

$$(N_1 \times M_2 \times \cdots \times M_k) \cap (M_1 \times N_2 \times M_3 \times \cdots \times M_k) \cap \cdots \cap (M_1 \times \cdots \times M_{k-1} \times N_k) \cong N_1 \times N_2 \times \cdots \times N_k = N.$$

Hence the result follows immediately from Lemma 2.13 and this completes the inductive step. □

3. Properties of Classical n-Absorbing Submodules and Classical n-Absorbing Avoidance Theorem

In this section, we prove the classical n-absorbing avoidance theorem for submodules.

In [11, Conjecture 1] the authors gave the following conjecture: Let R be a commutative ring and let M be an R-module. If N is an n-absorbing submodule of M, then $(N :_R M)$ is an n-absorbing ideal of R. In the following theorem we prove this conjecture for a classical n-absorbing submodule.

Theorem 3.1. *If N is a classical n-absorbing submodule of an R-module M, then $(N :_R M)$ is an n-absorbing ideal of R.*

Proof. Let $a_1,\dots,a_{n+1} \in R$, $\widehat{a_i} = a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{n+1}$, for every $i \in \{1,\dots, n+1\}$ and suppose that $a_1\cdots a_{n+1} \in (N :_R M)$. We show that $(N :_R M)$ is an n-absorbing ideal of R. For each $i \in \{1,\dots,n+1\}$, set

$$A_i = \{m \in M : \widehat{a_i}m \in N\} \text{ and } B_i = \{m \in M : \widehat{a_i}m \notin N\}.$$

It is easy to see that the sets A_i's,B_i's are submodules of M and $M = A_i \cup B_i$ for every $i \in \{1,\dots, n+1\}$. Hence either $M \subseteq A_i$ or $M \subseteq B_i$ for every $i \in \{1,\dots,n+1\}$, and so, either $M = A_i$ or $M = B_i$ for for every $i \in \{1,\dots,n+1\}$. If $M = A_i$ for some $i \in \{1,\dots,n+1\}$, then we are done. Hence assume that $M = B_i$ for every $i \in \{1,\dots,n+1\}$. Since N is a classical n-absorbing submodule of M and $a_1\cdots a_{n+1}m \in N$ for every $m \in M$, we must have $\widehat{a_i}m \in N$ for some $i \in \{1,\dots,n+1\}$, which is a contradiction, since $M = \bigcup_{i=1}^{n+1} B_i$. Hence $M = A_i$, for some $i \in \{1,\dots,n+1\}$. Thus $(N :_R M)$ is an n-absorbing ideal of R. □

Corollary 3.2. *Let K,N be submodules of the R-module M, $a_1,\dots,a_{n+1} \in R$, $\widehat{a_i} = a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{n+1}$, for every $i \in \{1,\dots,n+1\}$ and suppose that $a_1\cdots a_{n+1}K \subseteq N$. If N is a classical n-absorbing submodule of M, then $\widehat{a_i}K \subseteq N$, for some $i \in \{1,\dots,n+1\}$.*

Proof. The proof is similar to that of Theorem 3.1. □

Remark 3.3. Let R be a *um*-ring and N be a classical n-absorbing submodule of an R-module M. By Theorem 2.6 and Corollary 3.2, for every submodule K of M that is not contained in N, $(N :_R K)$ is an n-absorbing ideal of R. Hence, it follows from Corollary 2.7 (1) that every n-absorbing ideal of R is a strongly n-absorbing ideal of R, and this gives an affirmative answer to conjecture one in [6, page 41], when R is *um*-ring.

Theorem 3.4. *Let M be an R-module and N be a classical n-absorbing submodule of M such that $(N :_R M) \neq \sqrt{(N :_R M)}$. Then $N_{r^{t-1}} = (N :_M r^{t-1})$ is a classical $(n-t+1)$-absorbing submodule of M whenever $r \in \sqrt{(N :_R M)} \setminus (N :_R M)$ and t is the smallest positive integer in which $r^t \in (N :_R M)$. In particular, $N_{r^{n-1}} = (N :_M r^{n-1})$ is a classical prime submodule of M whenever $r^n \in (N :_R M)$ and $r^{n-1} \notin (N :_R M)$.*

Proof. Let N be a classical n-absorbing submodule of M, $a_1, \ldots, a_{n-t+2} \in R$ and $\widehat{a_i} = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{n-t+2}$ for every $i \in \{1, \ldots, n-t+2\}$ and let $m \in M$ such that $a_1 \cdots a_{n-t+2} m \in N_{r^{t-1}}$. Then $r^{t-1} a_1 \cdots a_{n-t+2} m \in N$. Since N is a classical n-absorbing submodule, we have either $r^{t-1} \widehat{a_i} m \in N$ for some $i \in \{1, \ldots, n-t+2\}$ or $r^{t-2} a_1 \cdots a_{n-t+2} m \in N$. The desired result is clear if $r^{t-1} \widehat{a_i} m \in N$ for some $i \in \{1, \ldots, n-t+2\}$. Assume that $r^{t-1} \widehat{a_i} m \notin N$ for every $i \in \{1, \ldots, n-t+2\}$ and $r^{t-2} a_1 \cdots a_{n-t+2} m \in N$. This means that $r r^{t-2} a_1 \cdots a_{n-t+1} (a_{n-t+2} + r) m \in N$, and since N is a classical n-absorbing submodule, we get

$$r^{t-2} a_1 \cdots a_{n-t+1} (a_{n-t+2} + r) m = (r^{t-2} a_1 \cdots a_{n-t+1} a_{n-t+2} m) + (r^{t-1} a_1 \cdots a_{n-t+1} m) \in N.$$

Therefore, $r^{t-1} a_1 \cdots a_{n-t+1} m \in N$, which is a contradiction. Hence $r^{t-1} \widehat{a_i} m \in N$ for some $i \in \{1, \ldots, n-t+2\}$ and thus $N_{r^{t-1}} = (N :_M r^{t-1})$ is a classical $(n-t+1)$-absorbing submodule of M.

The "in particular" statement is easily proved as above (case $t = n$). □

Lemma 3.5. *Let $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n (n \geq 2)$ be an efficient covering by submodules of the R-module M with the property that whenever N_j is classical n_j-absorbing, then there exists $r \in R$ such that $r^{n_j} \in (N_j :_R M)$, $r^{n_j - 1} \notin (N_j :_R M)$ and $(N_i :_R M) \nsubseteq ((N_j :_M r^{n_j-1}) :_R m)$ for every $m \in M \setminus N_j$ and $i \neq j$. Then no N_j is a classical n-absorbing submodule for $j \in \{1, \ldots, n\}$.*

Proof. Suppose that N_j is a classical n-absorbing submodule for some $j \in \{1, \ldots, n\}$ and look for a contradiction. It is easy to see that

$$N = (N \cap N_1) \cup (N \cap N_2) \cup \cdots \cup (N \cap N_n)$$

is an efficient union, otherwise $N \cap N_i \subseteq N \cap N_k$ for some $i \neq k$, and this implies $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_{k-1} \cup N_{k+1} \cup \cdots \cup N_n$ which is a contradiction. Thus, there exists an element $m_j \in N \setminus N_j$ for every $j \in \{1, \ldots, n\}$. Since $N = (N \cap N_1) \cup (N \cap N_2) \cup \cdots \cup (N \cap N_n)$ is an efficient union, we conclude that $(\bigcap_{i \neq j} N_i) \cap N \subseteq N_j \cap N$, by [18, Lemma 2.1]. Since N_j is a classical n-absorbing submodule of M and $m_j \in M \setminus N_j$, there exists $r \in R$ such that $r^{n_j} \in (N_j :_R M)$, $r^{n_j-1} \notin (N_j :_R M)$ and $(N_i :_R M) \nsubseteq ((N_j :_M r^{n_j-1}) :_R m_j)$ for every $i \neq j$. Therefore, there exists $y_i \in (N_i : M) \setminus ((N_j :_M r^{n_j-1}) :_R m_j)$ for every $i \neq j$. Let $y = \prod_{i \neq j} y_i$. Then $y = \prod_{i \neq j} y_i \in \bigcap_{i \neq j} (N_i : M)$ but $y = \prod_{i \neq j} y_i \notin ((N_j :_M r^{n_j-1}) :_R m_j)$, as $(N_j :_M$

r^{n_j-1}) is a classical prime submodule, by Theorem 3.4. Let $x = yr^{n_j-1}m_j$. Then $x \in N \cap (\bigcap_{i \neq j} N_i)$. But $x \notin N \cap N_j$, as $x \in N \cap N_j$ implies $y_i \in ((N_j :_M r^{n_j-1}) :_R m_j)$ for some $i \neq j$, a contradiction. Thus no N_j is a classical n-absorbing submodule of M. $\square$

Now, we prove the Classical n-Absorbing Avoidance Theorem.

Theorem 3.6. (*Classical n-Absorbing Avoidance Theorem for Submodules*) *Let $N_1, N_2, \dots, N_n$ be submodules of the R-module M with the property that whenever N_j is classical n_j-absorbing, then there exists $r \in R$ such that $r^{n_j} \in (N_j :_R M)$, $r^{n_j-1} \notin (N_j :_R M)$ and $(N_i :_R M) \nsubseteq ((N_j :_M r^{n_j-1}) :_R m)$ for every $m \in M \setminus N_j$ and $i \neq j$ and suppose that at most two of $N_1, N_2, \dots, N_n$ are not classical n-absorbing submodules. If N is a submodule of M such that $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$, then $N \subseteq N_j$ for some $j \in \{1, \dots, n\}$.*

Proof. Let $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ be a covering consisting of submodules of M such that at least $n-2$ of $N_1, N_2, \dots, N_n$ are classical n-absorbing submodules. If we reduce the covering to an efficient, the hypothesis remains valid. Then we may assume that the covering is efficient. If $n = 2$, then it is obvious. Suppose that $n > 2$. Since the covering is efficient, by Lemma 3.5 no N_j is a classical n-absorbing submodule, which is a contradiction. Therefore $n < 2$, and thus $N \subseteq N_j$ for some $j \in \{1, \dots, n\}$. $\square$

Corollary 3.7. (*n-Absorbing Avoidance Theorem for Submodules*) *Let $N_1, N_2, \dots, N_n$ be submodules of the R-module M with the property that whenever N_j is n_j-absorbing, then there exists $r \in R$ such that $r^{n_j} \in (N_j :_R M)$, $r^{n_j-1} \notin (N_j :_R M)$ and $(N_i :_R M) \nsubseteq ((N_j :_M r^{n_j-1}) :_R m)$ for every $m \in M \setminus N_j$ and $i \neq j$ and suppose that at most two of $N_1, N_2, \dots, N_n$ are not n-absorbing submodules. If N is a submodule of M such that $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$, then $N \subseteq N_j$ for some $j \in \{1, \dots, n\}$.*

Using the classical n-absorbing avoidance theorem, we state the following corollaries.

Corollary 3.8. *Let $N, N_1, \dots, N_n$ be submodules of the R-module M such that for every $i \in \{1, \dots, n\}$, N_i is classical n_i-absorbing and there exists $r \in R$ such that $r^{n_j} \in (N_j :_R M)$, $r^{n_j-1} \notin (N_j :_R M)$ and $(N_i :_R M) \nsubseteq ((N_j :_M r^{n_j-1}) :_R s)$ for every $s \in M \setminus N_j$ and $i \neq j$. If there exists $m \in M$ such that $Rm + N \nsubseteq \bigcup_{i=1}^{n} N_i$, then there exists $x \in N$ such that $m + x \notin \bigcup_{i=1}^{n} N_i$.*

Proof. Suppose that $m \in \bigcap_{i=1}^{k} N_i$ but $m \notin \bigcup_{i=k+1}^{n} N_i$. We distinguish between the cases $k = 0$ and $k \neq 0$. Suppose $k = 0$. Then $m = m + 0 \notin \bigcup_{i=1}^{n} N_i$ and so we are done. Thus, assume that $1 \leq k$. First, we show that $N \nsubseteq \bigcup_{i=1}^{k} N_i$, for otherwise by the classical n-absorbing avoidance theorem for submodules, $N \subseteq N_i$ for some $i \in \{1,\dots,k\}$, a contradiction. So, there exists $a \in N \setminus \bigcup_{i=1}^{k} N_i$. Next, we show that $\bigcap_{i=k+1}^{n}(N_i :_R M) \nsubseteq \bigcup_{j=1}^{k}((N_j :_M r^{n_j-1}) :_R s)$ for every $s \in M \setminus N_j$. Suppose that $\bigcap_{i=k+1}^{n}(N_i :_R M) \subseteq \bigcup_{j=1}^{k}((N_j :_M r^{n_j-1}) :_R s)$ for some $s \in M \setminus N_j$ and look for a contradiction. By Theorem 3.4, $(N_j :_M r^{n_j-1})$'s are classical prime submodules, then by [5, Lemma 2.1] $((N_j :_M r^{n_j-1}) :_R s)$'s are prime ideals of R. Thus by the prime avoidance theorem, $\bigcap_{i=k+1}^{n}(N_i :_R M) \subseteq ((N_j :_M r^{n_j-1}) :_R s)$ for some $j \in \{1,\dots,k\}$, and so $\bigcap_{i=k+1}^{n}\sqrt{(N_i :_R M)} \subseteq ((N_j :_M r^{n_j-1}) :_R s)$ for some $j \in \{1,\dots,k\}$. We have seen in Theorem 3.1 that $(N_i :_R M)$ is an n_i-absorbing ideal of R for every $i \in \{k+1,\dots,n\}$, then by [4, Theorem 2.5] there are at most n_i prime ideals of R minimal over $\sqrt{(N_i :_R M)}$ for every $i \in \{k+1,\dots,n\}$. We conclude by [17, Lemma 3.55] that one of these prime ideals which are minimal over $\sqrt{(N_i :_R M)}$ is contained in $((N_j :_M r^{n_j-1}) :_R s)$ for some $i \in \{k+1,\dots,n\}$. This means that $(N_i :_R M) \subseteq ((N_j :_M r^{n_j-1}) :_R s)$ for some $i \in \{k+1,\dots,n\}$, which contradicts the hypothesis. Hence, there exists $b \in \bigcap_{i=k+1}^{n}(N_i :_R M)$ such that $b \notin \bigcup_{j=1}^{k}((N_j :_M r^{n_j-1}) :_R s)$ for every $s \in M \setminus N_j$. Let $x = ab$. Then $x \in N$. We also have $x \in \bigcap_{i=k+1}^{n} N_i$, but $x \notin \bigcup_{j=1}^{k} N_j$, for otherwise $x = ab \in N_j$ for some $j \in \{1,\dots,k\}$, then $b \in (N_j : a) \subseteq \bigcup_{j=1}^{k}((N_j :_M r^{n_j-1}) :_R a)$ for some $j \in \{1,\dots,k\}$, a contradiction. Thus $x \in \bigcap_{i=k+1}^{n} N_i \setminus \bigcup_{i=1}^{k} N_i$. Since $m \in \bigcap_{i=1}^{k} N_i \setminus \bigcup_{i=k+1}^{n} N_i$, we conclude that $m + x \notin \bigcup_{i=1}^{n} N_i$. □

Corollary 3.9. *Let N be a finitely generated submodule of the R-module M such that $N = \langle r_1, r_2, \dots, r_s\rangle$. Let $N_1, N_2, \dots, N_n$ be submodules of M such that for every $i \in \{1,\dots,n\}$, N_i is classical n_i-absorbing, $N \nsubseteq N_i$ and there exists $r \in R$ such that $r^{n_j} \in (N_j :_R M)$, $r^{n_j-1} \notin (N_j :_R M)$ and $(N_i :_R M) \nsubseteq ((N_j :_M r^{n_j-1}) :_R m)$ for every $m \in M \setminus N_j$ and $i \neq j$. Then there exist $b_2, \dots, b_s \in R$ such that $\alpha = r_1 + b_2 r_2 + \dots + b_s r_s \notin \bigcup_{i=1}^{n} N_i$.*

Proof. We argue by induction on n. Suppose that $N \nsubseteq N_i$ and there exists $r \in R$ such that $r^{n_j} \in (N_j :_R M)$, $r^{n_j-1} \notin (N_j :_R M)$ and $(N_i :_R M) \nsubseteq ((N_j :_M r^{n_j-1}) :_R m)$ for every $m \in M \setminus N_j$ and $i \neq j$. The result being clear in the case $n = 1$. So, suppose inductively that $n > 1$ and the result has been proved for smaller values than n. Then there exist $a_2, \dots, a_s \in R$ such that $x = r_1 + a_2 r_2 + \dots + a_s r_s \notin \bigcup_{i=1}^{n-1} N_i$. If $x \notin N_n$, then $x \notin \bigcup_{i=1}^{n} N_i$ and there is nothing to prove. So suppose that $x \in N_n$. If $r_2, \dots, r_s \in N_n$, then $r_1 \in N_n$, and this contradicts the hypothesis that $N \nsubseteq N_n$. Thus for some i, we assume $r_i \notin N_n$. Without loss of generality, suppose that $r_2 \notin N_n$. By the hypothesis, there exists $r \in R$ such that $r^{n_n} \in (N_n :_R M)$, $r^{n_n-1} \notin (N_n :_R M)$ and $(N_i :_R M) \nsubseteq ((N_n :_M r^{n_n-1}) :_R r_2)$ for every $i \neq n$. Hence, there exists $y_i \in (N_i :_R M) \setminus ((N_n :_M r^{n_n-1}) :_R r_2)$ for every $i \neq n$. Let $y = \prod_{i=1}^{n-1} y_i$. Then $y \in (N_i :_R M)$ for every $i \neq n$ but $y \notin ((N_n :_M r^{n_n-1}) :_R r_2)$, since $(N_n :_M r^{n_n-1})$ is a classical prime submodule of M by Theorem 3.4. Therefore

$y \in (N_i :_R M) \setminus ((N_n :_M r^{n_n-1}) :_R r_2)$ for every $i \neq n$. Let $\alpha = r_1 + (a_2 + y)r_2 + \cdots + a_s r_s$. We consider two cases.

Case one: Suppose that $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$. So, by the classical n-absorbing Avoidance Theorem for submodules, $N \subseteq N_j$, for some $j \in \{1, \ldots, n\}$, which is a contradiction.

Case two: Suppose that $N \nsubseteq N_1 \cup N_2 \cup \cdots \cup N_n$. Then by a similar argument as above, we assume $r_2 \notin N_n$. Thus $\alpha = x + yr_2 \notin \bigcup_{i=1}^{n} N_i$. Hence, this completes the inductive step, and so the result is proved. □

4. Classical n-Absorbing Submodules in Specific Modules

In this section, we study classical n-absorbing submodules in some classes of modules.

There are interesting results in [4] on n-absorbing ideals. We extend some of them for classical n-absorbing submodules in the next few results.

We start with the following lemma.

Lemma 4.1. *Let I be an n-absorbing ideal of R such that R is a valuation ring, and let M be a faithful multiplication R-module. Then IM is a classical n-absorbing submodule of M.*

Proof. Let $a_1, \ldots, a_{n+1} \in R$, $\widehat{a_i} = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{n+1}$ for every $i \in \{1, \ldots, n+1\}$ and let $m \in M$ such that $a_1 \cdots a_n a_{n+1} m \in IM$. The result is clear if $(IM :_R \widehat{a_i} m) = R$, for some $i \in \{1, \ldots, n+1\}$. Suppose that $(IM :_R \widehat{a_i} m)$'s are proper ideals of R. We look for a contradiction. Since R is a valuation ring, $\bigcup_{i=1}^{n+1} (IM :_R \widehat{a_i} m)$ is a proper ideal of R, and so there is a maximal ideal P of R such that $\bigcup_{i=1}^{n+1} (IM :_R \widehat{a_i} m) \subseteq P$. Let $T_P(M) = \{m' \in M : (1-x)m' = 0$ for some $x \in P\}$ (as defined in [1, page 756]), and so $\widehat{a_i} m \notin T_P(M)$ for every $i \in \{1, \ldots, n+1\}$, for otherwise $\widehat{a_i} m \in T_P(M)$ for some $i \in \{1, \ldots, n+1\}$. This implies that $(1-x)\widehat{a_i} m = 0$ for some $x \in P$, which means that $(1-x)\widehat{a_i} m \in IM$ and hence $(1-x) \in (IM :_R \widehat{a_i} m) \subseteq P$. This contradicts the fact that P is maximal. It follows from [1, Theorem 1.2] that M is P-cyclic, and so, there exist $x \in P$ and $m' \in M$ such that $(1-x)M \subseteq Rm'$. This implies $(1-x)m = rm'$ and $(1-x)a_1 \cdots a_n a_{n+1} m = sm'$ for some $r \in R$ and $s \in I$, so $(a_1 \cdots a_n a_{n+1} r - s)m' = 0$ and since $(1-x)M \subseteq Rm'$, we get $(1-x)(a_1 \cdots a_n a_{n+1} r - s)M = 0$. Therefore, $(1-x)(a_1 \cdots a_n a_{n+1} r - s) \in (0 :_R M) = 0$ since M is a faithful module. Hence $(1-x)a_1 \cdots a_n a_{n+1} r = (1-x)s \in I$. But I is an n-absorbing ideal of R and $(IM :_R \widehat{a_i} m)$'s are proper ideals of R, then we have either $\widehat{a_i} \in I$ for some $i \in \{1, \ldots, n+1\}$ or $a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_{n+1} r \in I$ for some $i, j \in \{1, \ldots, n+1\}$ or $a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_{n+1}(1-x) \in I$ for some $i, j \in \{1, \ldots, n\}$ or $a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots \cdots a_{k-1} a_{k+1} \cdots a_{n+1}(1-x)r \in I$ for some $i, j, k \in$

$\{1,\ldots,n+1\}$. If $\widehat{a_i} \in I$ for some $i \in \{1,\ldots,n+1\}$, then we get a contradiction. If $a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{j-1}a_{j+1}\cdots a_{n+1}r \in I$ for some $i, j \in \{1,\ldots,n+1\}$, then $a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{j-1}a_{j+1}\cdots a_{n+1}(1-x)m \in IM$ and hence $1-x \in (IM :_R a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{j-1}a_{j+1}\cdots a_{n+1}m) \subseteq (IM :_R \widehat{a_i}m) \subseteq P$, a contradiction. If

$$a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{j-1}a_{j+1}\cdots a_{n+1}(1-x) \in I$$

for some $i, j \in \{1,\ldots,n+1\}$, then $a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{j-1}a_{j+1}\cdots a_{n+1}(1-x)m \in IM$ and hence $1-x \in (IM :_R \widehat{a_i}m) \subseteq P$, a contradiction. If

$$a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{j-1}a_{j+1}\cdots a_{k-1}a_{k+1}\cdots a_{n+1}(1-x)r \in I$$

for some $i, j, k \in \{1,\ldots,n+1\}$, then $(1-x)^2 \in (IM :_R \widehat{a_i}m) \subseteq P$, a contradiction. Thus IM is a classical n-absorbing submodule of M. $\square$

Theorem 4.2. *Let M be a finitely generated faithful multiplication module over the Noetherian integral domain R and N a submodule of M. If M is a Dedekind module, then for every classical n-absorbing submodule N of M, we have $N = N_1\cdots N_m$ where N_i's are maximal submodules of M and $1 \leq m \leq n$. The converse holds if R is a valuation ring.*

Proof. Let M be a Dedekind module and N a classical n-absorbing submodule of M. Since $(N :_R M)$ is an n-absorbing ideal of R by Theorem 3.1, it follows from [4, Theorem 5.1] and [16, Theorem 3.5] that $(N :_R M) = M_1\cdots M_m$ for maximal ideals $M_1,\ldots,M_m$ of R with $1 \leq m \leq n$. By [1, p. 756], $N = (N :_R M)M = M_1\cdots M_mM$. It again follows from [1, Theorem 2.5] that $N_i = M_iM$ is a maximal submodule of M for every $i \in \{1,\ldots,m\}$. Thus $N = N_1\cdots N_m$ where N_i's are maximal submodules of M with $1 \leq m \leq n$. We now turn to the converse statement, let R be a valuation ring and I an n-absorbing ideal of R. By Lemma 4.1, IM is a classical n-absorbing submodule of M, and so $IM = N_1\cdots N_m$ where N_i's are maximal submodules of M and $1 \leq m \leq n$. Since M is a non-zero multiplication R-module, by [1, Theorem 2.5] for every $i \in \{1,\ldots,m\}$, $N_i = P_iM$ where P_i is a maximal ideal of R. Therefore, $IM = P_1\cdots P_mM$. Since M is finitely generated, one may use [1, Theorem 3.1] to see that $I = P_1\cdots P_m$. Hence, by [4, Theorem 5.1], R is a Dedekind domain. Thus, by [16, Theorem 3.4], M is a finitely generated Dedekind R-module. $\square$

In the following, we extend these results to Prüfer modules.

Theorem 4.3. *Let M be a faithful multiplication Prüfer module over the ring R and N a submodule of M. If N is classical n-absorbing for some positive integer n, then N is a product of prime submodules of M. The converse is true if R is a valuation ring and M is finitely generated.*

Proof. Let N be a classical n-absorbing submodule of M. Then $(N :_R M)$ is an n-absorbing ideal of R by Theorem 3.1, and so $(N :_R M)$ is a product of prime ideals of R by [4, Theorem 5.7] and [13, Theorem 3.6]. Let $(N :_R M) = P_1 \cdots P_k$ where P_i's are prime and $k \in \mathbb{N}^*$. Hence, by [1, Corollary 2.11], $N = (N :_R M)M = P_1M \cdots P_kM$ and N is a product of prime submodules of M. Conversely, suppose that R is a valuation ring, M is finitely generated and N is a product of prime submodules of M. It follows from [1, Corollary 2.11] that $N = (N :_R M)M = P_1M \cdots P_kM$ where P_iM's are prime submodules and $k \in \mathbb{N}^*$. Since M is a Prüfer module, we can use [13, Theorem 3.6] to see that R is a Prüfer domain. Hence, by [4, Theorem 5.7] and [1, Theorem 3.1], $(N :_R M) = P_1 \cdots P_k$ and $(N :_R M)$ is an n-absorbing ideal of R for some positive integer n. Therefore, $N = (N :_R M)M$ is a classical n-absorbing submodule of M for some positive integer n, by Lemma 4.1. □

Next classical n-absorbing submodules of valuation modules are characterized.

Theorem 4.4. *Let M be a Bézout finitely generated faithful multiplication module over an integral domain R and N a classical n-absorbing submodule of M such that M-$rad(N) = P$ for a prime submodule P of M. Then $P^n \subseteq N$. In particular, this is also true if M is a valuation module.*

Proof. Let N be a classical n-absorbing submodule of M such that M-$rad(N) = P$ for a prime submodule P of M. It follows from [1, p. 756] that $N = (N :_R M)M$. Since M-$rad(N) = P$, [1, Corollary 2.11] follows that M-$rad(N) = P = pM$, for some prime ideal p of R. Since M is a multiplication module, by [1, Theorem 2.12] we get $\sqrt{(N :_R M)}M = pM$, and so $\sqrt{(N :_R M)} = p$, as M a is finitely generated faithful module [1, Theorem 3.1]. By Theorem 3.1, $(N :_R M)$ is an n-absorbing ideal of R and since M is a Bézout finitely generated faithful multiplication module over an integral domain R, by [2, Proposition 2.2] and [4, Theorem 5.1], $p^n = (\sqrt{(N :_R M)})^n \subseteq (N :_R M)$. Thus $P^n = p^nM = (\sqrt{(N :_R M)})^nM \subseteq (N :_R M)M = N$.

The"In particular" statement is clear. □

Theorem 4.5. *Let M be a valuation finitely generated faithful multiplication module over an integral domain R and N a submodule of M. Then the following statements are equivalent*

(1) *N is a classical n-absorbing submodule of M.*

(2) *N is a p-primary submodule of M for some prime ideal p of R with $p^nM \subseteq N$.*

(3) *$N = P^m$ for some prime submodule P $(= M$-$rad(N))$ of M such that $1 \leq m \leq n$.*

Proof. (1) $\Rightarrow$ (2) Let N be a classical n-absorbing submodule of M. Since M is a valuation module, it follows from [2, Proposition 2.2] that R is a valuation domain. This means that $\sqrt{(N :_R M)} = p$ where p is a divided prime ideal of R [14, Theorem 17.1 (2)]. By Theorem 3.1, $(N :_R M)$ is an n-absorbing ideal of R, and so $(N :_R M)$ is a p-primary ideal of R, by [4, Theorem 3.2]. Hence, N is a p-primary submodule of M and $p^n M \subseteq (N :_R M)M = N$, by [3, Corollary 1] and [4, Lemma 5.4], respectively.

(2) $\Rightarrow$ (3) Let N be a p-primary submodule of M for some prime ideal p of R with $p^n M \subseteq N$. Since the ideal $(N :_R M)$ is a p-primary ideal of R and as R is a valuation domain, it follows from [14, Theorem 17.3 (b)] that $(N :_R M) = p^m$ with $1 \leq m \leq n$. Thus $N = (N :_R M)M = p^m M = P^m$ with $1 \leq m \leq n$.

(3) $\Rightarrow$ (1) It follows from [11, Theorem 6].

□

Acknowledgement

The authors would like to thank the referees for the helpful suggestions.

REFERENCES

[1] Z. Abd El-Bast, P.F. Smith, *Multiplication modules*, Comm. Algebra, 16 (1988), 755–779.

[2] M.M. Ali, *Invertibility of multiplication modules*, New Zealand J. Math., 35 (2006), 17–29.

[3] S.E. Atani, F. Callalp, Ü. Tekir, *A short note on the primary submodules of multiplication modules*, Inter. J. Algebra, 8 (2007), 381–384.

[4] D.F. Anderson, A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Alg., 39 (5), 1646–1672, (2011).

[5] A. Azizi, *On prime and weakly prime submodules*, Vietnam J. Math., 36 (3) (2008), 315–325.

[6] A. Badawi, *n-Absorbing Ideals of Commutative Rings and Recent Progress on Three Conjectures: A Survey, In Rings, Polynomials, and Modules*. Springer, Cham, (2017), 33–52.

[7] A. Badawi, *On* 2*-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., 75 (2007), 417–429.

[8] M. Behboodi, *A generalization of Bears lower nilradical for modules*, J. Algebra Appl., 6 (2) (2007), 337–353.

[9] M. Behboodi, *On weakly prime radical of modules and semi-compatible modules*, Acta Math. Hungar., 113 (3) (2006), 239–250.

[10] H.S. Butts, P. Quartararo, *Finite unions of ideals and modules*, Proc. Amer. Math. Soc., 52 (1975), 91–96.

[11] A.Y. Darani, F. Soheilnia, *On n-absorbing submodules*, Math. Commun., 17 (2012), 547–557.

[12] M.K. Dubey, P. Aggarwal, *On n-absorbing submodules of modules over commutative rings*, Beitrge zur Algebra und Geometrie/Contributions to Algebra and Geometry, Springer Berlin, vol. 57, (2016), 679–690.

[13] Y. Hwan Cho, *On multiplication modules (II)*, Comm. Korean Math. Soc., 13 (1998), 727–733.

[14] R. Gilmer, *Multiplicative Ideal Theory*. Queen's Papers in Pure and Appl. Math. Vol. 90, Kingston, ON: Queens University, (1992).

[15] J. Moghaderi, R. Nekooei, *Valuation, discrete valuation and Dedekind modules*, Int. Electron. J. Algebra, 8 (2010), 18–29.

[16] A.G. Naoum, F.H. Al-Alwan, *Dedekind modules*, Comm. Algebra, 24 (1996), 397–412.

[17] R.Y. Sharp, *Steps in commutative algebra*, 2nd ed. London Mathematical Society Student Texts, 51. Cambridge University Press, Cambridge, 2000.

[18] C.P. Lu, *Unions of prime submodules*, Houston J. Math., 23(2) (1997), 203–213.

R. NIKANDISH
Department of Mathematics, Jundi-Shapur University of Technology
P.O. BOX 64615-334, Dezful, Iran
e-mail: r.nikandish@ipm.ir

M. J. NIKMEHR
Faculty of Mathematics, K.N. Toosi University of Technology
P.O. BOX 16315-1618, Tehran, Iran
e-mail: nikmehr@kntu.ac.ir

A. YASSINE
Faculty of Mathematics, K.N. Toosi University of Technology
P.O. BOX 16315-1618, Tehran, Iran
e-mail: yassine_ali@email.kntu.ac.ir